1. [10 pts.] Find the solution of \( x'' + 2x' + 2x = 0 \) with \( x(0) = 0 \) and \( x'(0) = 4 \).

Let \( x = e^{\lambda t} \) be a solution for some constant \( \lambda \). Thus
\[
\lambda^2 + 2\lambda + 2 = 0
\]
after putting into the eqn.

Thus \( \lambda = -1 \pm i \).
As a result,
\[
x = e^{t(-1+i)t} = e^{-t} e^{it} = e^{-t} (\cos t + i \sin t)
\]
is a soln. Its real part and imaginary part are soln as well. Thus
\[
x_1 = e^{-t} \cos t \quad \text{and} \quad x_2 = e^{-t} \sin t
\]
are solns.

Obviously \( x_2(t) \neq c \cdot x_1(t) \) for all \( t \), no matter what constant \( c \) we choose. Hence \( \{x_1, x_2\} \) is linearly independent. General soln is then:

\[
x = C_1 x_1 + C_2 x_2
= C_1 e^{-t} \cos t + C_2 e^{-t} \sin t
\]
for any constants \( C_1 \) & \( C_2 \).

Now \( x(0) = 0 = C_1 + C_2 \cdot 0 \), \( \therefore C_1 = 0 \).

Thus
\[
x = C_2 e^{-t} \sin t
\]
\[
x'(t) = C_2 (e^{-t} \cos t - e^{-t} \sin t)
\]
\[
x'(0) = 4 = C_2 (1 - 0) \quad \therefore C_2 = 4
\]

so,
\[
x = 4 e^{-t} \sin t.
\]

\( \star \)
2. [20 pts.]
(a) Write down the definition that two functions \( \{f, g\} \) are linearly independent.
(b) What is the definition of the Wronskian of \( \{f, g\} \)?
(c) Consider the functions \( f(t) = t^3 \) and \( g(t) = \sin t \) on the interval \((-\infty, \infty)\). Show that \( \{f, g\} \) are linearly independent.
(d) Evaluate the Wronskian of \( \{f, g\} \) at both \( t = 0 \) and \( t = \pi/2 \).
(e) In part (d), only one of the answers is zero. Does this violate Abel’s Theorem? Correct explanation is essential for any credit.

(a) \( \{f, g\} \) is linearly independent if whenever \( C_1 f(t) + C_2 g(t) = 0 \) for all \( t \), the only solution is \( C_1 = C_2 = 0 \).

(b) \[
W(f, g)(t) = \text{Wronskian} = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}
\]

(c) Clear that \( f(t) \neq c g(t) \) for all \( t \), no matter what constant \( c \) we choose. Similarly \( g(t) \neq c f(t) \). Hence \( \{f, g\} \) is linearly independent.

(d) \[
W(f, g)(t) = \begin{vmatrix} t^3 & \sin t \\ 3t^2 & \cos t \end{vmatrix} = t^3 \cos t - 3t^2 \sin t = t^2 (t \cos t - 3 \sin t)
\]

At \( t = 0 \), \( W(f, g)(0) = 0 \)

At \( t = \frac{\pi}{2} \), \( W(f, g)(\frac{\pi}{2}) = \left(\frac{\pi}{2}\right)^2 \left(\frac{\pi}{2} \cos \frac{\pi}{2} - 3 \sin \frac{\pi}{2}\right) \]

\[ = -\frac{3\pi^2}{4}. \]

(e) \( W = 0 \) at \( t = 0 \), but \( W \neq 0 \) at \( t = \frac{\pi}{2} \).

Since \( f, g \) are not solutions of \( x'' + p(t)x' + q(t) = 0 \), thus Abel’s Theorem does not apply.