MATH 3410 Midterm

Formula sheet

linear homogeneous equation (H): \( x'' + p(t)x' + q(t)x = 0 \)
linear non-homogeneous equation (NH): \( x'' + p(t)x' + q(t)x = f(t) \)

Abel's theorem: \( W(t) = ce^{-\int_{t_0}^{t} p(\xi) d\xi} \)

reduction of order: \( x_2(t) = x_1(t) \int \frac{e^{-\int_{t_0}^{t} p(\xi) d\xi}}{x_1^2(t)} dt \)

variation of parameter: \( x_p(t) = x_1(t) \int -\frac{x_2(t)f(t)}{W(t)} dt + x_2(t) \int \frac{x_1(t)f(t)}{W(t)} dt \)

Linear homogeneous system (H): \( \frac{dx}{dt} = A(t)x \).
Abel's theorem for a system: \( W(t) = W(t_0)e^{\int_{t_0}^{t} (a_{11}(\xi) + a_{22}(\xi)) d\xi} \).

Predator-prey system with positive constants \( a, b, c, d \):

\[
\begin{align*}
x' &= ax - bxy, \\
y' &= -cy + dxy.
\end{align*}
\]

Then \( H(x, y) = dx + by - c\log x - a\log y \) is constant along solution trajectories.

For \( x'' = f(x) \), the potential energy \( P(x) = -\int_{x_0}^{x} f(t) dt \) for any convenient \( x_0 \).

Trigonometric identities: \( \cos 2\theta = 1 - 2\sin^2 \theta = 2\cos^2 \theta - 1 \). This gives

\[
\int_{0}^{\pi} \sin^2 nx \, dx = \int_{0}^{\pi} \cos^2 nx \, dx = \int_{0}^{\pi} \sin^2 (n + 1/2)x \, dx = \int_{0}^{\pi} \cos^2 (n + 1/2)x \, dx = \pi/2
\]
for any \( n = 1, 2, 3, \ldots \).
(a) \( \text{Let } A = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix} \) Find its (real integer) eigenvalues and the corresponding eigenvectors.

\[
\det (A - \lambda I) = \begin{vmatrix} -4 - \lambda & 1 \\ 2 & -3 - \lambda \end{vmatrix} = 0
\]

\[
\lambda^2 + 7\lambda + 12 - 2 = 0
\]

\[
\lambda^2 + 7\lambda + 10 = 0 \quad \therefore \quad \lambda = -2 \text{ or } \lambda = -5
\]

For \( \lambda = -2 \),

\[
(A - \lambda I) \vec{v} = 0 \text{ gives } \begin{pmatrix} -4 + 2 & 1 \\ 2 & -3 + 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ take } v_1 = 1.
\]

so \( \text{eigenvector } = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

Next for \( \lambda = -5 \),

\[
(A - \lambda I) \vec{v} = 0 \text{ gives } \begin{pmatrix} -4 + 5 & 1 \\ 2 & -3 + 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ take } v_1 = 1
\]

so, \( \text{eigenvector } = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)
(b) (6 pts) Let $B$ be a real constant $2 \times 2$ matrix with a complex eigenvalue $\lambda = 1 + i$ with an eigenvector $\begin{pmatrix} 1 \\ i \end{pmatrix}$. Consider the system $\frac{dx}{dt} = Bx$. Find the general solution to the system.

\[
\vec{x} = e^{(1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ is a soln.}
\]

\[
= e^t e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 + i \\ 0 \end{pmatrix}
\]

\[
= e^t (\cos t + i \sin t) \begin{pmatrix} 1 \\ i \end{pmatrix}
\]

\[
= e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}
\]

**General solution:**

\[
\vec{x} = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}
\]
2. [25 pts.] Consider the nonlinear equation $x'' = -x + x^4$.

(a) (2 pts) Write the equation as a first order system.
(b) (3 pts) Find all the equilibrium points of this system.
(c) (4 pts) Find the potential energy $P$ for this system and draw its graph.
(d) (3 pts) Write down the total energy associated with this system; this should be constant along any solution trajectory.

(problem 2 continues on the next page.)

\[
\begin{align*}
\begin{cases}
    x' = y \\
    y' = -x + x^4
\end{cases}
\end{align*}
\]

For equilibrium points, solve \(y = 0\) and \(-x + x^4 = 0\), so $y = 0$ and $x = 0$ or $x = 1$.

Thus, equilibrium points are $(x, y) = (0, 0)$ and $(x, y) = (1, 0)$.

Now, $F(x) = \int_0^x f(\xi) d\xi = \int_0^x (-\xi + \xi^4) d\xi = -\frac{x^2}{2} + \frac{x^5}{5}$.

$P(x) = -F(x) = \frac{x^2}{2} - \frac{x^5}{5}$.

\[
\begin{align*}
\text{Total energy} &= \text{kinetic energy} + \text{potential energy} \\
&= \frac{1}{2} y^2 + P(x) \\
&= \frac{1}{2} y^2 + \frac{x^2}{2} - \frac{x^5}{5}
\end{align*}
\]
For parts (e) to (g), consider a different nonlinear equation \( x'' = f(x) \). Suppose the graph of its potential energy \( P \) is given by the following graph:

\[
Z = P(x) = \int_0^x f(s) \, ds.
\]

(e) (3 pts) Find all the equilibrium points.

(f) (7 pts) Draw the phase plane associated with this potential energy. Fill in all representative trajectories and the arrow direction for increasing \( t \).

(g) (3 pts) Describe what happens if \( x(0) = -1/2 \) and \( x'(0) = 0 \). Sketch \( x(t) \) versus \( t \) for this trajectory.

**Critical points of \( P \) is:**

\[
0 = P'(x) = -f(x) = -f(x)
\]

Thus they are equilibrium points of \( x'' = f(x) \).

Clear from the graph of \( P \) that \( x = 0 \) & \( x = -1 \) are equilibrium points, i.e. \((x, y) = (0, 0) \) & \((x, y) = (-1, 0) \) are equilibrium points.

\[
E(x, y) = \frac{1}{2} y^2 + P(x) = \text{total energy}
\]

\( E = 0 \) and \( E = -3 \) are special, as they correspond to energy level of equilibrium points.
Thus we plot all representative trajectories for:

(i) \( E > 0 \)
(ii) \( E = 0 \)
(iii) \( -3 < E < 0 \)
(iv) \( E < -3 \)

(f) With \( x(0) = -\frac{1}{2} \), \( x'(0) = 0 \), from the phase plane, we see that it will give rise to a periodic solution since its orbit in the phase plane is a closed loop without equilibrium point in the loop.

A plot of \( x(t) \) versus \( t \) is as follows:

\[
x
\]

\[
-\frac{1}{2}
\]
3. [8 pts.] Solve the nonlinear second order equation \( \frac{d^2 x}{dt^2} + x (\frac{dx}{dt})^3 = 0 \) with initial conditions \( x(0) = 1 \) and \( x'(0) = -2 \). Note that the equation is independent of \( t \) explicitly.

Let \( z = \frac{dx}{dt} \).

Then \( \frac{d^2 x}{dt^2} = \frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} = z \frac{dz}{dx} \). (we use the sloppy notation as in the text)

Thus, \( z \frac{dz}{dx} + x (\frac{dx}{dt})^3 = 0 \)

or \( z \frac{dz}{dx} + x z^3 = 0 \)

\( z = 0 \) or \( \frac{dz}{dx} + x z^2 = 0 \)

As \( x(10) = -2 \), so \( z = 0 \) can be discarded.

Thus we consider only: \( \frac{dz}{dx} + x z^2 = 0 \)

\[ \int \frac{dz}{z^2} + \int x \, dx = 0 \]

\[ -\frac{1}{z} + \frac{1}{2} x^2 = C \]

At \( t=0 \), \( -\frac{1}{z} + \frac{1}{2} \cdot 1^2 = C \), \( \therefore C = 1 \)

Thus \( -\frac{1}{z} + \frac{x^2}{2} = 1 \)

\( \frac{dx}{dt} = z = \frac{1}{(\frac{x^2}{2} - 1)} \)

\[ \int \left( \frac{x^2}{2} - 1 \right) dx = \int dt \]

\( \frac{x^3}{6} - x = t + C \)

At \( t=0 \), \( \frac{1}{6} - 1 = 0 + C \), \( \therefore C = -\frac{5}{6} \)

Thus \( \frac{x^3}{6} - x = t - \frac{5}{6} \).
4. [20 pts.] Let \(a, b, c, d\) be given positive constants and consider the predator-prey model
\[x' = ax - bxy,\]
\[y' = -cy + dxy.\]

(a) (13 pts) Show that \(H(x, y) = dx + by - c \log x - a \log y\) is constant along solution trajectories.

(b) (4 pts) Show that the only non-trivial equilibrium solutions of the model is \((x_0, y_0) = (c/d, a/b)\).

(c) (5 pts) Show that for any positive \((x, y) \neq (x_0, y_0)\), we have \(H(x, y) > H(x_0, y_0)\). (Study the minimum point of \(f(x) = dx - c \log x\) and \(g(y) = by - a \log y\).

(d) (3 pts) Take \(a = 1, b = 4, c = 2\) and \(d = 1\), show that there exists a periodic solution of the predator-prey system such that \(x + 4y - 4 = \log(x^2y)\).

(a) Let \((x(t), y(t))\) be a solution trajectory.

Then \[
\frac{d}{dt} H(x(t), y(t)) = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt}.
\]

\[
= (d - \frac{c}{x}) \frac{dx}{dt} + (b - \frac{a}{y}) \frac{dy}{dt}
\]

\[
= (d - \frac{c}{x}) (ax - bxy) + (b - \frac{a}{y}) (-cy + dxy)
\]

\[
= (dx - c)(a - by) + (by - a)(-c + dx)
\]

\[= 0\]

Hence \(H(x(t), y(t)) = constant\) along any solution trajectory.

(b) setting \(x' = y' = 0\), we have \(y(a - by) = 0\) \(\text{or} (b - y\log y) = 0\), \(x(\frac{c}{x}) = 0\) \(\text{or} (a - bx) = 0\).

Thus \(x = 0\) or \(y = \frac{a}{b}\) for \(1\)

and \(y = 0\) or \(x = \frac{c}{d}\) for \(2\).

The only equilibrium points are \((x, y) = (0, 0)\) and \((x, y) = \left(\frac{c}{d}, \frac{a}{b}\right)\).

(c) \(f(x) = dx - c \log x\). Then \(f'(x) = d - \frac{c}{x}\).

\(f' = 0\) when \(x = \frac{c}{d}\). And \(f\) attains its minimum at \(x = \frac{c}{d}\). Similarly \(g\) attains its minimum at \(y = \frac{a}{b}\).
Thus for \((x, y) \neq (x_0, y_0) = (\frac{c}{d}, \frac{a}{b})\), and \(x > 0, y > 0\),
\[
H(x, y) = f(x) + g(y)
\]
\[
> f(x_0) + g(y_0)
\]
\[= H(x_0, y_0).\]

(d) Take \(a = 1, b = 4, c = 2, d = 1\).

We have \((x_0, y_0) = (\frac{c}{d}, \frac{a}{b}) = (2, \frac{1}{4})\)

\[
H(x_0, y_0) = x_0 + 4y_0 - 2 \log x_0 - \frac{1}{2} \log y_0
\]
\[= 2 + 1 - 2 \log 2 - \log(\frac{1}{4})
\]
\[= 3.
\]

\[
H(x, y) = x + 4y - 2 \log x - \log y
\]
\[= x + 4y - \log(x^2y).
\]
\[= 4
\]

If \(x + 4y - \log(x^2y) = 4\)

Thus \(H(x, y) = 4 \geq H(x_0, y_0) = 3\)

Hence there exists a periodic solution with \(H(x, y) = 4\).