As of July 23, 2012 here are some more corrections and extensions of the 12-21-2007 paper in ARCH2008.1

Section (c) on pages 11 and 12 of the 8-12-2008 corrections needs to be replaced with the following:

(c) **Some of the incomplete gamma functions in section 3.3 for calculating $\mathcal{L}_{\bar{d}\wedge t}(x)$ and $\mathcal{L}_{\bar{d}}(x)$ are undefined when $x \leq 0$**

In this case, the definition of $\mathcal{L}_{\bar{d}\wedge t}(x)$ gives one of four results:

(i) If $x = 0$ L'Hôpital's rule appplied to the expressions in section 3.3 gives

$$\mathcal{L}_{\bar{d}\wedge t}(0) = 1 \text{ and } \mathcal{L}_{\bar{d}}(0) = 1.$$

(ii) If $-\frac{1}{\beta} < x < 0$ the expression in section 3.3 for $\mathcal{L}_{\bar{d}\wedge t}(x)$ remains valid and can be used in calculations for finite t.

$$\begin{aligned} \mathcal{L}_{\bar{d}\wedge t}(x) &= \frac{1}{\alpha\beta x}\left\{1 - e^{-xt}\left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right)\right]\right. \\ &\left. - (1+\beta x)^{-\alpha}\,\Gamma\left(\alpha; \frac{(1+\beta x)\,t}{\beta}\right)\right\} + e^{-xt}\mathbb{P}\left[\bar{d} \geq t\right]. \end{aligned} \tag{e.5}$$

An application of L'Hôpital's rule on the two exponential terms above is enough to show that

$$\mathcal{L}_{\bar{d}}(x) = \frac{1}{\alpha\beta x}\left\{1 - (1+\beta x)^{-\alpha}\right\}.$$

(iii) If $x = -\frac{1}{\beta}$ L'Hôpital's rule applied to the third term in brackets in (e.5) gives

$$\mathcal{L}_{\bar{d}\wedge t}(-\frac{1}{\beta}) = -\frac{1}{\alpha}\left\{1 - e^{\frac{1}{\beta}t}\left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right)\right] - \frac{t^{\alpha}}{\Gamma(\alpha+1)\,\beta^{\alpha}}\right\} + e^{\frac{1}{\beta}t}\mathbb{P}\left[\bar{d} \geq t\right] \tag{e.6}$$

which can be used in calculations for finite t and makes $\mathcal{L}_{\bar{d}}(-\frac{1}{\beta}) = \infty$ obvious.

(iv) If $x < -\frac{1}{\beta}$ the integral that defines the incomplete gamma function in (e.5) is still available (although it no longer defines an incomplete gamma function) and it can be integrated by expanding the exponential in a power series:

$$\mathcal{L}_{\bar{d}\wedge t}(x) = \frac{1}{\alpha\beta x}\left\{\begin{array}{c} 1 - e^{-xt}\left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right)\right] \\ -\int\limits_0^t \frac{s^{\alpha-1}e^{-s\left(x+\frac{1}{\beta}\right)}}{\Gamma(\alpha)\beta^{\alpha}}ds \end{array}\right\} + e^{-xt}\mathbb{P}\left[\bar{d} \geq t\right] \tag{e.7}$$

$$= \frac{1}{\alpha\beta x}\left\{\begin{array}{c} 1 - e^{-xt}\left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right)\right] \\ -\frac{t^{\alpha}}{\Gamma(\alpha)\beta^{\alpha}}\sum\limits_{n=0}^{\infty}\frac{\left[-\left(x+\frac{1}{\beta}\right)t\right]^n}{n!(\alpha+n)} \end{array}\right\} + e^{-xt}\mathbb{P}\left[\bar{d} \geq t\right] \tag{e.7.1}$$

Note that for $t = \infty$ and $x < -\frac{1}{\beta} < 0$ (e.7.1) shows that $\mathcal{L}_{\bar{d}}(x) = \infty$.

For finite t how many terms of the sum in (e.7.1) are needed to calculate $\mathcal{L}_{\bar{d} \wedge t}(x)$ within roughly an absolute accuracy of ϵ? The idea will be to require that n be large enough to achieve both (a) the additional term at n be smaller than $\frac{\epsilon}{2}$ and (b) the ratio of the term at $n+1$ to the term at n be less than $\frac{1}{2}$. A powers series argument then will force the entire tail of the summation from n to be smaller than ϵ:

$$\begin{aligned}
-\frac{1}{\alpha\beta x}\frac{t^{\alpha}}{\Gamma(\alpha)\beta^{\alpha}}\frac{\left[-\left(x+\frac{1}{\beta}\right)t\right]^{n}}{n!\,(\alpha+n)} &< \frac{\epsilon}{2} \\
\frac{\left[-\left(x+\frac{1}{\beta}\right)t\right]^{n}}{n!\,(\alpha+n)} &< -\frac{\alpha\beta x\Gamma(\alpha)\beta^{\alpha}}{t^{\alpha}}\frac{\epsilon}{2} \\
n\ln\left(-\left(x+\frac{1}{\beta}\right)t\right) - \ln(n!) - \ln(\alpha+n) &< \ln\left(-\frac{\alpha\beta x\Gamma(\alpha)\beta^{\alpha}}{t^{\alpha}}\frac{\epsilon}{2}\right) \qquad \text{(e.7.2)}
\end{aligned}$$

will achieve (a). Now taking the ratio of two terms:

$$\begin{aligned}
\frac{\left[-\left(x+\frac{1}{\beta}\right)t\right](\alpha+n)}{(n+1)(\alpha+n+1)} &< \frac{1}{2} \\
\frac{(\alpha+n)}{(n+1)(\alpha+n+1)} &< \frac{1}{2\left[-\left(x+\frac{1}{\beta}\right)t\right]} \\
n &> 2\left[-\left(x+\frac{1}{\beta}\right)t\right]\frac{(\alpha+n)}{(\alpha+n+1)} - 1
\end{aligned}$$

will achieve (b) and is satisfied when

$$n > 2\left[-\left(x+\frac{1}{\beta}\right)t\right] - 1 \qquad \text{(e.7.2)}$$

.

If n is chosen large enough to satisfy both (e.7.2) and (e.7.3) then the approximation in using n terms of the sum in (e.7.1) should be accurate within an absolute accuracy of ϵ.

Going back to (e.7) we can get some more information out of integrating by parts on the exponential rather than expanding the exponential:

$$\begin{aligned}
\mathcal{L}_{\bar{d}\wedge t}(x) &= \frac{1}{\alpha\beta x}\left\{\begin{array}{c} 1-e^{-xt}\left[1-\Gamma\left(\alpha;\frac{t}{\beta}\right)\right] \\ -\int\limits_0^t \frac{s^{\alpha-1}e^{-s\left(x+\frac{1}{\beta}\right)}}{\Gamma(\alpha)\beta^\alpha}ds \end{array}\right\} + e^{-xt}\mathbb{P}\left[\bar{d}\geq t\right] \\
&= \mathcal{L}_{\bar{d}\wedge t}(x) = \frac{1}{\alpha\beta x}\left\{\begin{array}{c} 1-e^{-xt}\left[1-\Gamma\left(\alpha;\frac{t}{\beta}\right)\right] \\ +\frac{1}{\Gamma(\alpha)\beta^\alpha}e^{-t\left(x+\frac{1}{\beta}\right)}\sum\limits_{j=0}^{k}\frac{(\alpha-1)\cdots(\alpha-j)}{\left(x+\frac{1}{\beta}\right)^{j+1}}t^{\alpha-j-1} \\ -\frac{(\alpha-1)\cdots(\alpha-k-1)}{\Gamma(\alpha)\beta^\alpha\left(x+\frac{1}{\beta}\right)^{k+1}}e^{-t\left(x+\frac{1}{\beta}\right)}\int\limits_0^t s^{n-k-2}e^{-(s-t)\left(x+\frac{1}{\beta}\right)}ds \end{array}\right\} + e^{-xt}\mathbb{P}\left[\bar{d}\geq t\right]
\end{aligned} \tag{e.8}$$

for any $k \leq \lfloor \alpha - 1 \rfloor$. If α happens to be an integer this provides an exact calculation for $\mathcal{L}_{\bar{d}\wedge t}(x)$ since the sum terminates with a zero error term when $k = \alpha - 1$. In this case, one need not use the approximation in (e.7.1) through (e.7.3). In addition to providing this exact calculation for integer α, (e.8) also helps to provide estimates for some later expressions involving $\mathcal{L}_{\bar{d}\wedge t}(x)$. For now, note that the integral in the error term in (e.8) is $O(t^{\alpha-k-2})$ because $-(s-t)(x+\frac{1}{\beta}) < 0$ and $e^{-(s-t)(x+\frac{1}{\beta})}$ makes the integrand irrelevant until s is close to t.

General Formulae To Calculate (2.8) and Related Expected Values

Corresponding to (2.10), (2.11) corrected, and page 7 of the 8-12-2008 corrections the following general formula is available:

$$\mathbb{E}\left[\left(\sum_{j=1}^{\infty} e_j^2\right)^n\right] =$$

$$= \sum_{\left\{i_l:\sum\limits_l l\cdot i_l=n\right\}} \frac{n!}{\prod\limits_l l!^{i_l}}\rho_{\{i_l\}} \left\{ \sum_{\left\{j_m,i_{l,m}:\sum\limits_m j_m\cdot i_{l,m}=i_l\right\}} \prod_m \frac{1}{j_m!}\left[(-1)^{\left(\sum\limits_l i_{l,m}-1\right)}\cdot \right.\right.$$
$$\left.\left. \cdot \frac{\left(\sum\limits_l i_{l,m}-1\right)!}{\prod\limits_l i_{l,m}!}\sum_{j=1}^{\infty}\left(\prod_l \mathbb{E}\left[e_j^{2l}\right]^{i_{l,m}}\right)\right]^{j_m}\right\} \tag{2.8}$$

$$\begin{aligned}
\text{defining } \rho_{\{i_l\}} &= \rho_{\ldots,l+1,\ldots,l+1,l,\ldots,l,l-1,\ldots,l-1,\ldots} \\
\text{where each } l\text{, for } n &\geq l \geq 1\text{, appears } i_l \text{ times.}
\end{aligned}$$

Corresponding to what's needed to compute (2.6) in the 8-12-2008 corrections:

$$\mathbb{E}\left[\left(\sum_{j=1}^{\infty} e_j^2\right)^n (1-F)^{-k\mathbf{t}_1\wedge t}\right] =$$

$$= \sum_{\left\{i_l : \sum_l l\cdot i_l = n\right\}} \frac{n!}{\prod_l l!^{i_l}} \rho_{\{i_l\},-\frac{k}{2}} \left\{ \sum_{\left\{j_m, i_{l,m} : \sum_m j_m \cdot i_{l,m} = i_l\right\}} \prod_m \frac{1}{j_m!} \left[(-1)^{\left(\sum_l i_{l,m}-1\right)} \cdot \right.\right.$$

$$\left.\left. \cdot \frac{\left(\sum_l i_{l,m}-1\right)!}{\prod_l i_{l,m}!} \sum_{j=0}^{\infty} \left(\prod_l \mathbb{E}\left[e_j^{2l}\right]^{i_{l,m}}\right)\right]^{j_m} \right\} \cdot \mathbb{E}\left[(1-F)^{-k\mathbf{t}_1\wedge t}\right] \tag{2.6.f}$$

By the rules for ρ at (4.5.8) $\rho_{\{i_l\},-\frac{k}{2}} = \rho_{\{i_l\}} \rho_{\sum_l l\cdot i_l,-\frac{k}{2}} = \rho_{\{i_l\}} \rho_{n,-\frac{k}{2}}$, so we can simplify (2.6.f) slightly to

$$\mathbb{E}\left[\left(\sum_{j=1}^{\infty} e_j^2\right)^n (1-F)^{-k\mathbf{t}_1\wedge t}\right] =$$

$$= \sum_{\left\{i_l : \sum_l l\cdot i_l = n\right\}} \frac{n!}{\prod_l l!^{i_l}} \rho_{\{i_l\}} \left\{ \sum_{\left\{j_m, i_{l,m} : \sum_m j_m \cdot i_{l,m} = i_l\right\}} \prod_m \frac{1}{j_m!} \left[(-1)^{\left(\sum_l i_{l,m}-1\right)} \cdot \right.\right.$$

$$\left.\left. \cdot \frac{\left(\sum_l i_{l,m}-1\right)!}{\prod_l i_{l,m}!} \sum_{j=0}^{\infty} \left(\prod_l \mathbb{E}\left[e_j^{2l}\right]^{i_{l,m}}\right)\right]^{j_m} \right\} \rho_{n,-\frac{k}{2}} \mathbb{E}\left[(1-F)^{-k\mathbf{t}_1\wedge t}\right] \tag{2.6.g}$$

A complete formula for computing (2.6)

If we take the formula for (2.6) in the 8-12-2008 corrections, expand the two binomials, and interchange the order of summations so that powers of $\sum_{j=1}^{\infty} e_j^2$ are summed over first, followed by powers of $(1-F)^{-\mathbf{t}_1 \wedge t}$, and finally substitute the expression (2.6.g) just derived into each occurence of $\mathbb{E}\left[\left(\sum_{j=1}^{\infty} e_j^2\right)^l (1-F)^{-k\mathbf{t}_1 \wedge t}\right]$, the resulting expression is:

$$
\begin{aligned}
&\mathbb{E}\left[(\ln(r_t) - \mathbb{E}[\ln(r_t)])^n\right] = \\
= \; & \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{l!} \sigma_T^{2l} \mathbb{E}\left[\left(\sum_{j=1}^{\infty} e_j^2 \right)^l \sum_{k=0}^{n-2l} \frac{1}{k!} (1-F)^{-k\mathbf{t}_1 \wedge t} \cdot \right. \\
& \sum_{m=0}^{\lfloor \frac{n-k-2l}{2} \rfloor} \frac{n!(2(m+l))?}{(n-2(m+l))!(2(m+l))!} (-1)^{n-2(m+l)-k} (\ln(T_0) - \mu_T)^{n-2(m+l)} \cdot \\
& (1-F)^{(n-2(m+l))t} \frac{(n-2(m+l))!}{(n-2(m+l)-k)!} \mathbb{E}\left[(1-F)^{-\mathbf{t}_1 \wedge t}\right]^{n-2(m+l)-k} \cdot \\
& \left. \frac{(m+l)!}{m!} \sigma^{2m} dt^m (1-F)^{2mdt} \left(\frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}} \right)^m \right] \\
= \; & \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \sigma_T^{2l} \left(\sum_{\left\{ i_g : \sum_g g \cdot i_g = l \right\}} \frac{1}{\prod_g g!^{i_g}} \rho_{\{i_g\}} \left\{ \sum_{\left\{ j_h, i_{g,h} : \sum_h j_h \cdot i_{g,h} = i_g \right\}} \prod_h \frac{1}{j_h!} \cdot \right. \right. \\
& \left. \left. \left[(-1)^{(\sum_g i_{g,h} - 1)} \frac{(\sum_g i_{g,h} - 1)!}{\prod_g i_{g,h}!} \sum_{j=1}^{\infty} \left(\prod_g \mathbb{E}\left[e_j^{2g}\right]^{i_{g,h}} \right) \right]^{j_h} \right\} \right) \left(\sum_{k=0}^{n-2l} \frac{1}{k!} \rho_{l,-\frac{k}{2}} \cdot \right. \\
& \mathbb{E}\left[(1-F)^{-k\mathbf{t}_1 \wedge t}\right] (\ln(T_0) - \mu_T)^k (1-F)^{kt} \left[\sum_{m=0}^{\lfloor \frac{n-k-2l}{2} \rfloor} \frac{n!(2(m+l))?(m+l)!}{(n-2(m+l)-k)!(2(m+l))!m!} \cdot \right. \\
& \left[-(\ln(T_0) - \mu_T)(1-F)^t \mathbb{E}\left[(1-F)^{-\mathbf{t}_1 \wedge \mathbf{t}}\right] \right]^{n-2(m+l)-k} \sigma^{2m} dt^m (1-F)^{2mdt} \cdot \\
& \left. \left. \left(\frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}} \right)^m \right] \right)
\end{aligned}
\tag{2.6}
$$

A Slight Restatement of (3.5) and (3.6)

The closed form just presented for (2.6) requires the ability to evaluate expressions such as

$$\sum_{j=1}^{\infty}\left(\prod_{k=1}^{n}\mathbb{E}\left[e_j^k\right]^{n_k}\right). \tag{3.5.a}$$

This can be evaluated by (3.5) in the original paper (or (3.6) for $t = \infty$). This is not immediately apparent because (3.5) is written in a slightly different form. Here is how to restate (3.5) to a form consistent with the form of (3.5.a).

Given the expression (3.5.a), for any variable x (whether random or not) define

$$\nu(x) = \prod_{k=1}^{n}\mathbb{E}\left[x^k\right]^{n_k}$$

Obviously if x is a non-random variable then

$$\nu(x) = \prod_{k=1}^{n} x^{k\cdot n_k}$$

This definition of $\nu(x)$ means that (3.5.a) looks like

$$\sum_{j=1}^{\infty}\left(\prod_{k=1}^{n}\mathbb{E}\left[e_j^k\right]^{n_k}\right) = \sum_{j=0}^{\infty}\nu(e_j).$$

This definition of $\nu(x)$ is equivalent to the one on page 11 of the original paper. The only change is that the original paper had

$$\mathbb{E}\left[\sum_{j=1}^{\infty}\left(e_j^n\prod_{k=1}^{n}\mathbb{E}\left[e_j^k\right]^{n_k}\right)\right] \text{ and } \nu(x) = x^n\prod_{k=1}^{n}\mathbb{E}\left[x^k\right]^{n_k}.$$

So long as monotone or dominated convergence applies (monotone applies in all of our applications) then

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^{\infty}\left(e_j^n\prod_{k=1}^{n}\mathbb{E}\left[e_j^k\right]^{n_k}\right)\right] &= \sum_{j=1}^{\infty}\left(\mathbb{E}\left[e_j^n\right]^{n_n+1}\prod_{k=1}^{n-1}\mathbb{E}\left[e_j^k\right]^{n_k}\right) \text{ and}\\
\mathbb{E}\left[x^n\prod_{k=1}^{n}\mathbb{E}\left[x^k\right]^{n_k}\right] &= \mathbb{E}\left[x^n\right]^{n_n+1}\prod_{k=1}^{n-1}\mathbb{E}\left[x^k\right]^{n_k} \text{ for random } x \text{ and}\\
x^n\prod_{k=1}^{n}x^{k\cdot n_k} &= x^{n\cdot(n_n+1)}\prod_{k=1}^{n-1}x^{k\cdot n_k} \text{ for non-random } x.
\end{aligned}$$

Therefore (3.5) and (3.5.a), together with the corresponding versions of $\nu(x)$, differ only in how to count the exponent on $\mathbb{E}\left[e_j^n\right]$ or $\mathbb{E}\left[x^n\right]$, where n is the highest power of e_j or x that appears.

With this notational understanding we can now rewrite (3.5) as

$$\sum_{j=1}^{\infty}\left(\prod_{k=1}^{n}\mathbb{E}\left[e_j^k\right]^{n_k}\right)=\sum_{j=1}^{\infty}\nu(e_j)$$

$$= \quad \left(1-(1-G)^t\,\frac{\mathbb{E}\left[(1-G)^{-\bar{\mathbf{d}}\wedge t}\right]-\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right](1-G)^{-t}}{\mathbb{E}\left[(1-G)^{\bar{\mathbf{d}}\wedge t}\right]-\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right](1-G)^{t}}\right)\cdot$$

$$\left\{\nu\left((1-F)^{\bar{\mathbf{d}}\wedge t}\right)-\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right]\nu\left((1-F)^t\right)\right\}\frac{\nu\left(1-(1-F)^{\mathbf{d}}\right)}{1-\nu\left((1-F)^{\mathbf{d}}\right)}+$$

$$+\nu\left(1-(1-F)^{\bar{\mathbf{d}}\wedge t}\right)-\mathbb{P}\left[\bar{\mathbf{d}}\geq t\right]\nu\left(1-(1-F)^t\right) \tag{3.5.a}$$

and (3.6) as

$$\lim_{t\to\infty}\sum_{j=1}^{\infty}\left(\prod_{k=1}^{n}\mathbb{E}\left[e_j^k\right]^{n_k}\right)=\lim_{t\to\infty}\sum_{j=1}^{\infty}\nu(e_j)$$

$$=\frac{\nu\left((1-F)^{\bar{\mathbf{d}}}\right)\nu\left(1-(1-F)^{\mathbf{d}}\right)}{1-\nu\left((1-F)^{\mathbf{d}}\right)}+\nu\left(1-(1-F)^{\bar{\mathbf{d}}}\right) \tag{3.6.a}$$

Now (3.5.a) and (3.6.a) are suitable for evaluating the expression above for (2.6).

On page 10 of the 8-12-2008 corrections there is an error in the asymptotic expression for drift compensation term D_t as $t\to\infty$. The correct formula is

$$\lim_{t\to\infty}D_t=$$

$$=-\frac{1}{2}\sigma^2\frac{(1-F)^{dt}}{1+(1-F)^{dt}}+\frac{1}{2}\sigma_T^2\frac{1-(1-F)^{dt}}{dt(1-F)^{dt}}\left(1-\lim_{t\to\infty}\mathbb{E}\left[\sum_{j=1}^{\infty}e_j^2\right]\right)$$