Answers to Chapter 10 H/W part 2

(7) \( t^2 x'' - 3 t x' + (4 - t) x = 0 \). \[ \mathbb{\circ} \]

Clear that \( t = 0 \) is a singular point of \( \mathbb{\circ} \).

Now \( \lim_{t \to 0} \left[ \frac{t^3 t}{t^2} \right] = -3 \) \& \( \lim_{t \to 0} t^2 \left( \frac{4-t}{t^2} \right) = 4 \)

\[ \therefore \text{ both } t \left( \frac{-3 t}{t^2} \right) \text{ \\& } t^2 \left( \frac{4-t}{t^2} \right) 	ext{ are analytic at } t = 0, \]

hence \( t = 0 \) is a regular singular point of \( \mathbb{\circ} \).

By the Frobenius method, there is at least one soln of the form

\[ \chi(t) = t^r \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} a_k t^{k+r} \] \[ \mathbb{\circ} \]

for some constant \( r \) \& \( a_k \). Put \( \mathbb{\circ} \) into \( \mathbb{\circ} \),

\[ t^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) t^{k+r-2} \]

\[ -3 t \sum_{k=0}^{\infty} a_k (k+r) t^{k+r-1} + (4-t) \sum_{k=0}^{\infty} \frac{a_k}{k} t^{k+r} = 0. \]

\[ \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) t^{k+r} \]

\[ -3 \sum_{k=0}^{\infty} a_k (k+r) t^{k+r} \]

\[ + 4 \sum_{k=0}^{\infty} \frac{a_k}{k} t^{k+r} \]

\[ - \sum_{k=0}^{\infty} a_k t^{k+r+1} = 0. \]

Replace \( k \) by \( k+1 \) in the last summation. (ie. \( j = k+1 \))

\[ \sum_{k=0}^{\infty} a_k \left[ (k+r)(k+r-1) - 3(k+r) + 4 \right] t^{k+r} \]

\[ - \sum_{j=1}^{\infty} a_{j-1} t^{j+r} = 0 \]

Next replace \( j \) by \( k \) in the last summation.
\[ k=0 : \quad r(r-1) - 3r + 4 = 0 \quad \text{(indicial eqn).} \]
\[ (r-2)^2 = 0, \quad \therefore \ r = 2 \quad \text{(double root).} \]

\[ k \geq 1 : \quad \text{Put } \ r = 2, \]
\[ \therefore \quad \left[ (k+2)(k+1) - 3(k+2) + 4 \right] A_k - A_{k-1} = 0 \]
\[ \therefore \quad k^2 A_k = A_{k-1} \]
\[ A_k = \frac{A_{k-1}}{k^2}. \]

\[ \therefore \quad A_1 = \frac{A_0}{1^2}, \quad A_2 = \frac{A_1}{2^2} = \frac{A_0}{2^2}, \quad A_3 = \frac{A_2}{3^2} = \frac{1}{2^2 \cdot 3^2} A_0 \]
\[ A_k = \frac{1}{1^2 \cdot 2^2 \cdots k^2} A_0 = \frac{1}{(k!)^2} A_0. \]

\[ \therefore \quad \chi(t) = t^r \sum_{k=0}^{\infty} \frac{A_k}{k!} t^k \]
\[ = A_0 t^2 \left( \sum_{k=0}^{\infty} \frac{1}{(k!)^2} t^k \right) \]

We will not seek the 2nd independent soln. From P.211 in the text, it is of the form
\[ \chi(t) \log t + t^r \sum_{k=0}^{\infty} b_k t^k \]
\[ = \left[ t^2 \sum_{k=0}^{\infty} \frac{1}{(k!)^2} t^k \right] \log t + t^2 \sum_{k=0}^{\infty} b_k t^k \]

where \( b_k \) needs to be determined by putting this form in the governing eqn.