As of 8–12-2008, here are some corrections and extensions of the 12-21-2007 paper in ARCH2008.1:

What need to be corrected are the definition at (1.1) and the formulae at (2.3), (2.6), (2.11), and (4.5.8). The extensions involve (a) (2.10)/(2.11) for $m = 4$, (b) calibration of the drift compensation in (1.1), (c) formulae for $\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(x)$ and $\mathcal{L}_{\bar{\mathbf{d}}}(x)$ for $x \leq 0$, (d) the value of (4.5.8) when $t \to \infty$, and (e) clarification of the proof of (3.6).

The definition at formula (1.1) and its explanatory material should be replaced by the following. This new definition accurately provides for the situation of a regime-switch occuring within the most recent model time step by pro-rating the mean-reversion among all of the random targets that arise within the most recent model time step. This new definition leads correctly to (1.2), (1.3) and the rest of the paper as originally published:

$$\begin{aligned} d\ln(\mathbf{r}_t) \quad &= \quad \left[1 - (1-F)^{dt}\right] [\, \ln(\mathbf{T}_t) - \ln(\mathbf{r}_{t-dt})] \\ &\quad + (1-F)^{dt} D_t dt + (1-F)^{dt} \sigma\sqrt{dt}\mathbf{N}_t \end{aligned} \tag{1.1}$$

where

$\mathbf{r}_t$	$=$	**the interest rate we want to model over time.**
F	$=$	**an annualized mean reversion factor between** 0 **and** 1.
dt	$=$	**a discrete time-step interval.**
$\mathbf{T}_t$	$=$	**a random mean reversion target for** $\mathbf{r}_t$ **at time** t **determined by**
$\mathbf{T}_t$	$=$	$\prod_{j=0}^{\infty} \mathbf{T}_j^{\frac{(1-F)^{dt\wedge(t-\mathbf{t_{j+1}})^+} - (1-F)^{dt\wedge(t-\mathbf{t_j})^+}}{1-(1-F)^{dt}}}$ **where**
$\{\mathbf{T}_j\}_{1\leq j}$	$=$	**i.i.d. lognormal mean reversion targets for the interest rate, thus characterizing each regime by a randomly chosen mean reversion target.** (T_0 **is a fixed target value for the first regime.)**
$\{\mathbf{t_{j+1}} - \mathbf{t_j}\}_{1\leq j}$	$=$	**a set of i.i.d. random variables with common law** $gamma(\alpha, \beta)$**, the inter-arrival intervals for regime-switches,**
$\mathbf{t_1}$	$=$	**a random variable independent of** $\{\mathbf{t_{j+1}} - \mathbf{t_j}\}_{1\leq j}$ **distributed as a randomly chosen point within a** $gamma(\alpha, \beta)$ **interval.**
$\mathbf{t}_0$	$=$	0 **launching the process at a random time within the first regime.**
D_t	$=$	**an annualized drift-compensation function available to be calibrated up front as part of the model.**
σ	$=$	**an annualized volatility parameter.**
$\{\mathbf{N}_t\}_{0\leq t}$	$=$	**i.i.d standard normal random variables independent of all the other random variables in the process.**

For a more concrete picture of the random target definition, for each t let k and k' be determined by the relations $\mathbf{t_k} < t \le \mathbf{t_{k+1}}$ and $\mathbf{t_{k'}} \le t - dt < \mathbf{t_{k'+1}}$. Then the foregoing definition of $\mathbf{T}_t$ works out to

$$\mathbf{T}_t = \mathbf{T}_k^{\frac{1-(1-F)^{dt\wedge(t-\mathbf{t_k})}}{1-(1-F)^{dt}}} \prod_{j=k'}^{k-1} \mathbf{T}_j^{\frac{(1-F)^{dt\wedge(t-\mathbf{t_{j+1}})}-(1-F)^{dt\wedge(t-\mathbf{t_j})}}{1-(1-F)^{dt}}}, \text{ where if}$$

$$\mathbf{t_k} \le t - dt < t \le \mathbf{t_{k+1}} \text{ then } \mathbf{T}_t = \mathbf{T}_k.$$

So $\mathbf{T}_t$ is a weighted geometric mean of values $\mathbf{T}_j$. Therefore $\ln(\mathbf{T}_t)$ is a weighted arithmetic average of values $\ln(\mathbf{T}_j)$:

$$\ln(\mathbf{T}_t) = \sum_{j=0}^{\infty} \ln(\mathbf{T}_j) \frac{(1-F)^{dt\wedge(t-\mathbf{t_{j+1}})_+} - (1-F)^{dt\wedge(t-\mathbf{t_j})_+}}{1-(1-F)^{dt}}$$

$$= \ln(\mathbf{T}_k)\frac{1-(1-F)^{dt\wedge(t-\mathbf{t_k})}}{1-(1-F)^{dt}} + \sum_{j=k'}^{k-1} \ln(\mathbf{T}_j)\frac{(1-F)^{dt\wedge(t-\mathbf{t_{j+1}})} - (1-F)^{dt\wedge(t-\mathbf{t_j})}}{1-(1-F)^{dt}},$$

where if $\mathbf{t_k} \le t - dt < t \le \mathbf{t_{k+1}}$ then $\ln(\mathbf{T}_t) = \ln(\mathbf{T}_k)$.

$\{\ln(\mathbf{T}_j)\}_{1\le j}$ = **i.i.d. normal random variables, independent of all the other random variables in the definitions. Further, define**

μ_T = **the common mean of the** $\{\ln(\mathbf{T}_j)\}_{1\le j}$.

σ_T^2 = **the common variance of the** $\{\ln(\mathbf{T}_j)\}_{1\le j}$.

The formula at (2.3) is an incorrect transcription of (2.4) when N=3. Correct is

$$\mathbb{E}\left[(\mathbf{r}_t)^l\right] \approx e^{l\mu+\frac{1}{2}(l\sigma)^2}\left\{1+\frac{l^4}{4!}\left[\mu_4-3\sigma^4\right]\left(1-\frac{1}{2!}(l\sigma)^2\right)+\frac{l^6}{6!}\left[\mu_6-15\sigma^6\right]\right\} \tag{2.3}$$

The formula at (2.6) is accurate only in the limit as $t \to \infty$. That's where we use it the most, so little harm is done, but we do want to use the correct formula for finite t when $n = 1$. **Here is the correct statement of all of this to replace (2.6) in the paper:**

$$\lim_{t\to\infty} \mathbb{E}\left[\{\ln(\mathbf{r}_t) - \mathbb{E}[\ln(\mathbf{r}_t)]\}^{2n+1}\right] = 0 \tag{2.6.a}$$

$$\lim_{t\to\infty} \mathbb{E}\left[\{\ln(\mathbf{r}_t) - \mathbb{E}[\ln(\mathbf{r}_t)]\}^{2n}\right] =$$

$$= (2n)?\mathbb{E}\left[\left\{\sigma^2 dt \frac{(1-F)^{2dt}}{1-(1-F)^{2dt}} + \sigma_T^2 \lim_{t\to\infty}\sum_{j=1}^{\infty}\mathbf{e}_j^2\right\}^n\right] \tag{2.6.b}$$

$$\mathbb{E}\left[\left\{\ln(\mathbf{r}_t) - \mathbb{E}\left[\ln(\mathbf{r}_t)\right]\right\}^2\right] =$$

$$\begin{aligned} &= \sigma^2 dt\,(1-F)^{2dt}\,\frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}} + \sigma_T^2\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right] \\ &\quad + \left(\ln\left(T_0\right)-\mu_T\right)^2(1-F)^{2t}\left\{\mathbb{E}\left[(1-F)^{-2\mathbf{t_1}\wedge t}\right]\right. \\ &\quad \left. - \left(\mathbb{E}\left[(1-F)^{-\mathbf{t_1}\wedge t}\right]\right)^2\right\} \end{aligned} \qquad (2.6.c)$$

where, as shown in section 3, $\mathbf{t_1}\wedge t = \bar{\mathbf{d}}\wedge t$ provides the means to calculate values.

The general correct version of (2.6) beyond the second moment is complicated:

$$\mathbb{E}\left[\left\{\ln(\mathbf{r}_t) - \mathbb{E}\left[\ln(\mathbf{r}_t)\right]\right\}^n\right] =$$

$$\begin{aligned} = &\sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{n!(2m)?}{(2m)!(n-2m)!}\left(\ln\left(T_0\right)-\mu_T\right)^{n-2m}(1-F)^{(n-2m)t}\cdot \\ &\mathbb{E}\left[\left((1-F)^{-\mathbf{t_1}\wedge t}-\mathbb{E}\left[(1-F)^{-\mathbf{t_1}\wedge t}\right]\right)^{n-2m}\cdot\right. \\ &\left.\left(\sigma^2 dt\,(1-F)^{2dt}\,\frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}}+\sigma_T^2\sum_{j=1}^{\infty}\mathbf{e}_j^2\right)^m\right] \end{aligned} \qquad (2.6)$$

where the covariances involved in terms of the form

$$\mathbb{E}\left[(1-F)^{-k\mathbf{t_1}\wedge t}\left(\sum_{j=1}^{\infty}\mathbf{e}_j^2\right)^l\right]$$

can be calculated using the techniques in section 4.5.

To prove all of this write

$$\mathbb{E}\left[\left(\left\{\ln(\mathbf{r}_t) - \mathbb{E}\left[\ln(\mathbf{r}_t)\right]\right\}^n\right)\right] =$$

$$\begin{aligned} &= \mathbb{E}\left[\mathbb{E}\left[\left\{\ln(\mathbf{r}_t)-\mathbb{E}\left[\ln(\mathbf{r}_t)\right]\right\}^n \mid \{\mathbf{t_j}\}\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\left\{\ln(\mathbf{r}_t)-\mathbb{E}\left[\ln(\mathbf{r}_t)\mid\{\mathbf{t_j}\}\right]\right.\right.\right. \\ &\quad \left.\left.\left.+\mathbb{E}\left[\ln(\mathbf{r}_t)\mid\{\mathbf{t_j}\}\right]-\mathbb{E}\left[\ln(\mathbf{r}_t)\right]\right\}^n \mid \{\mathbf{t_j}\}\right]\right] \end{aligned} \qquad (2.6.d)$$

and examine $\ln(\mathbf{r}_t)-\mathbb{E}\left[\ln(\mathbf{r}_t)\mid\{\mathbf{t_j}\}\right]$ and $\mathbb{E}\left[\ln(\mathbf{r}_t)\mid\{\mathbf{t_j}\}\right]-\mathbb{E}\left[\ln(\mathbf{r}_t)\right]$ separately:

First,

$$\ln(\mathbf{r}_t) - \mathbb{E}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t_j}\}\right] =$$

$$\begin{aligned} &= \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\,(1-F)^{sdt} \\ &\quad + \sum_{j=1}^{\infty} \left(\ln\left(\mathbf{T}_j\right) - \mu_T\right)\left((1-F)^{(t-\mathbf{t_{j+1}})_+} - (1-F)^{(t-\mathbf{t_j})_+}\right) \end{aligned}$$

by (1.3) and (2.5). Since $\left\{\mathbf{N}_{t-(s-1)dt}\right\}$ and $\{(\ln(\mathbf{T}_j) - \mu_T)\}$ are independent mean-zero normal distributions conclude that, conditional on $\{\mathbf{t_j}\}$, $\ln(\mathbf{r}_t) - \mathbb{E}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t_j}\}\right]$ is a mean-zero normal distribution with odd central moments 0 and even central moments

$$\mathbb{E}\left[\{\ln(\mathbf{r}_t) - \mathbb{E}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t_j}\}\right]\}^{2m} \mid \{\mathbf{t_j}\}\right] =$$

$$\begin{aligned} &= (2m)?\left\{\mathbb{E}\left[\{\ln(\mathbf{r}_t) - \mathbb{E}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t_j}\}\right]\}^2 \mid \{\mathbf{t_j}\}\right]\right\}^m \\ &= (2m)?\left\{\sigma^2 dt \sum_{s=1}^{\frac{t}{dt}} (1-F)^{2sdt} + \sigma_T^2 \sum_{j=1}^{\infty} \mathbf{e}_j^2\right\}^m \\ &= (2m)?\left\{\sigma^2 dt\,(1-F)^{2dt}\,\frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}} + \sigma_T^2 \sum_{j=1}^{\infty} \mathbf{e}_j^2\right\}^m \end{aligned}$$

So

$$\lim_{t\to\infty} \mathbb{E}\left[\{\ln(\mathbf{r}_t) - \mathbb{E}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t}_j\}\right]\}^{2m} \mid \{\mathbf{t_j}\}\right] =$$

$$= (2m)?\left\{\sigma^2 dt \frac{(1-F)^{2dt}}{1-(1-F)^{2dt}} + \sigma_T^2 \lim_{t\to\infty} \sum_{j=1}^{\infty} \mathbf{e}_j^2\right\}^m$$

Next, look at

$$\mathbb{E}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t_j}\}\right] - \mathbb{E}\left[\ln(\mathbf{r}_t)\right] = \mathbb{E}\left[\ln(\mathbf{r}_t) - \mathbb{E}\left[\ln(\mathbf{r}_t)\right] \mid \{\mathbf{t_j}\}\right]$$

$$\begin{aligned} &= (\ln(T_0) - \mu_T)\left((1-F)^{(t-\mathbf{t_1})_+} - \mathbb{E}\left[(1-F)^{(t-\mathbf{t_1})_+}\right]\right) \\ &= (\ln(T_0) - \mu_T)(1-F)^t\left((1-F)^{-\mathbf{t_1}\wedge t} - \mathbb{E}\left[(1-F)^{-\mathbf{t_1}\wedge t}\right]\right) \end{aligned}$$

by (1.3) and (2.5). Conclude (see (e) at the end of this note for the limit) that (i)

$$\lim_{t\to\infty} \left(\mathbb{E}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t_j}\}\right] - \mathbb{E}\left[\ln(\mathbf{r}_t)\right]\right) = 0$$

on almost all paths $\{\mathbf{t_j}\}$ for the switching regimes.

(ii)

$$\mathbb{E}\left[\left(\mathbb{E}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t}_j\}\right] - \mathbb{E}\left[\ln(\mathbf{r}_t)\right]\right)^2 \mid \{\mathbf{t}_j\}\right] =$$

$$= \quad \left(\ln\left(T_0\right) - \mu_T\right)^2 \left(1-F\right)^{2t} \left(\mathbb{E}\left[\left(1-F\right)^{-2\mathbf{t_1}\wedge t}\right] - \left(\mathbb{E}\left[\left(1-F\right)^{-\mathbf{t_1}\wedge t}\right]\right)^2\right) \tag{2.6.e}$$

and (iii)

$$\mathbb{E}\left[\left(\mathbb{E}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t_j}\}\right] - \mathbb{E}\left[\ln(\mathbf{r}_t)\right]\right)^m \mid \{\mathbf{t_j}\}\right] =$$

$$= \left(\ln\left(T_0\right) - \mu_T\right)^m \left(1-F\right)^{mt} \mathbb{E}\left[\left\{\left(1-F\right)^{-\mathbf{t_1}\wedge t} - \mathbb{E}\left[\left(1-F\right)^{-\mathbf{t_1}\wedge t}\right]\right\}^m\right]$$

from which all the conclusions about (2.6) follow, using (2.6.d).

The formula at (2.11) is in error and should read instead

$$\mathbb{E}\left[\left(\sum_{j=1}^{\infty}\mathbf{e}_j^2\right)^3\right] =$$

$$\begin{aligned}
= \quad & \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^6\right] + 3\rho_{2,1}\left\{\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4\right]\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right] - \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4\mathbb{E}\left[e_j^2\right]\right]\right\} \\
& +\rho_{1,1,1}\left\{\left(\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\right)^3 - 3\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\mathbb{E}\left[\mathbf{e}_j^2\right]\right]\right. \\
& \left. +2\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\left(\mathbb{E}\left[\mathbf{e}_j^2\right]\right)^2\right]\right\}
\end{aligned} \tag{2.11}$$

To prove this write

$$\left(\sum_{j=1}^{\infty}\mathbf{e}_j^2\right)^3 =$$

$$= \left[\begin{array}{c}
\sum_{j=1}^{\infty}\mathbf{e}_j^6 + 3\sum_{j=1}^{\infty}\mathbf{e}_j^4\left\{\left(\sum_{i=1}^{\infty}\mathbf{e}_i^2\right) - \mathbf{e}_j^2\right\} \\
+\sum_{j=1}^{\infty}\mathbf{e}_j^2\left\{\begin{array}{c}\left(\sum_{i=1}^{\infty}\mathbf{e}_i^2\left[\left(\sum_{k=1}^{\infty}\mathbf{e}_k^2\right) - \mathbf{e}_i^2 - \mathbf{e}_j^2\right]\right) \\ -\mathbf{e}_j^2\left[\left(\sum_{k=1}^{\infty}\mathbf{e}_k^2\right) - 2\mathbf{e}_j^2\right]\end{array}\right\}
\end{array}\right]$$

and take expected values on both sides (carefully.)

(4.5.8) has a factor missing and should read

$$\begin{aligned}
\rho_{a,b} &= \frac{1-\mathbb{P}\left[\bar{\mathbf{d}} \geq t\right]}{D}\left\{\mathbb{E}\left[(1-F)^{2(a-b)\bar{\mathbf{d}}\wedge t}\right] - \mathbb{P}\left[\bar{\mathbf{d}} \geq t\right](1-F)^{2(a-b)t}\right\} \text{ where} \\
D &= \left\{\mathbb{E}\left[(1-F)^{2a\bar{\mathbf{d}}\wedge t}\right] - \mathbb{P}\left[\bar{\mathbf{d}} \geq t\right](1-F)^{2at}\right\}\cdot \\
&\quad \cdot\left\{\mathbb{E}\left[(1-F)^{-2b\bar{\mathbf{d}}\wedge t}\right] - \mathbb{P}\left[\bar{\mathbf{d}} \geq t\right](1-F)^{-2bt}\right\}, \text{ by (4.5.2)-(4.5.5)} \\
&\quad \text{and (4.5.7), and} \\
\rho_{a_1,\ldots,a_k} &= \rho_{a_1,a_2+\ldots+a_k}\rho_{a_2,\ldots,a_k} \quad \text{recursively.}
\end{aligned} \tag{4.5.8}$$

This follows from observing that each of (4.5.3), (4.5.4) and (4.5.5) are missing a factor $\frac{1}{1-\mathbb{P}\left[\bar{\mathbf{d}} \geq t\right]}$

Extending the results of the paper,
(a) here is the extension of (2.10) and (2.11) to the case $m = 4$.

$$\mathbb{E}\left[\left(\sum_{j=1}^{\infty}\mathbf{e}_j^2\right)^4\right] =$$

$$\begin{aligned}
= \quad & \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^8\right] + 4\rho_{3,1}\left\{\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^6\right]\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right] - \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^6\mathbb{E}\left[e_j^2\right]\right]\right\} \\
& +3\rho_{2,2}\left\{\left(\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4\right]\right)^2 - \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4\mathbb{E}\left[e_j^4\right]\right]\right\} \\
& +6\rho_{2,1,1}\left\{\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4\right]\left(\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\right)^2 - 2\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4\mathbb{E}\left[e_j^2\right]\right]\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\right. \\
& \left. -\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4\right]\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\mathbb{E}\left[e_j^2\right]\right] + 2\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4\left(\mathbb{E}\left[e_j^2\right]\right)^2\right]\right\} \\
& +\rho_{1,1,1,1}\left\{\left(\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\right)^4 - 6\left(\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\right)^2\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\mathbb{E}\left[e_j^2\right]\right]\right. \\
& +8\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\left(\mathbb{E}\left[e_j^2\right]\right)^2\right] + 3\left(\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\mathbb{E}\left[e_j^2\right]\right]\right)^2 \\
& \left. -6\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\left(\mathbb{E}\left[e_j^2\right]\right)^3\right]\right\}
\end{aligned}$$

To prove this write (next page)

$$\left(\sum_{j=1}^{\infty}\mathbf{e}_j^2\right)^4 =$$

$$= \left[\begin{array}{l}
\sum_{j=1}^{\infty}\mathbf{e}_j^8 + 4\sum_{j=1}^{\infty}\mathbf{e}_j^6\left\{\left(\sum_{i=1}^{\infty}\mathbf{e}_i^2\right) - \mathbf{e}_j^2\right\} + 3\sum_{j=1}^{\infty}\mathbf{e}_j^4\left\{\left(\sum_{i=1}^{\infty}\mathbf{e}_i^4\right) - \mathbf{e}_j^4\right\} \\
+6\sum_{j=1}^{\infty}\mathbf{e}_j^4\left\{\left(\sum_{i=1}^{\infty}\mathbf{e}_i^2\left[\left(\sum_{k=1}^{\infty}\mathbf{e}_k^2\right) - \mathbf{e}_i^2 - \mathbf{e}_j^2\right]\right) - \mathbf{e}_j^2\left[\left(\sum_{k=1}^{\infty}\mathbf{e}_k^2\right) - 2\mathbf{e}_j^2\right]\right\} \\
+\sum_{j=1}^{\infty}\mathbf{e}_j^2\left\{\begin{array}{l}
\left(\sum_{i=1}^{\infty}\mathbf{e}_i^2\left[\begin{array}{c}
\left(\sum_{k=1}^{\infty}\mathbf{e}_k^2\left(\left(\sum_{l=1}^{\infty}\mathbf{e}_l^2\right) - \mathbf{e}_k^2 - \mathbf{e}_i^2 - \mathbf{e}_j^2\right)\right) \\
-\mathbf{e}_i^2\left(\left(\sum_{l=1}^{\infty}\mathbf{e}_l^2\right) - 2\mathbf{e}_i^2 - \mathbf{e}_j^2\right) \\
-\mathbf{e}_j^2\left(\left(\sum_{l=1}^{\infty}\mathbf{e}_l^2\right) - 2\mathbf{e}_j^2 - \mathbf{e}_i^2\right)
\end{array}\right]\right) \\
-\mathbf{e}_j^2\left[\begin{array}{c}
\left(\sum_{k=1}^{\infty}\mathbf{e}_k^2\left(\left(\sum_{l=1}^{\infty}\mathbf{e}_l^2\right) - \mathbf{e}_k^2 - 2\mathbf{e}_j^2\right)\right) \\
-2\mathbf{e}_j^2\left(\left(\sum_{l=1}^{\infty}\mathbf{e}_l^2\right) - 3\mathbf{e}_j^2\right)
\end{array}\right]
\end{array}\right\}
\end{array}\right]$$

and take expected values on both sides (very carefully).

(b) here is a calibration of the drift compensation using (2.5) and the following intuitively appealing condition:

$$\begin{aligned}
\mathbb{E}\left[\mathbf{r}_t \mid \{\mathbf{t}_j\}\right] &= r_0^{(1-F)^t} T_0^{(1-F)^{(t-\mathbf{t}_1)_+}-(1-F)^t} \bar{T}^{1-(1-F)^{(t-\mathbf{t}_1)_+}} \text{ where} \\
\bar{T} &= \mathbb{E}\left[\mathbf{T}_j\right] = e^{\mu_T + \frac{1}{2}\sigma_T^2} \text{ by def. of } \mathbf{T}_j \text{ lognormal} \qquad \text{(e.1)}
\end{aligned}$$

Since, conditional on $\{\mathbf{t}_j\}$, $\mathbf{r}_t$ is lognormal

$$\begin{aligned}
\mathbb{E}\left[\mathbf{r}_t \mid \{\mathbf{t}_j\}\right] &= e^{\mathbb{E}[\ln(\mathbf{r}_t)\mid\{\mathbf{t}_j\}] + \frac{1}{2}\mathbb{V}[\ln(\mathbf{r}_t)\mid\{\mathbf{t}_j\}]} \text{ so} \\
\mathbb{E}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t}_j\}\right] &= \ln\left(\mathbb{E}\left[\mathbf{r}_t \mid \{\mathbf{t}_j\}\right]\right) - \frac{1}{2}\mathbb{V}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t}_j\}\right]
\end{aligned}$$

$$\begin{aligned}
&= \ln(r_0)(1-F)^t + \ln(T_0)\left[(1-F)^{(t-\mathbf{t}_1)_+} - (1-F)^t\right] \\
&\quad + \left(\mu_T + \frac{1}{2}\sigma_T^2\right)\left[1 - (1-F)^{(t-\mathbf{t}_1)_+}\right] - \frac{1}{2}\mathbb{V}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t}_j\}\right] \text{ by (e.1)} \\
&= \left[\ln(r_0) - \ln(T_0)\right](1-F)^t \\
&\quad + \left[\ln(T_0) - \left(\mu_T + \frac{1}{2}\sigma_T^2\right)\right](1-F)^t(1-F)^{-t\wedge\mathbf{t}_1} \\
&\quad + \left(\mu_T + \frac{1}{2}\sigma_T^2\right) - \frac{1}{2}\mathbb{V}\left[\ln(\mathbf{r}_t) \mid \{\mathbf{t}_j\}\right] \qquad \text{(e.2)}
\end{aligned}$$

But $\mathbb{E}\left[\mathbb{V}\left[\ln\left(\mathbf{r}_t\right) \mid \{\mathbf{t}_j\}\right]\right] = \mathbb{V}\left[\ln\left(\mathbf{r}_t\right)\right] - \mathbb{V}\left[\mathbb{E}\left[\ln\left(\mathbf{r}_t\right) \mid \{\mathbf{t}_j\}\right]\right]$ by the usual rule on conditioned variances, so using (e.2)

$$\mathbb{E}\left[\ln\left(\mathbf{r}_t\right)\right] = \mathbb{E}\left[\mathbb{E}\left[\ln\left(\mathbf{r}_t\right) \mid \{\mathbf{t}_j\}\right]\right]$$

$$\begin{aligned} = \quad & \left[\ln\left(r_0\right) - \ln\left(T_0\right)\right]\left(1-F\right)^t \\ & + \left[\ln\left(T_0\right) - \left(\mu_T + \frac{1}{2}\sigma_T^2\right)\right]\left(1-F\right)^t \mathbb{E}\left[\left(1-F\right)^{-t\wedge\mathbf{t}_1}\right] \\ & + \left(\mu_T + \frac{1}{2}\sigma_T^2\right) - \frac{1}{2}\left(\mathbb{V}\left[\ln\left(\mathbf{r}_t\right)\right] - \mathbb{V}\left[\mathbb{E}\left[\ln\left(\mathbf{r}_t\right) \mid \{\mathbf{t}_j\}\right]\right]\right) \end{aligned}$$

Now, use (2.6.c) and (2.6.e) on the last term above to get

$$\mathbb{E}\left[\ln\left(\mathbf{r}_t\right)\right] =$$

$$\begin{aligned} = \quad & \left[\ln\left(r_0\right) - \ln\left(T_0\right)\right]\left(1-F\right)^t \\ & + \left[\ln\left(T_0\right) - \left(\mu_T + \frac{1}{2}\sigma_T^2\right)\right]\left(1-F\right)^t \mathbb{E}\left[\left(1-F\right)^{-t\wedge\bar{\mathbf{d}}}\right] \\ & + \mu_T + \frac{1}{2}\sigma_T^2\left[1 - \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\right] \\ & - \frac{1}{2}\sigma^2 dt\left(1-F\right)^{2dt}\frac{1-\left(1-F\right)^{2t}}{1-\left(1-F\right)^{2dt}} \end{aligned} \qquad \text{(e.3)}$$

(Remember that $\mathbf{t}_1 \simeq t \wedge \bar{\mathbf{d}}$ by (4.1.1) so $t \wedge \mathbf{t}_1 \simeq t \wedge \bar{\mathbf{d}}$.) This is the easiest expression to use for calculating $\mathbb{E}\left[\ln\left(\mathbf{r}_t\right)\right]$ to apply in calculating (2.2)-(2.4).

Comparing (e.3) with (2.5) allows us to express the cumulative drift compensation as

$$dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\left(1-F\right)^{sdt} =$$

$$\begin{aligned} = \quad & \frac{1}{2}\sigma_T^2\left\{1 - \left(1-F\right)^t \mathbb{E}\left[\left(1-F\right)^{-t\wedge\bar{\mathbf{d}}}\right] - \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\right\} \\ & - \frac{1}{2}\sigma^2 dt\left(1-F\right)^{2dt}\frac{1-\left(1-F\right)^{2t}}{1-\left(1-F\right)^{2dt}} \end{aligned} \qquad \text{(e.4)}$$

(compare this with formula (2.2.7) in the Nov. 3, 2006 ARCH2007.1 paper)

Asymptotically,

$$dt \sum_{s=1}^{\infty} D_{t-(s-1)dt} \left(1-F\right)^{sdt}$$

$$= \frac{1}{2}\sigma_T^2 \left\{ 1 - \lim_{t\to\infty} \mathbb{E}\left[\sum_{j=1}^{\infty} \mathbf{e}_j^2\right] \right\} - \frac{1}{2}\sigma^2 dt \frac{\left(1-F\right)^{2dt}}{1-\left(1-F\right)^{2dt}}$$

with reference to (e) at the end of this note for

$$\lim_{t\to\infty} \left(1-F\right)^t \mathbb{E}\left[\left(1-F\right)^{-t\wedge\bar{\mathbf{d}}}\right] = 0.$$

To calculate D_t (in applying (1.1) in a Monte Carlo model, for example) the easiest thing probably is to calculate (e.4) at t and again at $t-dt$ and subtract:

$$D_t = \frac{\text{(e.4) at } t}{dt\left(1-F\right)^{dt}} - \frac{\text{(e.4) at } t-dt}{dt}$$

Written out, this difference works out to (compare (2.2.8) in the Nov. 3, 2006 ARCH2007.1 paper):

$$D_t =$$

$$\begin{aligned} = \quad & -\frac{1}{2}\sigma^2 \frac{\left(1-F\right)^{dt}}{1+\left(1-F\right)^{dt}} \left(1+\left(1-F\right)^{2t-dt}\right) \\ & +\frac{1}{2}\sigma_T^2 \frac{1}{dt\left(1-F\right)^{dt}} \left\{1-\left(1-F\right)^{dt}\right. \\ & -\left(1-F\right)^t \left(\mathbb{E}\left[\left(1-F\right)^{-t\wedge\bar{\mathbf{d}}}\right] - \mathbb{E}\left[\left(1-F\right)^{-(t-dt)\wedge\bar{\mathbf{d}}}\right]\right) \\ & \left. -\left(\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]_t - \left(1-F\right)^{dt}\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]_{t-dt}\right)\right\} \end{aligned}$$

Asymptotically

$$\lim_{t\to\infty} D_t =$$

$$= -\frac{1}{2}\sigma^2 \frac{\left(1-F\right)^{dt}}{1+\left(1-F\right)^{dt}} + \frac{1}{2}\sigma_T^2 \frac{1-\left(1-F\right)^{dt}}{dt\left(1-F\right)^{dt}} \lim_{t\to\infty}\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]$$

with reference to (e) at the end of this note for

$$\lim_{t\to\infty} \left(1-F\right)^t \mathbb{E}\left[\left(1-F\right)^{-t\wedge\bar{\mathbf{d}}}\right] = 0.$$

(c) Some of the incomplete gamma functions in section 3.3 for calculating $\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(x)$ and $\mathcal{L}_{\bar{\mathbf{d}}}(x)$ are undefined when $x \leq 0$.

In this case, integrating by parts in the definition of $\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(x)$ gives one of four results:

(i) If $x = 0$

$$\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(0) = -\frac{1}{\alpha\beta}\left\{t\left(1-\Gamma(\alpha;\frac{t}{\beta})\right)-\alpha\Gamma(\alpha+1;\frac{t}{\beta})\right\}+\mathbb{P}[\bar{\mathbf{d}}\geq t]$$

which can be used in calculations for finite t and, using l'Hôpital's rule on the first term, shows that $\mathcal{L}_{\bar{\mathbf{d}}}(0) = \frac{1}{\beta}$.

(ii) If $-\frac{1}{\beta} < x < 0$ the expression in section 3.3 for $\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(x)$ remains valid and can be used in calculations for finite t

$$\begin{aligned}\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(x) = & \frac{1}{\alpha\beta x}\left\{1-e^{-xt}\left[1-\Gamma\left(\alpha;\frac{t}{\beta}\right)\right]\right. \\ & \left. -(1+\beta x)^{-\alpha}\,\Gamma\left(\alpha;\frac{(1+\beta x)\,t}{\beta}\right)\right\}+e^{-xt}\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right].\end{aligned} \tag{e.5}$$

An application of l'Hôpital's rule on the two exponential terms is enough to show that

$$\mathcal{L}_{\bar{\mathbf{d}}}(x) = \frac{1}{\alpha\beta x}\left\{1-(1+\beta x)^{-\alpha}\right\}$$

(iii) If $x = -\frac{1}{\beta}$

$$\mathcal{L}_{\bar{\mathbf{d}}\wedge t}\left(-\frac{1}{\beta}\right) = -\frac{1}{\alpha}\left\{1-e^{\frac{1}{\beta}t}\left(1-\Gamma(\alpha;\frac{t}{\beta})\right)-\frac{t^{\alpha}}{\Gamma(\alpha+1)\beta^{\alpha}}\right\}+e^{\frac{1}{\beta}t}\mathbb{P}[\bar{\mathbf{d}}\geq t] \tag{e.6}$$

which can be used in calculations for finite t and makes $\mathcal{L}_{\bar{\mathbf{d}}}(-\frac{1}{\beta}) = \infty$ obvious.

(iv) If $x < -\frac{1}{\beta}$

$$\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(x) = \frac{1}{\alpha\beta x}\left\{\begin{array}{c}1-e^{-xt}\left(1-\Gamma(\alpha;\frac{t}{\beta})\right)\\ -\int_{0}^{t}\frac{s^{\alpha-1}e^{-s(x+\frac{1}{\beta})}}{\Gamma(\alpha)\beta^{\alpha}}ds\end{array}\right\}+e^{-xt}\mathbb{P}[\bar{\mathbf{d}}\geq t] \tag{e.7}$$

and further integration by parts gives

$$\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(x) = \frac{1}{\alpha\beta x}\left\{\begin{matrix} 1 - e^{-xt}\left(1-\Gamma(\alpha;\frac{t}{\beta})\right) \\ +\frac{1}{\Gamma(\alpha)\beta^\alpha}e^{-t\left(x+\frac{1}{\beta}\right)}\sum\limits_{j=0}^{k}\frac{(\alpha-1)\cdots(\alpha-j)}{\left(x+\frac{1}{\beta}\right)^{j+1}}t^{\alpha-j-1} \\ -\frac{(\alpha-1)\cdots(\alpha-k-1)}{\Gamma(\alpha)\beta^\alpha\left(x+\frac{1}{\beta}\right)^{k+1}}e^{-t\left(x+\frac{1}{\beta}\right)}\int\limits_0^t s^{\alpha-k-2}e^{-(s-t)\left(x+\frac{1}{\beta}\right)}ds \end{matrix}\right\} + e^{-xt}\mathbb{P}[\bar{\mathbf{d}} \geq t] \tag{e.8}$$

for any $k \leq \lfloor \alpha - 1 \rfloor$. Note that when α is an integer the sum terminates at $k = \alpha - 1$ with no remainder and provides an exact calculation. Otherwise, the remaining integral is of order $O(t^{\alpha-k-2})$, so for $k = \lfloor \alpha - 1 \rfloor$ and large t the sum would provide a reasonable approximate calculation (it certainly converges as $t \to \infty$).

For small t I believe (but have not validated) that inserting a reasonable number of terms of

$$\int\limits_0^t \frac{s^{\alpha-1}e^{-s(x+\frac{1}{\beta})}}{\Gamma(\alpha)\beta^\alpha}ds = \frac{t^\alpha}{\Gamma(\alpha)\beta^\alpha}e^{-t\left(x+\frac{1}{\beta}\right)}\sum_{n=0}^{\infty}\frac{\left(x+\frac{1}{\beta}\right)^n t^n}{\alpha\,(\alpha+1)\cdots(\alpha+n)}$$

into (e.6) would be more accurate.

At any rate, $\mathcal{L}_{\bar{\mathbf{d}}}(x) = \infty$ follows directly from $\mathcal{L}_{\bar{\mathbf{d}}}(-\frac{1}{\beta}) = \infty$ and the definition of $\mathcal{L}_{\bar{\mathbf{d}}}(x)$, sparing us the need to take the limit in (e.8) or (e.7).

(d) The value of formula (4.5.8) as $t \to \infty$ depends critically on the behavior of $\mathbb{P}\left[\bar{\mathbf{d}} \geq t\right](1-F)^{-2bt}$ as $t \to \infty$ which in turn depends critically on the distribution of $\bar{\mathbf{d}}$. In our case, with the distribution of $\bar{\mathbf{d}}$ determined by the assumption that $\mathbf{d}$ follows a $gamma(\alpha, \beta)$ distribution,

$$\begin{aligned}
\lim_{t\to\infty} \rho_{a,b} &= \frac{\mathbb{E}[\,(1-F)^{2(a-b)\bar{\mathbf{d}}}]}{\mathbb{E}[\,(1-F)^{2a\bar{\mathbf{d}}}]\mathbb{E}[\,(1-F)^{-2b\bar{\mathbf{d}}}]} \\
\text{when } 2b &< -\frac{1}{\beta \ln(1-F)} \\
\lim_{t\to\infty} \rho_{a,b} &= 0 \\
\text{when } 2b &\geq -\frac{1}{\beta \ln(1-F)} \qquad \text{(limit 4.5.8)}
\end{aligned}$$

(Remember, we are assuming that $a \geq b$, using the fact that $\rho_{a,b} = \rho_{b,a}$ to handle the opposite case). For higher order covariances,

$$\lim_{t\to\infty} \rho_{a_1,\ldots,a_k} = \lim_{t\to\infty} \rho_{a_1,a_2+\ldots+a_k} \lim_{t\to\infty} \rho_{a_2,\ldots,a_k} \text{ recursively.}$$

To Prove This: by formula (4.5.8) in the paper (including with the correction earlier in this note),

$$\lim_{t\to\infty} \rho_{a,b} = \frac{\mathbb{E}\left[(1-F)^{2(a-b)\bar{\mathbf{d}}}\right]}{\mathbb{E}\left[(1-F)^{2a\bar{\mathbf{d}}}\right] \lim_{t\to\infty}\left\{\mathbb{E}\left[(1-F)^{-2b\bar{\mathbf{d}}\wedge t}\right] - \mathbb{P}\left[\bar{\mathbf{d}} \geq t\right](1-F)^{-2bt}\right\}}$$

Using the approach of section 3.3 (including the assumption that $\mathbf{d}$ follows a $gamma(\alpha, \beta)$ distribution) and the results from (c) above

$$\begin{aligned}
\mathbb{E}\left[(1-F)^{-2b\bar{\mathbf{d}}}\right] &= \mathcal{L}_{\bar{\mathbf{d}}}(2b\ln(1-F)) \\
&= \infty \left(\text{for } -\frac{1}{\beta} \geq 2b\ln(1-F)\right) \\
&= \frac{1}{\alpha\beta 2b\ln(1-F}\left\{1-(1+\beta 2b\ln(1-F)^{-\alpha}\right\} \\
&\quad \left(\text{for } -\frac{1}{\beta} < 2b\ln(1-F)\right)
\end{aligned}$$

since $\ln(1-F) < 0$. So the critical issue is

$$\lim_{t\to\infty}\left(\mathbb{P}\left[\bar{\mathbf{d}} \geq t\right](1-F)^{-2bt}\right).$$

Using the approach of section 3.3 and applying l'Hôpital's rule twice

$$\begin{aligned}\lim_{t\to\infty}\left(\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right](1-F)^{-2bt}\right) &= \lim_{t\to\infty}\frac{-f_{\bar{\mathbf{d}}}(t)}{2b\ln(1-F)(1-F)^{2bt}}\\ &= \lim_{t\to\infty}\frac{-\mathbb{P}\left[\mathbf{d}\geq t\right]}{\mathbb{E}\left[\mathbf{d}\right]2b\ln(1-F)(1-F)^{2bt}} \qquad \text{by (3.7)}\\ &= \lim_{t\to\infty}\frac{f_{\mathbf{d}}(t)}{\mathbb{E}\left[\mathbf{d}\right](2b\ln(1-F))^2(1-F)^{2bt}}.\end{aligned}$$

With the assumption that $\mathbf{d}$ follows a $gamma(\alpha,\beta)$ distribution section 3.3 now gives

$$\begin{aligned}\lim_{t\to\infty}\left(\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right](1-F)^{-2bt}\right) &= \lim_{t\to\infty}\frac{t^{\alpha-1}e^{-\frac{t}{\beta}}}{\alpha\Gamma(\alpha)\beta^{\alpha+1}(2b\ln(1-F))^2(1-F)^{2bt}}\\ &= \lim_{t\to\infty}\frac{t^{\alpha-1}e^{-t\left(\frac{1}{\beta}+2b\ln(1-F)\right)}}{\Gamma(\alpha+1)\beta^{\alpha+1}(2b\ln(1-F))^2}. \qquad \text{(e.9)}\end{aligned}$$

So if $\frac{1}{\beta}+2b\ln(1-F)>0$ the last limit is 0 and

$$\lim_{t\to\infty}\left\{\mathbb{E}\left[(1-F)^{-2b\bar{\mathbf{d}}\wedge t}\right]-\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right](1-F)^{-2bt}\right\}=\mathbb{E}\left[(1-F)^{-2b\bar{\mathbf{d}}}\right]$$

$$\text{(thereby making } \lim_{t\to\infty}\rho_{a,b}=\frac{\mathbb{E}[(1-F)^{2(a-b)\bar{\mathbf{d}}}]}{\mathbb{E}[(1-F)^{2a\bar{\mathbf{d}}}]\mathbb{E}[(1-F)^{-2b\bar{\mathbf{d}}}]}\text{).}$$

But if $\frac{1}{\beta}+2b\ln(1-F)\leq 0$, it will be necessary to work at a finer level of detail. Again from section 3.3

$$\mathbb{E}\left[(1-F)^{-2b\bar{\mathbf{d}}\wedge t}\right]=\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(2b\ln(1-F))$$

$$\begin{aligned}&= \frac{1}{\mathbb{E}\left[\mathbf{d}\right]}\int_0^t e^{-2b\ln(1-F)s}\mathbb{P}\left[\mathbf{d}\geq s\right]ds+e^{-2b\ln(1-F)t}\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right] \quad \text{and use (e.8)}\\ &= \frac{1}{\alpha\beta 2b\ln(1-F)}\left\{1-e^{-2b\ln(1-F)t}\left(1-\Gamma(\alpha;\frac{t}{\beta})\right)\right.\\ &\quad +\frac{1}{\Gamma(\alpha)\beta^\alpha}e^{-t\left(2b\ln(1-F)+\frac{1}{\beta}\right)}\sum_{j=0}^{k}\frac{(\alpha-1)\cdots(\alpha-j)}{\left(2b\ln(1-F)+\frac{1}{\beta}\right)^{j+1}}t^{\alpha-j-1}\\ &\quad \left.-\frac{(\alpha-1)\cdots(\alpha-k-1)}{\Gamma(\alpha)\beta^\alpha\left(2b\ln(1-F)+\frac{1}{\beta}\right)^{k+1}}e^{-t\left(2b\ln(1-F)+\frac{1}{\beta}\right)}\int_0^t s^{\alpha-k-2}e^{-(s-t)\left(2b\ln(1-F)+\frac{1}{\beta}\right)}ds\right\}\\ &\quad +e^{-2b\ln(1-F)t}\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right]\end{aligned}$$

where k is $\lfloor\alpha-1\rfloor$. (If $\frac{1}{\beta}+2b\ln(1-F)=0$ use the comparable expression based on (e.6) instead of (e.8))

Combine this with expression (e.9) above for $\lim_{t\to\infty}\left(\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right](1-F)^{-2bt}\right)$ and note that $\lim_{t\to\infty}\left\{\mathbb{E}\left[(1-F)^{-2b\bar{\mathbf{d}}\wedge t}\right]-\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right](1-F)^{-2bt}\right\}$ will be determined completely by the terms containing factors of $e^{-t\left(\frac{1}{\beta}+2b\ln(1-F)\right)}$ and $e^{-2b\ln(1-F)t}$. Leaving out all other terms, for simplicity,

$$\lim_{t\to\infty}\left\{\mathbb{E}\left[(1-F)^{-2b\bar{\mathbf{d}}\wedge t}\right]-\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right](1-F)^{-2bt}\right\}=$$

$$\begin{aligned}
=\quad & \lim_{t\to\infty}e^{-t2b\ln(1-F)}\left\{-\frac{\left[1-\Gamma\left(\alpha;\frac{t}{\beta}\right)\right]}{\alpha\beta 2b\ln(1-F)}+\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right]\right\}\\
& +\lim_{t\to\infty}e^{-t\left(\frac{1}{\beta}+2b\ln(1-F)\right)}\left\{\frac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}2b\ln(1-F)}\sum_{j=0}^{k}\frac{(\alpha-1)\cdots(\alpha-j)}{\left(2b\ln(1-F)+\frac{1}{\beta}\right)^{j+1}}t^{\alpha-j-1}\right.\\
& -\frac{(\alpha-1)\cdots(\alpha-k-1)}{\Gamma(\alpha+1)\beta^{\alpha+1}2b\ln(1-F)\left(2b\ln(1-F)+\frac{1}{\beta}\right)^{k+1}}\int_0^t s^{\alpha-k-2}e^{-(s-t)\left(2b\ln(1-F)+\frac{1}{\beta}\right)}ds\\
& \left.-\frac{t^{\alpha-1}}{\Gamma(\alpha+1)\,\beta^{\alpha+1}\left(2b\ln(1-F)\right)^2}\right\}.
\end{aligned}$$

Now the first limit is always non-negative because $\ln(1-F)<0$. The second limit will be determined completely by the terms involving $t^{\alpha-1}$ because the integral is of order $O(t^{\alpha-k-2})$. These terms combine to give

$$\frac{-t^{\alpha-1}}{\Gamma(\alpha+1)\,\beta^{\alpha+1}\left(2b\ln(1-F)\right)^2\left(1+\beta 2b\ln(1-F)\right)}$$

(in the case $\frac{1}{\beta}+2b\ln(1-F)=0$ use of (e.6) instead of (e.8) will give

$\dfrac{t^{\alpha}}{\alpha\Gamma(\alpha)\,\beta^{\alpha}}$ as the dominant term in the second limit.)

So the second limit diverges to $+\infty$ whenever $\frac{1}{\beta}+2b\ln(1-F)\leq 0$, making the entire limit diverge

$$\lim_{t\to\infty}\left\{\mathbb{E}\left[(1-F)^{-2b\bar{\mathbf{d}}\wedge t}\right]-\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right](1-F)^{-2bt}\right\}=\infty.$$

(thereby making $\lim_{t\to\infty}\rho_{a,b}=0$).

(e) Finally, (3.6) requires an argument similar to this in order to prove that

$$\lim_{t\to\infty}(1-G)^t\mathbb{E}\left[(1-G)^{-\bar{\mathbf{d}}\wedge t}\right]=0$$

For a sufficiently long-tailed distribution of $\bar{\mathbf{d}}$ this limit could fail to vanish, adding an additional factor in (3.6). But assuming $\mathbf{d}$ is *gamma*(α,β):

If $-\frac{1}{\beta}<\ln(1-G)$, (e.5) above gives

$$\lim_{t\to\infty}(1-G)^t\mathbb{E}\left[(1-G)^{-\bar{\mathbf{d}}\wedge t}\right]=\lim_{t\to\infty}e^{t\ln(1-G)}\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(\ln(1-G))$$

$$\begin{aligned}
&= \lim_{t\to\infty}\frac{1}{\alpha\beta\ln(1-G)}\left\{e^{t\ln(1-G)}-\left[1-\Gamma\left(\alpha;\frac{t}{\beta}\right)\right]\right.\\
&\quad\left.-e^{t\ln(1-G)}\left(1+\beta\ln(1-G)\right)^{-\alpha}\Gamma\left(\alpha;\frac{(1+\beta\ln(1-G))\,t}{\beta}\right)\right\}+\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right]\\
&= 0
\end{aligned}$$

because $\ln(1-G)<0$.

If $\ln(1-G)\leq-\frac{1}{\beta}$, (e.6) or (e.8) above give (ignoring terms in $\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(\ln(1-G))$ not of exponential order)

$$\begin{aligned}
\lim_{t\to\infty}(1-G)^t\mathbb{E}\left[(1-G)^{-\bar{\mathbf{d}}\wedge t}\right] &= \lim_{t\to\infty}e^{t\ln(1-G)}\mathcal{L}_{\bar{\mathbf{d}}\wedge t}(\ln(1-G))\\
&= \lim_{t\to\infty}e^{t\ln(1-G)}\left\{c_1e^{-t\left(\frac{1}{\beta}+\ln(1-G)\right)}+c_2e^{-t\ln(1-G)}\right\}
\end{aligned}$$

for expressions c_1 and c_2 where c_1 is of polynomial order as $t\to\infty$ and $\lim_{t\to\infty}c_2=0$. So

$$\begin{aligned}
\lim_{t\to\infty}(1-G)^t\mathbb{E}\left[(1-G)^{-\bar{\mathbf{d}}\wedge t}\right] &= \lim_{t\to\infty}c_1e^{-t\frac{1}{\beta}}\\
&= 0
\end{aligned}$$

This same argument gives

$$\lim_{t\to\infty}(1-F)^t\mathbb{E}\left[(1-F)^{-\bar{\mathbf{d}}\wedge t}\right]=0$$

in the asymptotic versions of the drift correction and cumulative drift correction earlier in this note and

$$\lim_{t\to\infty}(1-F)^t\mathbb{E}\left[(1-F)^{-\mathbf{t}_1\wedge t}\right]=0$$

in the proof of the correction to (2.6.) earlier in this note.

Since $(1-F)^t\,(1-F)^{-\mathbf{t}_1\wedge t}\geq 0$ this also implies that

$$\lim_{t\to\infty}(1-F)^t\,(1-F)^{-\mathbf{t}_1\wedge t}=0$$

on almost all paths $\{\mathbf{t_j}\}$ for the switching regimes.