STATIONARY IMMUNIZATION THEORY

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Summary

Classical immunization theory studies conditions under which present-value relationships are immune to adverse development if interest rates change. It can be characterized as "active immunization" because maintenance of an immunized position over time can require active trading - rebalancing - in response to interest rate movements or to just the evolution of the portfolio.

This paper explores a concept that could be called "stationary immunization" because of an analogy with the stationary population theory in life contingencies. It deals with conditions under which the volatility of annual interest rate spreads in a portfolio managed on a buy-and-hold strategy is modulated relative to the volatility of market interest rates. Since active immunization suboptimizes yields, there is merit to a concept of immunization built around a buy-and-hold model, avoiding transaction costs and undue exposure to the short end of the yield curve.

All of the information about maturity patterns of assets and liabilities, and their interaction, is contained in the moments of their respective schedules of principal repayments, considered as generalized density functions. Based on this analogy, complex formulas express interest rate spreads in terms of asset and liability principal repayment schedules interacting with the evolution of market interest rates. The formulas resolve into (1) "stationary" components, which prevail in a mature portfolio and reflect generalized moments of the repayment schedules, plus (2)"transient" components, which reflect start-up anomalies and disappear over time in a stably growing portfolio.

The resulting formulas describe the response characteristics of a stably growing insurance company viewed as an "antenna" or "tuner" that modulates an incoming signal of market interest rates over time into an output signal of interest rate spreads in the portfolio over time. The task of immunization is (or should be) to maintain in the portfolio a modulation structure that will dampen the volatility of the output response to any such input signal. Mathematically, a Fourier transform expresses on-going sensitivity of portfolio interest rate spreads to market interest rate cyclicity across the entire spectrum, replacing the Taylor's series coefficients by which classical immunization expresses sensitivity of present values just to one-time pulses in interest rates.

The reason to explore such a highly idealized model is similar to the reason to explore stationary populations in mortality theory. Each provides a framework and a touchstone to identify what might prove to be systematically important aspects of real world relationships. In addition, we glimpse the possibility of a model of the insurance enterprise complementary, in the sense of a mathematical duality, to the balance sheet focused present-value model that until now has dominated life actuarial practice.

TEORIA DE INMUNIZACION ESTACIONARIA

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Resumen

La teoría clásica de inmunización estudia las condiciones bajo las cuales relaciones de valor presente quedan inmunes a desarrollos adversos en caso de cambios en las tasas de interés. Se puede caracterizar como "inmunización activa" porque el mantenimiento de una posición inmunizada puede requerir transacciones activas - balanceo - en respuesta a movimientos en tasas de interés o a la evolución del portafolio.

Este ensayo explora un concepto que podría ser llamado "inmunización estacionaria" por una analogía con la teoría de población estacionaria en contingencias de vida. El ensayo explora las condiciones bajo las cuales la volatilidad de la diferencia en tasas de interés anuales en un portafolio administrado bajo una estrategia de "comprar y mantener" es modulada en relación a la volatilidad de tasas de interés del mercado. Dado a que la inmunización activa sub-optimiza los rendimientos, existe mérito en un concepto de inmunización construido alrededor de un modelo de "comprar y mantener".

Toda la información sobre patrones de vencimiento de activos y pasivos y su interacción, está contenida en los momentos de sus respectivas planillas de re-pagos a la suma principal, considerada como funciones de densidad generalizadas. Basada en esta analogía, fórmulas complejas expresan las diferencias en tasas de interés en términos de planillas de re-pago a la suma principal de activos y pasivos interactuando con la evolución de tasas de interés en le mercado. Las fórmulas resuelven hacia (1) componentes "estacionarios", que prevalecen en un portafolio maduro, más (2) componentes "transitorios", que desaparecen a través del tiempo en un portafolio de crecimiento estable.

Las resultantes fórmulas describen las características de respuesta de una compañía de seguros creciendo establemente siendo vista como una "antena" o "sintonizador" que modula una señal entrante de tasas de interés de mercado a través del tiempo hacia una señal saliente de diferencias en tasas de interés del portafolio a través del tiempo. La tarea de inmunización es (o debería ser) el mantener en el portafolio una estructura de modulación que disminuya la volatilidad de las respuestas a cualquiera de estas señales entrantes. Matemáticamente, una transformación Fourier expresa la sensibilidad corriente hacia la ciclicalidad de tasas de interés a través del todo el espectro, reemplazando los coeficientes de la serie de Taylor de inmunización clásica.

Adicionalmente, vemos la posibilidad de un modelo de la compañía de seguros complementaria, en el sentido matemático de dualidad, al balance general enfocado al modelo de valor presente que hasta ahora ha dominado la práctica actuarial.

I. Introduction

Classical asset/liability immunization theory works with present values of assets and liabilities. It studies the theoretical conditions governing relationships between present values of asset cash flows in a portfolio and present values of liability cash flows in a related portfolio. In particular, it establishes the theoretical conditions under which such relationships are immune to adverse development if interest rates should change.

As any such theory, classical immunization theory has limitations which suggest exploration of alternative theoretical points of view, not to replace classical immunization theory but to complement it in a richer total framework. In particular, classical immunization theory can be characterized as requiring "active" immunization. Maintenance of an immunized position over time requires active trading of the portfolio - rebalancing - in response **both** to interest rate movements **and** to just the simple evolution of the portfolio over time. If rebalancing fails to occur for any reason, the supposed immunization will turn out to have been a chimera.

Transaction costs and the possible need to increase exposure at times to the short end of the yield curve, usually at lower yields, suggest that the rebalancing implicit in classical immunization practice suboptimizes returns over time, perhaps beyond a reasonable risk premium for the protection afforded. A little more fundamentally, the theoretical model to calculate present values in the first place reflects an enormously unrealistic oversimplification of the true complex economics of the insurance enterprise. The fact that the oversimplification involved in the present value concept has served so many useful purposes so very well over time does not necessarily imply that it will do so well for the asset/liability mismatch problem, or that (at a minimum) another model of insurance economics would not provide a useful complement for the purpose.

At an even deeper level, to focus exclusively on present value relationships creates vulnerability to the fundamental paradox that no single accounting model can present simultaneously a completely accurate and timely balance sheet and a completely accurate and timely income statement. This is directly analogous to the uncertainty principle in physics. Concepts such as reserves, asset values, margins and profits radically interpenetrate and distort each other. But just as the mutual distortions of positions and momenta are related by disciplined principles of complementarity in quantum mechanics, one can hope to develop disciplined principles of complementarity between the balance sheet focused present value account of the insurance enterprise, on the one hand, and on the other hand ... what?

This paper explores the behavior of an enormously unrealistic oversimplification of the true complex economics of the insurance enterprise. But not a present value appears anywhere. Instead, the paper focuses on the ***interest rate spread*** between the interest currently available from the assets versus the interest currently required on the liabilities, as the enterprise evolves over time. Put another way, this model develops the "momentum" picture of the insurance enterprise, in contrast to the "position" picture at the foundation of classical immunization theory.

The gross simplifications this model makes will create a parallel to the stationary population theory of life contingencies. Here we model two portfolios (populations), one of assets and another of liabilities, and the interaction between the two over time and with changing interest rates. Nevertheless, the close parallel determines the title "stationary" immunization theory. (Actually, the somewhat more general stable growth concept governs the model.)

The model portfolios grow from the assumption that at each point in time:

1. The model takes on new liabilities at an exponentially growing rate.
2. The principal value of liabilities taken on at each prior point in time grows and/or runs-off according to a schedule that is the same for each such prior generation of liabilities.
3. The principal value of assets purchased at each prior point in time grows and/or runs-off according to a schedule that is the same for each such prior generation of assets.
4. New assets are purchased with the net cash flow, defined as the net of (1) through (3), borrowing on the same terms if negative.
5. Each generation of assets earns interest at the market rate of interest that prevailed when it was purchased.
6. Each generation of liabilities requires interest at the market rate of interest that prevailed when it was taken on.
7. The **interest rate spread** is the difference between (5) and (6), in the net premium sense that if all goes according to plan the spread would be zero. (If the model handles all this well, addition of an explicit margin later would be no problem.)

The formulas that result are forbidding, but generally resolve into a "stationary" component that has some intuitive appeal and prevails once the portfolios mature into a stable system, plus thorny "transient" components that reflect start-up anomalies and disappear over time. Occasionally, a "residual" crops up, which is the remnant of past transients that have not disappeared, but that change vanishingly less over time.

There is no such thing as an insurance enterprise quite this simple, but then there is no such thing as a present value, either. Together one could hope that they provide more rich and complementary a view than does either alone. Before that can happen, the interest-rate-spread point of view requires elaboration perhaps comparable to that the present-value view has experienced over the decades.

Section II. below develops mathematical notations and results that allow a simplification of most of the analytic complexities that would otherwise arise in the subsequent model development to algebraic manipulations. Sections III., IV., and V. then apply that machinery to develop the liability, asset, and interest rate spread components, respectively, of the general model in largely algebraic fashion. Section VI. recasts the interest rate spread component of the model into a form that explicitly displays its dependence on changes in market interest rates. This step, unfortunately, uses some ponderous calculus to supplement the algebraic logic. Section VII. specializes the general model to two specific applications, a stochastic model and a model of the effects of an interest rate jump, which bring to a focus almost all of the prior developments. Finally, Section VIII. applies a Fourier transform to analyze the characteristics of the general model of Section VI. more deeply, followed by some concluding remarks in Section IX.

Especially for core formulas that underlie the work, explicit expressions for the error terms involved, usually identified as "transients," accompany the simplifying approximations developed in this paper. This complicates things, but can be anticipated to help with future attempts to relax some of the simplifying assumptions made here.

Source references made by authors' names within the text of the paper are listed at the end. DeVylder's monograph came to hand only after completing this work. Its initial chapters offer convenient reference for the manner in which some of the purely mathematical ground in Section II. of this paper has been covered already in risk theory, but the application to asset/liability modeling appears to be new, as do some of the generalizations in Section II. involving exponential growth.

II. Notation and Mathematical Preliminaries

We will model the maturity schedules for assets and liabilities according to an analogy with probability density and distribution functions. Means and higher moments of the maturity schedules, together with some generalizations of those concepts, encode important information about sensitivity to interest rate changes. We will need shorthand notation to model and manipulate such information. Portfolio growth rates add further complexity to the model. Convolution notation and delta-functions help to manage the complexity.

CONVOLUTIONS

Given two functions $f(x)$ and $g(x)$, their **convolution** $(f * g)(y)$ is a new function defined by $(f*g)(y) = \int_{-\infty}^{\infty} f(y-x)g(x)dx$. Risk theorists (see DeVylder, sec. I.1.3.2, or Wooddy) prefer to write $f * G$ for this integral instead of $f * g$, where G is a distribution with density $G' = g$. The $f * g$ used here follows the more traditional notation of real analysis and physics (see sec. 7.13 of Rudin, ch. 4 of Brigham, or sec. 2.3 and 2.4 of James. The latter two also show the connection between convolutions and delta functions, introduced below.)

Convolutions follow the rules $f * g = g * f$, $f * (g * h) = (f * g) * h$, and $f * (g + h) = f * g + f * h$. A property that we will need later is that if $e(y - x) = e(y) / e(x)$, e.g. if $e(x)$ is an exponential function, then

$$(e \cdot f) * (e \cdot g) = e \cdot (f * g) \qquad \textbf{(II.1)}$$

where $(e \cdot g)(x) = e(x)\, g(x)$. The proofs follow directly from the definition of convolution.

DELTA FUNCTIONS

Let $\Delta(x)$ be the function defined by $\Delta(x) = 1$ for $x \geq 0$ and $\Delta(x) = 0$ for $x < 0$. Then for any function $f(x)$, $(\Delta * f)(y) = \int_{-\infty}^{y} f(x)dx$ follows directly from the definition of convolution, which will simplify many formulas. In fact, the technical content of this paper is a fugue on **integration by parts** (if $f(-\infty) = g(-\infty) = 0$ then

$$\Delta * (f \cdot g) = (\Delta * f) \cdot g - \Delta * ((\Delta * f) \cdot g'), \text{ where } g' \text{ is the derivative of } g)$$

in counterpoint with II.1. If we define $I(x) = x$ for $x \geq 0$, and $I(x) = 0$ for $x < 0$, then it follows immediately from the definitions of Δ and of convolution that

$$\Delta * \Delta = I. \qquad \textbf{(II.2)}$$

The **Dirac delta-function** $\delta(x)$ is a generalized "function," special rules for the use of which this paper will reflect scrupulously, but without explicit recitation. It is defined by $\delta(x) = 0$ for $x \neq 0$ and $(\delta * f)(x) = f(x)$ for all functions $f(x)$. $\delta(x)$ is the derivative of $\Delta(x)$, which we will write as $\Delta'(x) = \delta(x)$.

Sec. 15 of Dirac is still the best short introduction to delta functions and their properties. For a rigorous development, unfortunately, none of the sources is easy. Sec. 5.3 of Robinson may be the most coherent of the rigorous treatments, but its rigor is the (ultra)product of a highly subtle framework from technical model theory. Dirac denotes by $\varepsilon(x)$ what this paper calls $\Delta(x)$. Elsewhere in the physics literature $H(x)$ or $\theta(x)$ may be found, depending upon the author. This paper uses $\Delta(x)$ for consistency with the upper case/lower case notation convention for distributions/densities followed for all other functions that appear in the paper.

DENSITIES AND DISTRIBUTIONS

Let $f(x)$ be a function such that $\int_{-\infty}^{\infty} f(x)dx = 1$. Then $f(x)$ is analogous to a **probability density function**, except that we allow $f(x) < 0$ and $\int f(x)dx > 1$ over subintervals of the real line to occur so long as the full integral is still unity. This paper always assumes that $f(x) = 0$ for $x < 0$, and some of the results require that $f(x)$ be of bounded variation, which is hereby assumed. Neither assumption impairs asset and liability maturity schedule modeling.

Define $F(y) = (\Delta * f)(y)$, so that $F'(x) = f(x)$. Then $F(y)$ is analogous to a **probability distribution function** corresponding to $f(x)$, except that $F(y)$ might not be monotonic, and in fact $F(y) < 0$ and $F(y) > 1$ are possible for some values of y, so long as $F(y) \to 1$ as $y \to \infty$.

PARTIAL MOMENTS

Define the **mean, or first moment**, of a distribution F by ${}_F\mu = \int_{-\infty}^{\infty} xf(x)dx = \lim_{y\to\infty}\left(\Delta * (I \cdot f)\right)(y)$. Now generalize this concept by defining the function ${}_F\mu(y) = (\Delta * (\Delta - F))(y)$. To see that this function deserves the notation suggesting a mean,

$$
\begin{aligned}
{}_F\mu(y) &= (\Delta * (\Delta \cdot (\Delta - F)))(y), \text{ trivially, and integrating by parts} \\
&= y\,(\Delta - F)(y) - (\Delta * (I \cdot (\delta - f)))(y) \\
&= y\,(\Delta - F)(y) + (\Delta * (I \cdot f))(y) \qquad \textbf{(II.3)}
\end{aligned}
$$

because $\Delta * \Delta = I$, $(\Delta - F)' = (\delta - f)$, and $I \cdot \delta = 0$ everywhere. Since $(\Delta - F)(y) \to 0$ as $y \to \infty$, II.3 implies that ${}_F\mu(y) \to {}_F\mu$ as $y \to \infty$. (This is one of the steps requiring bounded variation.) To avoid confusion between ${}_F\mu(y)$ the function and ${}_F\mu$ the value of the mean, we will always write ${}_F\mu^{\infty}$ for the value of the mean. Looking further, ${}_F\,\mu(y) = y\,(\Delta - F)(y) + [\Delta * (I \cdot (f\,/\,F(y)))](y)\,F(y)$, so ${}_F\,\mu(y)$ can be viewed as a sort of "**partial mean**," a weighted average of the amount y (for values of $x \geq y$) with an amount equal to a truncated mean of F (over only values of $x<y$).

Except for a normalizing factor, ${}_F\,\mu(y)$ is what risk theory calls the "concave transform" (sec. I.3.2.2 of DeVylder), but the word "concave" no longer may be apt since this paper does not restrict $F(x)$ to the range of a true probability distribution (we allow $F(x) < 0$ and $F(x) > 1$.) II.3 is the "surface interpretation of the first moment" (sec. I.3.2.1 of DeVylder). Both concepts extend to higher moments as follows.

Define the **second moment** of a distribution F by

${}_F m_2 = \int_{-\infty}^{\infty} x^2 f(x)dx = \lim_{y\to\infty}\left(\Delta * \left(I^2 \cdot f\right)\right)\ (y)$. Generalize this concept by defining the

function ${}_F\,m_2(y) = 2\,(\Delta * (I \cdot (\Delta - F)))(y)$. Integrating by parts,

$$\begin{aligned} {}_F\,m_2(y) &= y^2\,(\Delta - F)(y) - (\Delta * (I^2 \cdot (\delta - f)))(y) \\ &= y^2\,(\Delta - F)(y) + (\Delta * (I^2 \cdot f))(y) \end{aligned} \qquad \textbf{(II.4)}$$

just as in the case of the first moment and ${}_F\,m_2(y) \to {}_F\,m_2$ as $y \to \infty$. To avoid confusion between ${}_F\,m_2(y)$ the function and ${}_F\,m_2$ the value of the second moment, we will always write ${}_F\,m_2^{\infty}$ for the value of the second moment.

Similarly, ${}_F\,m_2(y) = y^2\,(\Delta - F)(y) + [\Delta * (I^2 \cdot (f\,/\,F(y)))](y)\,F(y)$, and ${}_F\,m_2(y)$ can be viewed as a sort of "**partial second moment**," the weighted average of the amount y^2 (for values of $x \geq y$) with a truncated second moment of F (over only values of $x < y$).

An extraordinarily useful relationship (which will be recognized from stationary population theory) arises between the first and second partial moment concepts:

$$\begin{aligned} \Delta * {}_F\,\mu &= \Delta * (\Delta \cdot {}_F\,\mu)\text{, trivially, and integrating by parts} \\ &= I \cdot {}_F\,\mu - \Delta * (I \cdot (\Delta - F))\text{, using } \Delta * \Delta = I \\ &\qquad\qquad \text{and } {}_F\,\mu' = (\Delta * (\Delta - F))' = (\Delta - F), \\ &= I \cdot {}_F\,\mu - (^1/_2)\,{}_F\,m_2\text{, by definition of } {}_F\,m_2 \end{aligned} \qquad \textbf{(II.5)}$$

(See ch. 8, sec. 1 of Jordan. Our ${}_F\,\mu(x)$ is his $T_0 - T_x$ in the stationary population and our ${}_F\,m_2(x)$ is his $2\int_{0,x} y\,l_y\,dy = 2(Y_0 - Y_x - x\,T_x)$. II.5 also can be viewed as an extension of DeVylder, sec. I.3.2.2 Theorem 1, after applying both the concave transform and the surface interpretation back upon the concave transform itself.)

MEANS AND CONVOLUTIONS

Define $f^{*(0)} = \delta$

$f^{*(1)} = f$

$f^{*(2)} = f * f$

$f^{*(3)} = f * f * f$, and so on. For short let $\left(\sum f^{**}\right) = \sum_{n=0}^{\infty} f^{*(n)}$.

For a broad range of density functions *f*, including most of those that represent realistic asset or liability portfolio maturity schedules, $(\sum f^{**})(x) \to$ some definite limit as $x \to \infty$. (However, we mention below one class of density functions for which this is not true that includes some simple maturity schedules.) Assuming that $(\sum f^{**})(x)$ has a limit as $x \to \infty$, then the value of the limit is

$$(\sum f^{**})(x) \to 1 /_{\mathrm{F}}\, \mu^{\infty} \text{ as } x \to \infty \qquad \textbf{(II.6)}$$

To prove (II.6) note that if $(\sum f^{**})(x)$ has a limit as $x \to \infty$, then

$$\lim_{y\to\infty}\left(\left(\Delta - F\right) * \left(\Sigma f^{**}\right)\right)(y) = {}_{F}\,\mu^{\infty} \lim_{x\to\infty}\left(\Sigma f^{**}\right)(x) \qquad \textbf{(II.7)}$$

because in the convolution on the left, as $y \to \infty$, $(\Delta - F)(y - x)$ becomes negligible unless *x* is very large, in which case $(\sum f^{**})(x)$ approaches its assumed limiting value. In that case, the entire convolution on the left approaches $((\Delta - F) * \Delta)(y)$ times that limiting value. But $((\Delta - F) * \Delta)(y) = {}_{\mathrm{F}}\,\mu(y)$ and ${}_{\mathrm{F}}\,\mu(y) \to {}_{\mathrm{F}}\,\mu^{\infty}$ as $y \to \infty$, by II.3, proving II.7. On the other hand, the entire convolution on the left of II.7 must equal simply 1 for all positive *y* because

$$(\Delta - F) * (\sum f^{**}) = \Delta * (\delta - f) * (\sum f^{**}) = \Delta * ((\sum f^{**}) - f * (\sum f^{**})) = \Delta * f^{*(0)} = \Delta * \delta = \Delta.$$

Since the left side of II.7 is 1, the limit on the right of II.7 must be $1 /_{\mathrm{F}}\, \mu^{\infty}$, proving II.6.

When $(\sum f^{**})(x)$ does not have a limiting value it is because it oscillates endlessly nearer to a repeated cycle of accumulation points as $x \to \infty$. For this class of density functions, the limiting value $1 /_{\mathrm{F}}\, \mu^{\infty}$ still holds on average over the cycle of accumulation points for large *x*. This exception is a resonance phenomenon that requires such precise tuning that a typically complicated portfolio of assets or liabilities is unlikely to have a maturity schedule that falls into this exceptional class.

Variations on the theme of II.6 and the surrounding discussion provide some of the cornerstones for risk theory (see sec. I.1.4, sec. I.1.5 and app. A of DeVylder.) The nearly algebraic proof afforded in this paper using delta functions and partial mean functions, however, seems especially perspicuous.

STABLE GROWTH

To introduce a **constant growth rate** g into the mathematical tools outlined above, define ${}^{g}\Delta(x) = (1+g)^{x}\,\Delta(x)$. For any distribution F define ${}_{F}{}^{g}s$ to be the limit of $(\Delta * (f / {}^{g}\Delta))(y)$ as $y \to \infty$. We will see that ${}_{F}{}^{g}s$ functions as a kind of scaling factor in many formulas. It is the expected value of $(1 / {}^{g}\Delta)(x)$ on the density function $f(x)$. Now define a new density function ${}_{g}f = f / ({}_{F}{}^{g}s \cdot {}^{g}\Delta)$ with distribution ${}_{g}F = \Delta * {}_{g}f$. From the definition of ${}_{F}{}^{g}s$, we see that ${}_{g}F(y) \to 1$ as $y \to \infty$, so these meet the definitions of density and distribution functions. The notations ${}_{gF}\,\mu(y)$, ${}_{gF}\,\mu^{\infty}$, ${}_{gF}\,m_2(y)$, and ${}_{gF}\,m_2{}^{\infty}$ for the partial mean, mean, partial second moment and second moment of this new distribution ${}_{g}F$ follow the definitions set out and relationships derived in the preceding pages.

So far, this looks formally like the "exponential transformation" process in risk theory (sec. I.1.5.3 and sec. I.3.2.4 of DeVylder) with "adjustment coefficient ρ" equal to $-\ln(1+g)$ and "renewal equation coefficient q" equal to $({}_{F}\mu^{\infty}\ \ln(1+g))/(1-{}_{F}{}^{g}s)$ in the couple with the concave transform of F. To fit the requirements of an asset/liability model with growth, however, there will be some new concepts required that apparently have no precise counterparts yet in risk theory, to generalize the earlier partial mean and partial second moment concepts. Before proceeding to the generalizations, this is a convenient spot to prove a lemma about ${}^{g}\Delta$ that will be needed in developing the asset and liability models:

$$
\begin{aligned}
(\delta - f) * {}^{g}\Delta &= {}^{g}\Delta - f * {}^{g}\Delta \\
&= {}^{g}\Delta - {}^{g}\Delta \cdot ((f / {}^{g}\Delta) * \Delta)), \text{ by II.1.} \\
&= {}^{g}\Delta \cdot (\Delta - {}_{F}{}^{g}s({}_{g}f * \Delta)), \text{ factoring and using the definition of } {}_{g}f \\
&= {}^{g}\Delta \cdot (\Delta - {}_{F}{}^{g}s \cdot {}_{g}F), \text{ by definition of } {}_{g}F \\
&= {}^{g}\Delta \cdot ((1 - {}_{F}{}^{g}s)\,\Delta + {}_{F}{}^{g}s(\Delta - {}_{g}F)) \qquad \textbf{(II.8)}
\end{aligned}
$$

This implies that $[(\delta - f) * {}^{g}\Delta](x) \to (1 - {}_{F}{}^{g}s)\cdot {}^{g}\Delta(x)$ as $x \to \infty$.

Another way to look at II.8 is to say that $(\delta - f) * {}^{g}\Delta$ splits into:

a "**stationary**" (i.e. stably growing) component $(1 - {}_{F}{}^{g}s)\cdot {}^{g}\Delta(x)$ that characterizes the situation after x becomes large,

plus

a "**transient**" component ${}_{F}{}^{g}s({}^{g}\Delta \cdot (\Delta - {}_{g}F))(x)$ that reflects start up relationships and disappears over time as the situation matures.

Now to the generalizations of the partial mean and partial second moment concepts.

A GENERALIZED PARTIAL MEAN

By analogy with ${}_F\mu(y)$, define ${}_F^g\mu(y) = (\Delta * ({}^g\Delta \cdot (\Delta - {}_g F)))(y)$. An analysis similar to (II.3), integrating by parts, provides some intuitive meaning to this definition:

$$\begin{aligned} {}_F^g\mu &= (\Delta * {}^g\Delta) \cdot (\Delta - {}_g F) - \Delta * ((\Delta * {}^g\Delta) \cdot (\delta - {}_g f)) \\ &= (\Delta * {}^g\Delta) \cdot (\Delta - {}_g F) + \Delta * ((\Delta * {}^g\Delta) \cdot {}_g f) \end{aligned} \qquad \textbf{(II.9)}$$

So ${}_F^g\mu(y)$ is a "**partial expected value**" of the quantity $(\Delta * {}^g\Delta)(x)$ over the density function ${}_g f(x)$. As $y \to \infty$, ${}_F^g\mu(y) \to {}_F^g\mu^\infty$, the expected value of $(\Delta * {}^g\Delta)(x)$ over the density ${}_g f(x)$. The form (II.9) will prove useful in some manipulations, but it also will be useful to insert the calculation (basic calculus)

$$(\Delta * {}^g\Delta) = ({}^g\Delta - \Delta) / ln(1 + g) \qquad \textbf{(II.10)}$$

(As $g \to 0$, L'Hôspital's rule verifies that the expression in II.10 for $(\Delta * {}^g\Delta) \to I$ which equals $(\Delta * \Delta)$, consistent with II.2, and validating the use of notation suggesting a mean.)

Using (II.9), (II.10) and the definition of ${}_g f$ allows the calculation

$${}_F^g\mu^\infty = (1 - {}_F^g s) / ({}_F^g s \, ln(1 + g)) \qquad \textbf{(II.11)}$$

(now L'Hôspital's rule applied to II.11 verifies that ${}_F^g\mu^\infty \to {}_F\mu^\infty$ as $g \to 0$, so we are, in fact, dealing with a generalization of the concept of a mean.)

TWO GENERALIZED PARTIAL SECOND MOMENTS

There are three different ways to generalize the useful formula II.5 into this stable growth model. They will provide two different ways to generalize the partial second moment concept, both of which will appear in the asset and liability modeling. First,

$\Delta * {}_F^g\mu = \Delta * (\Delta \cdot {}_F^g\mu)$, trivially, and integrating by parts gives

$$\begin{aligned} &= I \cdot {}_F^g\mu - \Delta * (I \cdot ({}^g\Delta \cdot (\Delta - {}_g F))), \text{ since } {}_F^g\mu' = {}^g\Delta \cdot (\Delta - {}_g F). \\ &= I \cdot {}_F^g\mu - \Delta * ((I \cdot {}^g\Delta) \cdot (\Delta - {}_g F)). \end{aligned} \qquad \textbf{(II.12)}$$

Comparing with II.5, this suggests a definition of the **generalized partial second moment** function as ${}_F^g m_2 = 2\,[\Delta * ((I \cdot {}^g\Delta) \cdot (\Delta - {}_g F))]$, which integrates by parts and simplifies to give

$${}_F^g m_2 = 2[(\Delta * (I \cdot {}^g\Delta)) \cdot (\Delta - {}_g F) + \Delta * ((\Delta * (I \cdot {}^g\Delta)) \cdot {}_g f)] \qquad \textbf{(II.13)}$$

The form II.13 will prove useful in manipulations, but it also will be useful to insert the calculation $((\Delta * (I \cdot {}^g\Delta)) = (\Delta * {}^g\Delta) \cdot I - \Delta * \Delta * {}^g\Delta$, integrating by parts. Now repeated use of II.10 & II.2 gives

$$((\Delta * (I \cdot {}^{g}\Delta)) = (({}^{g}\Delta - \Delta) / ln(1 + g)) \cdot I \; - \; \Delta * (({}^{g}\Delta - \Delta) / ln(1 + g))$$
$$= ({}^{g}\Delta \cdot I \cdot ln(1 + g) - ({}^{g}\Delta - \Delta)) / (ln(1 + g))^2 \qquad \textbf{(II.14)}$$

(As $g \rightarrow 0$, L'Hôspital's rule verifies that II.14 $\rightarrow (\,^1/_2\,)\, I^2$, validating the use of notation suggesting a second moment.) Using II.13, II.14, and the definition of $_g\, f$ allows the calculation

$$_F{}^{g}m_2{}^{\infty} = 2\,[(_F\,\mu^{\infty} \cdot ln(1 + g) - (1 - {}_F{}^{g}s)) / ({}_F{}^{g}s \cdot (ln(1 + g))^2)] \qquad \textbf{(II.15)}$$

(which $\rightarrow {}_F\, m_2{}^{\infty}$ as $g \rightarrow 0$, using L'Hôspital's rule.)

Next, a second way to generalize II.5 is

$${}^{g}\Delta *{}_F{}^{g}\mu \;= {}^{g}\Delta * \Delta * ({}^{g}\Delta \cdot (\Delta - {}_g\, F)), \text{ by definition of } {}_F{}^{g}\mu$$
$$= \Delta * {}^{g}\Delta * ({}^{g}\Delta \cdot (\Delta - {}_g\, F)), \text{ which by II.1 becomes}$$
$$= \Delta * ({}^{g}\Delta \cdot (\Delta * (\Delta - {}_g\, F)). \text{ Now integrate by parts to get}$$
$$= (\Delta * {}^{g}\Delta) \cdot {}_{gF}\, \mu \; - \Delta * ((\Delta * {}^{g}\Delta) \cdot (\Delta - {}_g\, F)) \qquad \textbf{(II.16)}$$

Comparing with II.5 (and II.2) this suggests an **alternative** definition of a **generalized partial second moment** function as $_{gF}{}^{g}m_2 = 2\,[\Delta * ((\Delta * {}^{g}\Delta) \cdot (\Delta - {}_g\, F))]$, which integrates by parts and simplifies to give

$$_{gF}{}^{g}m_2 = 2\,[(\Delta * \Delta * {}^{g}\Delta) \cdot (\Delta - {}_g\, F) + \Delta * ((\Delta * \Delta * {}^{g}\Delta) \cdot {}_g\, f)] \qquad \textbf{(II.17)}$$

The form II.17 will prove useful in manipulations, but it also will be useful to insert the calculation

$$(\Delta * \Delta * {}^{g}\Delta) = \Delta * (({}^{g}\Delta - \Delta)/ln(1+ g)) = (({}^{g}\Delta - \Delta) - I \cdot ln(1+g))/(ln(1+g))^2 \qquad \textbf{(II.18)}$$

based on repeated use of II.10 & II.2. (As $g \rightarrow 0$, L'Hôspital's rule verifies that II.18 $\rightarrow (\,^1/_2\,)\, I^2$, validating the use of notation suggesting a second moment.) Using II.17, II.18, and the definition of $_g\, f$ allows the calculation

$$_{gF}{}^{g}m_2{}^{\infty} = 2\,[((1 - {}_F{}^{g}s) - {}_F{}^{g}s \cdot {}_{gF}\, \mu^{\infty} \cdot ln(1 + g)) / ({}_F{}^{g}s \cdot ln(1 + g)^2)] \qquad \textbf{(II.19)}$$

(which $\rightarrow {}_F\, m_2{}^{\infty}$ as $g \rightarrow 0$, using L'Hôspital's rule.)

Finally, the generalized partial second moment function $_{gF}{}^{g}m_2$ figures in one more useful analogue of II.5:

${}^{g}\Delta *_{F}{}^{g}\mu = {}^{g}\Delta \cdot (\Delta * ((\Delta / {}^{g}\Delta) \cdot_{F}{}^{g}\mu))$ by II.1. Then an integration by parts gives

$= {}^{g}\Delta \cdot [(\Delta * (\Delta / {}^{g}\Delta)) \cdot_{F}{}^{g}\mu - \Delta * ((\Delta * (\Delta / {}^{g}\Delta)) \cdot {}^{g}\Delta \cdot (\Delta -_{g} F))]$ which by II.1

$= {}^{g}\Delta \cdot [(\Delta * (\Delta / {}^{g}\Delta)) \cdot_{F}{}^{g}\mu - \Delta * ((\Delta * {}^{g}\Delta) \cdot (\Delta -_{g} F))]$ and, by definition of ${}_{gF}{}^{g}m_2$,

$= {}^{g}\Delta \cdot [(\Delta * (\Delta / {}^{g}\Delta)) \cdot_{F}{}^{g}\mu - (1/2)_{gF}{}^{g}m_2]$ **(II.20)**

SUMMARY OF PARTIAL MOMENT DEFINITIONS AND RELATIONSHIPS

The following display organizes the notational pattern and definitions for the various partial mean and partial second moment functions and generalizations that have been defined in this section.

${}_{F}\mu = \Delta * (\Delta - F)$	${}_{F}m_2 = 2\, \Delta * (I \cdot (\Delta - F))$ ${}_{F}m_2 = 2\, \Delta * ((\Delta * \Delta) \cdot (\Delta - F))$
${}_{gF}\mu = \Delta * (\Delta -_{g} F)$	${}_{gF}m_2 = 2\, \Delta * (I \cdot (\Delta -_{g} F))$ ${}_{gF}m_2 = 2\, \Delta * ((\Delta * \Delta) \cdot (\Delta -_{g} F))$
${}_{F}{}^{g}\mu = \Delta * ({}^{g}\Delta \cdot (\Delta -_{g} F))$	${}_{F}{}^{g}m_2 = 2\, \Delta * ((I \cdot {}^{g}\Delta) \cdot (\Delta -_{g} F))$ ${}_{gF}{}^{g}m_2 = 2\, \Delta * ((\Delta * {}^{g}\Delta) \cdot (\Delta -_{g} F))$

The various relations among partial mean and partial moment functions and their generalizations that motivated the definitions are summarized in:

$$\Delta *_{F} \mu = I \cdot_{F} \mu - ({}^{1}/_{2})_{F} m_2 \quad \textbf{(II.5)}$$

$$\Delta *_{gF} \mu = I \cdot_{gF} \mu - ({}^{1}/_{2})_{gF} m_2 \quad \text{(II.5) applied to } {}_{g}F$$

$$\Delta *_{F}{}^{g}\mu = I \cdot_{F}{}^{g}\mu - ({}^{1}/_{2})\, {}_{F}{}^{g}m_2 \quad \textbf{(II.12)}$$

$${}^{g}\Delta *_{F}{}^{g}\mu = (\Delta * {}^{g}\Delta) \cdot_{gF} \mu - ({}^{1}/_{2})\, {}_{gF}{}^{g}m_2 \quad \textbf{(II.16)}$$

$${}^{g}\Delta *_{F}{}^{g}\mu = {}^{g}\Delta \cdot [(\Delta * (\Delta / {}^{g}\Delta)) \cdot_{F}{}^{g}\mu - ({}^{1}/_{2})_{gF}{}^{g}m_2] \quad \textbf{(II.20)}$$

When only the ultimate generalized mean and moment values are needed, rather than the partial mean and partial moment functions, their expressions directly in terms of ${}_{F}{}^{g}s$ are summarized in:

$${}_{F}{}^{g}\mu^{\infty} = (1 - {}_{F}{}^{g}s) / ({}_{F}{}^{g}s\, ln(1 + g)) \quad \textbf{(II.11)}$$

$${}_{F}{}^{g}m_2{}^{\infty} = 2\, [({}_{F}\, \mu^{\infty} \cdot ln(1 + g) - (1 - {}_{F}{}^{g}s)) / ({}_{F}{}^{g}s \cdot (ln(1 + g))^2)] \quad \textbf{(II.15)}$$

$${}_{gF}{}^{g}m_2{}^{\infty} = 2\, [((1 - {}_{F}{}^{g}s) - {}_{F}{}^{g}s \cdot {}_{gF}\, \mu^{\infty} \cdot ln(1 + g)) / ({}_{F}{}^{g}s \cdot ln(1 + g)^2)] \quad \textbf{(II.19)}$$

III. Model of Liability Portfolio Structure

At each point in time $x \geq 0$, assume that **new liabilities** are taken on at the incremental rate ${}^{g}\Delta(x)$. **(III.1)**

The new liabilities modeled in ${}^{g}\Delta(x)$ **include** any rollover of prior liabilities into new, current interest rate guarantees, pricing, or reserves at time x. They **exclude** any increment to the liabilities to which prior interest rate guarantees, pricing, or reserves still attach at time x. (Such increments will be modeled as negative "amounts maturing" in the prior generations of liabilities).

Assume that each such generation of liabilities matures (or rolls over into a new, current interest rate) according to a maturity schedule for principal represented by the density function $b(x)$, with corresponding distribution function $B(y)=(\Delta * b)(y)$. At any time y, the rate at which liability principal originally taken on at time x is maturing is thus $b(y - x)\cdot{}^{g}\Delta(x)$. By the definition of convolution, therefore, integrating over all x the total rate of **maturing liabilities** in the portfolio at time y is

$(b * {}^{g}\Delta)(y)$ **(III.2)**

Use of the same maturity schedule for all liability generations gives the model its "stationary" character.

For some values of $y - x$, the maturity schedule $b(y - x)$ for principal can be negative if guarantees, pricing, or reserves attach interest rates at issue (i.e. time x) to increments of liabilities occurring later (i.e. time y). For many applications, such in fact will be the case (compound interest, multiple premium reserves, etc.) The concept is that the remaining principal at time y belonging to generation x is the amount to which the interest rate from time x still applies. The rate of decrease in that remaining principal at y (compared to the original amount at x) is what is represented by $b(y - x)$, so an increase is represented by a negative $b(y - x)$.

At any time z, the **total liabilities in force** equal the integral of all liabilities taken on in the past, less the integral of all liabilities that have already matured. By III.1 and III.2 this is:

$$\begin{aligned}(\Delta * {}^{g}\Delta)(z) - (\Delta * (b * {}^{g}\Delta))(z) &= ((\Delta * {}^{g}\Delta) - (B * {}^{g}\Delta))(z) \\ &= ((\Delta - B) * {}^{g}\Delta))(z) \qquad \textbf{(III.3)}\end{aligned}$$

In words, III.3 says that total liabilities in force at any time in the model equal the remaining survivors of total liabilities originally taken on at all prior times.

The stationary character of the model is revealed by the following calculation for **total liabilities in force**:

$$\begin{aligned}(\Delta - B) * {}^{g}\Delta &= \Delta * (\delta - b) * {}^{g}\Delta \\ &= \Delta * ({}^{g}\Delta \cdot ((1 - {}_{B}{}^{g}s)\,\Delta + {}_{B}{}^{g}s(\Delta - {}_{g}B))), \text{ by II.8.} \\ &= ((1 - {}_{B}{}^{g}s) / \ln(1 + g))({}^{g}\Delta - \Delta) + {}_{B}{}^{g}s\,{}_{B}{}^{g}\mu, \text{ by II.10 \& the def. of } {}_{B}{}^{g}\mu. \\ &= {}_{B}{}^{g}s\,({}_{B}{}^{g}\mu^{\infty}\,({}^{g}\Delta - \Delta) + {}_{B}{}^{g}\mu), \text{ by II.11.} \\ &= {}_{B}{}^{g}s\,({}_{B}{}^{g}\mu^{\infty}\,{}^{g}\Delta - ({}_{B}{}^{g}\mu^{\infty}\,\Delta - {}_{B}{}^{g}\mu)). \qquad \textbf{(III.4)}\end{aligned}$$

Now since ${}_{B}{}^{g}\mu(z) \to {}_{B}{}^{g}\mu^{\infty}$ as $z \to \infty$, III.4 shows that the **total liabilities in force** $(\Delta - B) * {}^{g}\Delta$ consist of:

a "**stationary**" (i.e. stably growing) component ${}_{B}{}^{g}s\ {}_{B}{}^{g}\mu^{\infty}\ {}^{g}\Delta$ that prevails in the mature portfolio,

plus

a "**transient**" component $- {}_{B}{}^{g}s\ ({}_{B}{}^{g}\mu^{\infty}\ \Delta - {}_{B}{}^{g}\mu)$ that disappears over time as the portfolio matures.

In other words, $(\Delta - B) * {}^{g}\Delta \approx {}_{B}{}^{g}s\ {}_{B}{}^{g}\mu^{\infty}\ {}^{g}\Delta$ in a mature portfolio. **(III.5)**

IV. Model of Asset Portfolio Structure

Model the asset portfolio corresponding to the liability portfolio modeled in section III according to the net premium principle. At each point in time $x \geq 0$, **net new assets** are purchased at an incremental rate equal to the net rate of liability flows, III.1 less III.2: ${}^{g}\Delta(x) - (b * {}^{g}\Delta)(x) = ((\delta - b) * {}^{g}\Delta)(x)$ (The model treats negative net new asset purchases, should they occur, as borrowings on the same terms as would be available for asset purchases. This can be considered an extension of the of the simplifying stationary portfolio assumption set.)

Assume that each such purchased generation of net new assets matures (or rolls over into a new, current interest rate) according to a maturity schedule for principal represented by the density function $a(x)$, with corresponding distribution function $A(y) = (\Delta * a)(y)$. At any time y, the rate at which net new asset principal originally purchased at time x is maturing is thus $a(y - x)((\delta - b) * {}^{g}\Delta)(x)$. By the definition of convolution, therefore, integrating over all x the total rate of **maturing net new assets** in the portfolio at time y is $(a * (\delta - b) * {}^{g}\Delta)(y)$. Use of the same maturity schedule for all asset generations gives the model its "stationary" character.

For some values of $y - x$, the maturity schedule $a(y - x)$ for principal can be negative if investment terms attach interest rates at the time of original investment or commitment (i.e. time x) to increments of funding or accrual occurring later (i.e. time y). This is less likely to be the case than for liabilities, but it can occur (zero coupons, forward commitments.) The concept is that the remaining principal at time y belonging to generation x is the amount to which the interest rate from time x still applies. The rate of decrease in that remaining principal at y (compared to the original amount at x) is what is represented by $a(y - x)$, so an increase is represented by a negative $a(y - x)$.

Now maturing net new asset principal $(a * (\delta - b) * {}^g\Delta)(y)$ also must be used to purchase assets since the original "net new asset" definition already accounted for all liability flows. These purchases, in turn, mature according to schedule $a(x)$ at a total rate at time y of $(a * a * (\delta - b) * {}^g\Delta)(y)$. This requires yet another generation of asset purchases, etc. In total, therefore, the rate of all **asset purchases** in the model at time x is

$$((\Sigma a^{**}) * (\delta - b) * {}^g\Delta)(x) \qquad \textbf{(IV.1)}$$

In words, IV.1 says that total asset purchases at any point in time in the model come from the new liabilities taken on at that time, offset by any prior liabilities maturing at that time, plus reinvestment of all maturing amounts of prior asset purchases, including maturities of previously reinvested maturities to any order of iteration. Similarly, the rate of all **asset maturities** in the model at time y is

$$(a * ((\Sigma a^{**}) * (\delta - b) * {}^g\Delta))(y) \qquad \textbf{(IV.2)}$$

At any time z, the integral of the difference between IV.1 and IV.2 over all past times should give the **total assets in force**:

$$\begin{aligned}
\Delta * (((\Sigma a^{**}) * (\delta - b) * {}^g\Delta) - (a * ((\Sigma a^{**}) * (\delta - b) * {}^g\Delta))) &= \\
&= \Delta * ((\delta - a) * (\Sigma a^{**}) * (\delta - b) * {}^g\Delta) \\
&= \Delta * (\delta * (\delta - b) * {}^g\Delta), \text{ since } (\delta - a) * (\Sigma a^{**}) = a^{*(0)} = \delta. \\
&= \Delta * ((\delta - b) * {}^g\Delta) \\
&= ((\Delta - B) * {}^g\Delta)
\end{aligned}$$

The last expression agrees with III.3, as it should (because of the net premium principle underlying the asset model.)

The stationary character of the asset model is revealed by the following thorny calculation for IV.1, the rate of **all asset purchases** in the model at a given time:

$$\begin{aligned}
(\Sigma a^{**}) * (\delta - b) * {}^g\Delta &= (\Sigma a^{**}) * (((1 - {}_B{}^g s)\, \Delta + {}_B{}^g s(\Delta - {}_g B)) \cdot {}^g\Delta), \text{ by II.8.} \\
&= (1 - {}_B{}^g s)\, (\Sigma a^{**}) * {}^g\Delta + {}_B{}^g s\, (\Sigma a^{**}) * ((\Delta - {}_g B) \cdot {}^g\Delta) \qquad \textbf{(IV.3)}
\end{aligned}$$

But $(\Sigma a^{**}) * (\delta - a) = \delta$ implies that ${}^g\Delta = (\Sigma a^{**}) * (\delta - a) * {}^g\Delta$, so

$${}^g\Delta = (1 - {}_A{}^g s)\, (\Sigma a^{**}) * {}^g\Delta + {}_A{}^g s(\Sigma a^{**}) * ((\Delta - {}_g A) \cdot {}^g\Delta), \qquad \textbf{(IV.4)}$$

by IV.3 with $b = a$.

Now solve IV.4:

$$(\Sigma a^{**}) * {}^{g}\Delta = (1/(1 - {}_{A}{}^{g}s))({}^{g}\Delta - {}_{A}{}^{g}s(\Sigma a^{**}) * ((\Delta - {}_{g} A) \cdot {}^{g}\Delta)) \qquad \textbf{(IV.5)}$$

and substitute the expression from IV.5 into IV.3 to get

$$(\Sigma a^{**}) * (\delta - b) * {}^{g}\Delta =$$

$$= ((1 - {}_{B}{}^{g}s)/(1 - {}_{A}{}^{g}s))({}^{g}\Delta - {}_{A}{}^{g}s(\Sigma a^{**}) * ((\Delta - {}_{g} A) \cdot {}^{g}\Delta))) + {}_{B}{}^{g}s(\Sigma a^{**}) * ((\Delta - {}_{g} B) \cdot {}^{g}\Delta),$$

and, finally, apply II.11 and collect terms to get

$$= (({}_{B}{}^{g}s \cdot {}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s \cdot {}_{A}{}^{g}\mu^{\infty})) \cdot {}^{g}\Delta - {}_{B}{}^{g}s(\Sigma a^{**}) * [(({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta - {}_{g} A) - (\Delta - {}_{g} B)) \cdot {}^{g}\Delta] \qquad \textbf{(IV.6)}$$

But $(\Delta - {}_{g} A)(x) \to 0$ and $(\Delta - {}_{g} B)(x) \to 0$ as $x \to \infty$, and $(\Sigma a^{**})(y - x) \to (1/{}_{A}\mu^{\infty})$ as $(y - x) \to \infty$, by II.6, so $((\Sigma a^{**}) * ((\Delta - {}_{g} A) \cdot {}^{g}\Delta))(y) \to (1/{}_{A}\mu^{\infty})(\Delta * ((\Delta - {}_{g} A) \cdot {}^{g}\Delta))(y) =$
$= (1/{}_{A}\mu^{\infty}){}_{A}{}^{g}\mu(y) = (1/{}_{A}\mu^{\infty})\, {}_{A}{}^{g}\mu^{\infty}$ as $y \to \infty$. (Like the proof of II.7.)
Similarly, $((\Sigma a^{**}) * ((\Delta - {}_{g} B) \cdot {}^{g}\Delta))(y) \to (1/{}_{A}\mu^{\infty})\, {}_{B}{}^{g}\mu^{\infty}$ as $y \to \infty$.

Thus $[(\Sigma a^{**}) * ((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta - {}_{g} A) - (\Delta - {}_{g} B)) \cdot {}^{g}\Delta)](y) \to 0$ as $y \to \infty$.

Therefore, IV.6 shows that the **total asset purchases** IV.1 at any given time in the model consist of:

a "**stationary**" (i.e. stably growing) component $(({}_{B}{}^{g}s \cdot {}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s \cdot {}_{A}{}^{g}\mu^{\infty})) \cdot {}^{g}\Delta$ that prevails in the mature portfolio,

plus

a "**transient**" component $- {}_{B}{}^{g}s(\Sigma a^{**}) * ((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta - {}_{g} A) - (\Delta - {}_{g} B)) \cdot {}^{g}\Delta)$ that disappears over time as the portfolio matures.

In other words, in a mature portfolio,

$$(\Sigma a^{**}) * (\delta - b) * {}^{g}\Delta \approx (({}_{B}{}^{g}s \cdot {}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s \cdot {}_{A}{}^{g}\mu^{\infty})) \cdot {}^{g}\Delta \qquad \textbf{(IV.7)}$$

V. Model of Interest Rate Spreads

Taking a net premium point of view, the model should have no interest rate spread between the interest rate available from new assets purchased at any point in time and the interest rate required to attract new liabilities at that same point in time. Any interest rate spread that develops in the model should reflect solely the evolution of model components prior or subsequent to the point when interest rates are set on particular assets and liabilities.

Let $\rho(x)$ stand for this common **market interest rate** available/required at each time x. If $P(z)$ stands for the **interest rate spread** in the model at time z, then the net premium principle requires that

$P(z)$ = integral over all prior x of
$\rho(x)$ times rate of **asset purchases** at time x (from IV.1), minus
rate of **new liabilities** taken on at time x (from III.1), minus
integral over all y subsequent to x and prior to z of
rate of **asset maturities** at y from purchases at x (from IV.2), minus
rate of **liability maturities** at y from x (from III.2). **(V.1)**

In the compact convolution notation V.1 is simply:

$$P = \Delta * (((\Sigma a^{**}) * (\delta - b) * {}^{g}\Delta) \cdot \rho) - \Delta * ({}^{g}\Delta \cdot \rho) -$$

$$- [\Delta * a * (((\Sigma a^{**}) * (\delta - b) * {}^{g}\Delta) \cdot \rho) - \Delta * b * ({}^{g}\Delta \cdot \rho)] \qquad \textbf{(V.2)}$$

which further simplifies to

$$P = (\Delta - A) * (((\Sigma a^{**}) * (\delta - b) * {}^{g}\Delta) \cdot \rho) - (\Delta - B) * ({}^{g}\Delta \cdot \rho) \qquad \textbf{(V.3)}$$

In words, V.3 just says (with reference to IV.1 and III.1) that at any point in time the **interest rate spread** is the difference between (a) the remaining survivors of the assets purchased at each past point in time, times the interest rate at the time of purchase, and (b) the remaining survivors of the liabilities taken on at each past point in time, times the interest rate at the time taken on. Thus, the model does reconcile to the intuitive result. V.3 does not, however, reveal the stationary character of the interest rate spread.

As a first step toward a more revealing analysis of the interest rate spread, go back to V.1 and ask at each given point in time x for the difference between (a) the rate of asset purchases at time x (from IV.1), and (b) the rate at which new liabilities are taken on at time x (from III.1):

$$\text{(IV.1) - (III.1)} = ((\Sigma a^{**}) * (\delta - b) * {}^{g}\Delta - {}^{g}\Delta)(x)$$

is the **initial net investment/obligation** at rate $\rho(x)$. Using IV.6 this can be expressed

$$(\Sigma a^{**}) * (\delta - b) * {}^{g}\Delta - {}^{g}\Delta = ((({}_{B}{}^{g}s \cdot {}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s \cdot {}_{A}{}^{g}\mu^{\infty})) - 1) \cdot {}^{g}\Delta -$$

$$- {}_{B}{}^{g}s(\Sigma a^{**}) * ((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta - {}_{g}A) - (\Delta - {}_{g}B)) \cdot {}^{g}\Delta) \qquad \textbf{(V.4)}$$

Following exactly the same reasoning as after IV.6, as $y \rightarrow \infty$ both

$$((\Sigma a^{**}) * ((\Delta -_{g} A) \cdot {}^{g}\Delta))(y) \rightarrow (1/{}_{A}\mu^{\infty})\, {}_{A}{}^{g}\mu^{\infty} \;\&\; ((\Sigma a^{**}) * ((\Delta -_{g} B) \cdot {}^{g}\Delta))(y) \rightarrow (1/{}_{A}\mu^{\infty})\, {}_{B}{}^{g}\mu^{\infty}$$

So V.4 shows that the **initial net investment/obligation** at any given rate in the model consists of:

a "**stationary**" (i.e. stably growing) component $((({}_{B}{}^{g}s \cdot {}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s \cdot {}_{A}{}^{g}\mu^{\infty})) - 1) \cdot {}^{g}\Delta$ that prevails in the mature portfolio,

plus

a "**transient**" component $- {}_{B}{}^{g}s(\Sigma a^{**}) * ((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta - {}_{g}A) - (\Delta - {}_{g}B)) \cdot {}^{g}\Delta)$ that disappears over time as the portfolio matures.

In other words, the **initial net investment/obligation** at any given rate in a mature portfolio

$$\approx ((({}_{B}{}^{g}s \cdot {}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s \cdot {}_{A}{}^{g}\mu^{\infty})) - 1) \cdot {}^{g}\Delta \qquad \textbf{(V.5)}$$

The full stationary character of the interest rate spread model is revealed by analyzing all of V.2 (i.e. **including** the maturities subsequent to the initial net investment/obligation at a given rate) with the help of IV.6. First, collecting terms in V.2 the opposite from the way they were collected for V.3,

$$P = \Delta * (((\Sigma a^{**}) * (\delta - b) * {}^{g}\Delta - {}^{g}\Delta) \cdot \rho) - [A * (((\Sigma a^{**}) * (\delta - b) * {}^{g}\Delta) \cdot \rho) - B * ({}^{g}\Delta \cdot \rho)]$$

Now substitute for both occurrences of $(\sum a^{**}) * (\delta - b) * {}^{g}\Delta$ the expression given by IV.6:

$$P=\Delta*[\{(({}_{B}{}^{g}s\cdot{}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s\cdot{}_{A}{}^{g}\mu^{\infty}))\cdot{}^{g}\Delta -$$
$$-{}_{B}{}^{g}s(\sum a^{**})*((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta -{}_{g} A)-(\Delta -{}_{g}B))\cdot{}^{g}\Delta)-{}^{g}\Delta\}\cdot\rho]-$$

$$-[A*\{[(({}_{B}{}^{g}s\cdot{}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s\cdot{}_{A}{}^{g}\mu^{\infty}))\cdot{}^{g}\Delta-{}_{B}{}^{g}s(\sum a^{**})*(({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta-{}_{g}A)-(\Delta -{}_{g}B))\cdot{}^{g}\Delta]\cdot\rho\}$$

$$- B*({}^{g}\Delta\cdot\rho)],$$

and now factor ${}^{g}\Delta$ and ρ terms as far out of brackets as possible and collect terms,

$$P=((({}_{B}{}^{g}s\cdot{}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s\cdot{}_{A}{}^{g}\mu^{\infty}))-1)(\Delta*({}^{g}\Delta\cdot\rho)) -$$

$$-{}_{B}{}^{g}s\Delta*[\{(\sum a^{**})*((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta-{}_{g}A)-(\Delta-{}_{g}B))\cdot{}^{g}\Delta)\}\cdot\rho] -$$

$$-((({}_{B}{}^{g}s\cdot{}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s\cdot{}_{A}{}^{g}\mu^{\infty}))A-B)*({}^{g}\Delta\cdot\rho) +$$

$$+ {}_{B}{}^{g}sA*[\{(\sum a^{**})*((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta -{}_{g} A)-(\Delta -{}_{g} B))\cdot{}^{g}\Delta)\}\cdot\rho].$$

Finally, collect terms one more time around ${}^{g}\Delta \cdot \rho$ and ρ to get,

$$P = ((({}_{B}{}^{g}s\cdot{}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s\cdot{}_{A}{}^{g}\mu^{\infty}))\ (\Delta - A) - (\Delta - B)) * ({}^{g}\Delta \cdot \rho) -$$

$$- {}_{B}{}^{g}s(\Delta - A) * [\{(\sum a^{**}) * ((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta - {}_{g} A) - (\Delta - {}_{g} B)) \cdot {}^{g}\Delta)\}\cdot \rho] \qquad \textbf{(V.6)}$$

Referring to V.4 and V.5, we can interpret V.6 in words to say that the **interest rate spread** at any point in time is equal to:

> the surviving amounts of the "**stationary**" (i.e. stably growing) components of the net investments/obligations at all prior interest rates,
>
> $$((({}_{B}{}^{g}s\cdot{}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s\cdot{}_{A}{}^{g}\mu^{\infty}))\ (\Delta - A) - (\Delta - B)) * ({}^{g}\Delta \cdot \rho)$$
>
> plus
>
> the surviving amounts of the "**transient**" components of the net investments/obligations at all prior interest rates
>
> $$- {}_{B}{}^{g}s(\Delta - A) * [\{(\sum a^{**}) * ((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta - {}_{g} A) - (\Delta - {}_{g} B)) \cdot {}^{g}\Delta)\}\cdot \rho]$$
>
> (where, interestingly, the formula displays that "transients" mature on the asset portfolio schedule, and are reinvested and rematured to all orders of iteration.)

The surviving transients
$-{}_{B}{}^{g}s(\Delta - A) * [\{(\sum a^{**}) * ((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta - {}_{g}A) - (\Delta - {}_{g}B)) \cdot {}^{g}\Delta)\} \cdot \rho] \rightarrow 0$ in the mature portfolio (because **both** $(\Delta - A)(z - y) \rightarrow 0$ as $(z - y) \rightarrow \infty$ **and** $[\{(\sum a^{**}) * ((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta - {}_{g}A) - (\Delta - {}_{g}B)) \cdot {}^{g}\Delta)\} \cdot \rho](y) \rightarrow 0$ as $y \rightarrow \infty$. The reasoning following IV.6 establishes the latter fact, since the ρ at the end of the $[\ ](y)$ expression here is just a multiplicative factor.) In other words, in a mature portfolio, the **interest rate spread**

$$P \approx ((({}_{B}{}^{g}s \cdot {}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s \cdot {}_{A}{}^{g}\mu^{\infty}))\ (\Delta - A) - (\Delta - B)) * ({}^{g}\Delta \cdot \rho) \qquad \textbf{(V.7)}$$

The model equation V.6 (or its approximation V.7 for the mature portfolio) describes the response characteristics of the stationary (i.e. stably growing) insurance company viewed as an "antenna" or "tuner" that modulates an incoming signal of market interest rates $\rho(x)$ over time into an output signal $P(z)$ of interest rate spreads in the portfolio over time (in a net premium sense.) The task of immunization is (or should be) to maintain over time in the portfolio a modulation structure that will dampen the output response to any such input signal, so that the company's interest rate spread output signal (in the gross premium sense) will reflect to an acceptable degree the deliberate margins set in the pricing and/or reserving, not the extraneous market interest rate signal.

One more application of the convolution formalism should be recorded. The **total interest rate spread over time** accumulated in the model over all times z from 0 to some time w is just the integral $(\Delta * P)(w)$. Directly from V.6, using the definitions of the partial mean functions ${}_{A}\mu$ and ${}_{B}\mu$:

$$\Delta * P = ((({}_{B}{}^{g}s \cdot {}_{B}{}^{g}\mu^{\infty})/({}_{A}{}^{g}s \cdot {}_{A}{}^{g}\mu^{\infty}))\ {}_{A}\mu - {}_{B}\mu) * ({}^{g}\Delta \cdot \rho) -$$

$$- {}_{B}{}^{g}s\ {}_{A}\mu * [(\sum a^{**}) * ((({}_{B}{}^{g}\mu^{\infty}/{}_{A}{}^{g}\mu^{\infty})(\Delta - {}_{g}A) - (\Delta - {}_{g}B)) \cdot {}^{g}\Delta) \cdot \rho] \qquad \textbf{(V.8)}$$

VI. Sensitivity to Interest Rate Changes

While the modulation equations V.6 or V.7 that express evolving portfolio interest rate spreads $P(z)$ in terms of the history of market interest rate levels $\rho(x)$ are interesting to behold, there is something inherently unsatisfying about them. The model arose from net premium assumptions, so the history of market interest rate **levels** should not be the real driver. The history of **changes** in market interest rates is what should count.

If interest rates never change, V.6 should read $P = 0$ and the approximation V.7 should read $\approx P(z) \rightarrow 0$ as $z \rightarrow \infty$. Both statements can be proved (an interesting test of ability to manipulate convolutions and partial means!) but they are not transparent (unless such manipulation has become second nature.) We would like to have a model in which the history of market interest rate changes, ρ', the derivative of ρ, manifestly drives the interest rate spread P.

To get such a model, simply **integrate by parts** in the integrals that define the convolutions in V.6, resulting in:

$$P(z)=\left\{\begin{array}{l}\left[\dfrac{{}^g_B s\cdot{}^g_B\mu^\infty}{{}^g_A s\cdot{}^g_A\mu^\infty}(\Delta-A)-(\Delta-B)\right]*{}^g\Delta-\\[2ex]-{}^g_B s(\Delta-A)*\left[\left(\sum a^{**}\right)*\left(\left[\dfrac{{}^g_B\mu^\infty}{{}^g_A\mu^\infty}(\Delta-_g A)-(\Delta-_g B)\right]{}^g\Delta\right)\right]\end{array}\right\}(z)\rho(\infty)$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{y}\left\{\begin{array}{l}\left[\dfrac{{}^g_B s\cdot{}^g_B\mu^\infty}{{}^g_A s\cdot{}^g_A\mu^\infty}(\Delta-A)-(\Delta-B)\right](z-x)\cdot{}^g\Delta(x)-\\[2ex]-{}^g_B s(\Delta-A)(z-x)\left[\left(\sum a^{**}\right)*\left(\left[\dfrac{{}^g_B\mu^\infty}{{}^g_A\mu^\infty}(\Delta-_g A)-(\Delta-_g B)\right]{}^g\Delta\right)\right](x)\end{array}\right\}dx\rho'(y)dy$$

The first { } term is 0 upon substituting III.4 (and its analogue for A) into the first term within it and using $(\Delta-A)*\left(\sum a^{**}\right)=\Delta$ and the definitions of ${}^g_A\mu$ and ${}^g_B\mu$ in the second term within it. A factor $\Delta(y)$ can be inserted into the integral over dy because for $x<0$ each term in the integral over dx is 0 anyway. Finally, the integral over dx would be 0 if taken to ∞ for the same reasons that the first { } term is 0; so the integral over dx can be replaced by the negative of its complementary integral, all of which gives :

$$P(z)=\int_{-\infty}^{\infty}\int_{y}^{\infty}\left\{\begin{array}{l}\left[\dfrac{{}^g_B s\cdot{}^g_B\mu^\infty}{{}^g_A s\cdot{}^g_A\mu^\infty}(\Delta-A)-(\Delta-B)\right](z-x)\cdot{}^g\Delta(x)-\\[2ex]-{}^g_B s(\Delta-A)(z-x)\left[\left(\sum a^{**}\right)*\left(\left[\begin{array}{l}\dfrac{{}^g_B\mu^\infty}{{}^g_A\mu^\infty}(\Delta-_g A)-\\-(\Delta-_g B)\end{array}\right]{}^g\Delta\right)\right](x)\end{array}\right\}dx\Delta(y)\rho'(y)dy$$

Now make the change of variables $z-x$ for x :

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{z-y}\left\{\begin{array}{l}\left[\dfrac{{}^g_B s\cdot{}^g_B\mu^\infty}{{}^g_A s\cdot{}^g_A\mu^\infty}(\Delta-A)-(\Delta-B)\right](x)\cdot{}^g\Delta(z-x)-\\[2ex]-{}^g_B s(\Delta-A)(x)\left[\left(\sum a^{**}\right)*\left(\left[\begin{array}{l}\dfrac{{}^g_B\mu^\infty}{{}^g_A\mu^\infty}(\Delta-_g A)-\\-(\Delta-_g B)\end{array}\right]{}^g\Delta\right)\right](z-x)\end{array}\right\}dx\Delta(y)\rho'(y)dy$$

Since $x\le z-y$, the relation ${}^g\Delta(z-x)\Delta(y)={}^g\Delta(z-y-x)\cdot{}^g\Delta(y)$ will hold, and also a factor $\Delta(z-y-x)$ can be inserted into the integral over dx, so

$$P(z)=\left\{\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{z-y}\left[\frac{{}^{\mathrm{g}}_{\mathrm{B}}s\cdot{}^{g}_{B}\mu^{\infty}}{{}^{\mathrm{g}}_{\mathrm{A}}s\cdot{}^{g}_{A}\mu^{\infty}}(\Delta-A)-(\Delta-B)\right](x)\cdot{}^{g}\Delta(z-y-x)dx\cdot{}^{g}\Delta(y)\rho'(y)dy-\\&-{}^{\mathrm{g}}_{\mathrm{B}}s\int_{-\infty}^{\infty}\int_{-\infty}^{z-y}\left[\begin{aligned}&(\Delta-A)(x)\cdot\\&\cdot\left[\left(\sum a^{**}\right)*\left(\left[\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}(\Delta-{}_{g}A)-(\Delta-{}_{g}B)\right]{}^{g}\Delta\right)\right](z-x)\cdot\\&\cdot\Delta(z-y-x)\end{aligned}\right]dx\Delta(y)\rho'(y)dy\end{aligned}\right\}$$

For $x>z-y$, ${}^{g}\Delta(z-y-x)=0$ and $\Delta(z-y-x)=0$, so the integrals over dx can be taken to ∞, resulting in :

$$=\left\langle\begin{aligned}&\left\{\left[{}^{g}\Delta*\left(\frac{{}^{\mathrm{g}}_{\mathrm{B}}s\cdot{}^{g}_{B}\mu^{\infty}}{{}^{\mathrm{g}}_{\mathrm{A}}s\cdot{}^{g}_{A}\mu^{\infty}}(\Delta-A)-(\Delta-B)\right)\right]*\left({}^{g}\Delta\rho'\right)\right\}(z)-\\&-{}^{\mathrm{g}}_{\mathrm{B}}s\left\{(\Delta-A)*\left(\left[\left(\sum a^{**}\right)*\left(\left[\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}(\Delta-{}_{g}A)-(\Delta-{}_{g}B)\right]{}^{g}\Delta\right)\right]\cdot[\Delta*(\Delta\cdot\rho')]\right)\right\}(z)\end{aligned}\right\rangle$$

Now apply III.4 (and its analogue for A) to the first term:

$$=\left\langle\begin{aligned}&\left\{\left[\begin{aligned}&\frac{{}^{\mathrm{g}}_{\mathrm{B}}s\cdot{}^{g}_{B}\mu^{\infty}}{{}^{\mathrm{g}}_{\mathrm{A}}s\cdot{}^{g}_{A}\mu^{\infty}}{}^{\mathrm{g}}_{\mathrm{A}}s\left({}^{g}_{A}\mu^{\infty}\cdot{}^{g}\Delta-\left({}^{g}_{A}\mu^{\infty}\cdot\Delta-{}^{g}_{A}\mu\right)\right)-\\&-{}^{\mathrm{g}}_{\mathrm{B}}s\left({}^{g}_{B}\mu^{\infty}\cdot{}^{g}\Delta-\left({}^{g}_{B}\mu^{\infty}\cdot\Delta-{}^{g}_{B}\mu\right)\right)\end{aligned}\right]*\left({}^{g}\Delta\rho'\right)\right\}(z)-\\&-{}^{\mathrm{g}}_{\mathrm{B}}s\left\{(\Delta-A)*\left(\left[\left(\sum a^{**}\right)*\left(\left[\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}(\Delta-{}_{g}A)-(\Delta-{}_{g}B)\right]{}^{g}\Delta\right)\right]\cdot[\Delta*(\Delta\cdot\rho')]\right)\right\}(z)\end{aligned}\right\rangle$$

Finally, just some algebra gives :

$$\begin{aligned}P(z)=&\ {}^{\mathrm{g}}_{\mathrm{B}}s\left[\left(\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}{}^{g}_{A}\mu-{}^{g}_{B}\mu\right)*\left({}^{g}\Delta\rho'\right)\right](z)-\\&-{}^{\mathrm{g}}_{\mathrm{B}}s\left\{(\Delta-A)*\left(\left[\left(\sum a^{**}\right)*\left(\left[\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}(\Delta-{}_{g}A)-(\Delta-{}_{g}B)\right]{}^{g}\Delta\right)\right]\cdot[\Delta*(\Delta\cdot\rho')]\right)\right\}(z)\end{aligned}$$

(VI.1)

The second term in VI.1 is a “transient” that vanishes as $z\rightarrow\infty$ by exactly the same reasoning as that preceding V.7.

Thus VI.1 shows that the **interest rate spread** *P* consists of :

a "**stationary**" (i.e. stably growing) component that characterizes the mature portfolio

$${}_{B}^{g}s\left(\left({}_{B}^{g}\mu^{\infty}/{}_{A}^{g}\mu^{\infty}\right)\,{}_{A}^{g}\mu - {}_{B}^{g}\mu\right) * \left({}^{g}\Delta\cdot\rho'\right),$$

plus

a "**transient**" component that disappears over time as the portfolio matures

$$-{}_{B}^{g}s(\Delta - A)*\left\{\left[\left(\sum a^{**}\right)*\left(\left[\frac{{}_{B}^{g}\mu^{\infty}}{{}_{A}^{g}\mu^{\infty}}(\Delta -{}_{g}A)-(\Delta -{}_{g}B)\right]{}^{g}\Delta\right)\right]\cdot[\Delta*(\Delta\cdot\rho')]\right\}$$

In the mature portfolio

$$P \approx {}_{B}^{g}s\left(\left({}_{B}^{g}\mu^{\infty}/{}_{A}^{g}\mu^{\infty}\right)\,{}_{A}^{g}\mu - {}_{B}^{g}\mu\right) * \left({}^{g}\Delta\cdot\rho'\right) \qquad \textbf{(VI.2)}$$

VI.1 (or VI.2 in the mature state) describes the response characteristics of the stationary (i.e. stably growing) insurance company to an input signal ρ' of **changes** to market interest rates. If market interest rates do not change, then the interest rate spread manifestly vanishes. The generalized partial mean functions ${}_{A}^{g}\mu$ and ${}_{B}^{g}\mu$ together with the related scale factor ${}_{B}^{g}s$ tell the whole story of asset/liability mismatch in the mature portfolio (given the highly idealized model constructed here.)

Taking $\Delta *$ (VI.1), and using II.12 in the first term and the definition of ${}_{A}\mu$ in the second, we can express V.8, the **total interest rate spread over time** $(\Delta * P)(w)$ accumulated in the portfolio up to any given time *w*, in terms of ρ':

$$(\Delta * P)(w) = {}_{B}^{g}s\left\{\left[I\cdot\left(\frac{{}_{B}^{g}\mu^{\infty}}{{}_{A}^{g}\mu^{\infty}}\,{}_{A}^{g}\mu - {}_{B}^{g}\mu\right) - \left(\frac{1}{2}\right)\left(\frac{{}_{B}^{g}\mu^{\infty}}{{}_{A}^{g}\mu^{\infty}}\,{}_{A}^{g}m_2 - {}_{B}^{g}m_2\right)\right]*\left({}^{g}\Delta\rho'\right)\right\}(w) -$$
$$-{}_{B}^{g}s\left\{{}_{A}\mu*\left(\left[\left(\sum a^{**}\right)*\left(\left[\frac{{}_{B}^{g}\mu^{\infty}}{{}_{A}^{g}\mu^{\infty}}(\Delta -{}_{g}A)-(\Delta -{}_{g}B)\right]{}^{g}\Delta\right)\right]\cdot[\Delta*(\Delta\cdot\rho')]\right)\right\}(w)$$

(VI.3)

The second term in VI.3 **does not** disappear as $w \to \infty$ because it represents the accumulated "**residual**" of transients from the pre-mature portfolio that may not cancel out over time. The "residual" term **does** remain uniformly limited as $w\to\infty$, and it **does** vanish uniformly in respect to contributions from $\rho'(y)$ as $y \to \infty$. That is, if $\rho'(y) = 0$ for all $y < y'$, then the residual term as $w \to \infty$ vanishes uniformly as $y'\to\infty$. In other words, the transient term in VI.1 tends not only to disappear at each time *z* in the mature portfolio, but also to accumulate across all times (in total to ∞) a disappearing total effect from all $\rho'(y)$ occurring in the mature portfolio.

To prove all of this, let $w \to \infty$ in the residual term in VI.3 and apply reasoning similar to the proof of II.7 (with the reasoning following IV.6 supplying the necessary $\to 0$) to conclude that the residual term:

$$-{}^{g}_{B}s\left\{{}_{A}\mu * \left(\left[\left(\sum a^{**}\right) * \left(\left[\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}\left(\Delta -_{g} A\right) - \left(\Delta -_{g} B\right)\right]^{g}\Delta\right)\right] \cdot \left[\Delta * \left(\Delta \cdot \rho'\right)\right]\right)\right\} \approx$$

$$\approx -{}^{g}_{B}s \cdot {}_{A}\mu^{\infty}\left\{\Delta * \left(\left[\left(\sum a^{**}\right) * \left(\left[\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}\left(\Delta -_{g} A\right) - \left(\Delta -_{g} B\right)\right]^{g}\Delta\right)\right] \cdot \left[\Delta * \left(\Delta \cdot \rho'\right)\right]\right)\right\}$$

in the mature portfolio. In fact, treating the two $\Delta * (\ )$ terms as integrals, the big integrand $\rightarrow 0$ uniformly by the reasoning following IV.6; and the inner integral $[\Delta * (\Delta \cdot \rho')](y) = \rho(y) - \rho(0)$ for $y \geq 0$ and $= 0$ for $y < 0$, by simple calculus; so for a y' chosen large enough, the residual term

$$\approx -{}^{g}_{B}s \cdot {}_{A}\mu^{\infty}\int_{0}^{y'}\left[\left(\sum a^{**}\right) * \left(\left[\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}\left(\Delta -_{g} A\right) - \left(\Delta -_{g} B\right)\right]^{g}\Delta\right)\right](y)\left[\rho(y) - \rho(0)\right]dy$$

in the mature portfolio. Now integrate by parts, again use the identity $\int^{y'}(\ )dy \equiv \Delta * (\ )(y')$, and use the definitions of ${}^{g}_{B}\mu$ and ${}^{g}_{A}\mu$ to get the residual term

$$\approx -{}^{g}_{B}s \cdot {}_{A}\mu^{\infty}\left\{\begin{array}{l}\left[\left(\sum a^{**}\right) * \left(\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}{}^{g}_{A}\mu - {}^{g}_{B}\mu\right)\right](y')\left[\rho(y') - \rho(0)\right] - \\ -\left[\Delta * \left(\left[\left(\sum a^{**}\right) * \left(\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}{}^{g}_{A}\mu - {}^{g}_{B}\mu\right)\right] \cdot \rho'\right)\right](y')\end{array}\right\}$$

Finally, for y' chosen large enough, reasoning similar to the proof of II.7 (using II.6 to get the necessary limiting value) gives

$$\left[\left(\sum a^{**}\right) * \left(\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}{}^{g}_{A}\mu - {}^{g}_{B}\mu\right)\right](y') \approx \frac{1}{{}_{A}\mu^{\infty}}\left[\Delta * \left(\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}{}^{g}_{A}\mu - {}^{g}_{B}\mu\right)\right](y')$$

and using II.12 and the definitions of ${}^{g}_{B}\mu^{\infty}$, ${}^{g}_{A}\mu^{\infty}$, ${}^{g}_{B}m_{2}^{\infty}$, and ${}^{g}_{A}m_{2}^{\infty}$ makes the residual term in the mature portfolio approach

$$\begin{aligned}\approx\ & \frac{1}{2}{}^{g}_{B}s\left(\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}{}^{g}_{A}m_{2}^{\infty} - {}^{g}_{B}m_{2}^{\infty}\right)\left[\rho(y') - \rho(0)\right] + \\ & + {}^{g}_{B}s \cdot {}_{A}\mu^{\infty}\left\{\Delta * \left(\left[\left(\sum a^{**}\right) * \left(\frac{{}^{g}_{B}\mu^{\infty}}{{}^{g}_{A}\mu^{\infty}}{}^{g}_{A}\mu - {}^{g}_{B}\mu\right)\right] \cdot \rho'\right)\right\}(y')\end{aligned} \qquad \textbf{(VI.4)}$$

for y' chosen large enough, which validates all the claims made about the "residual" term in VI.3.

Summarizing, to put the complete approximation for the **total interest rate spread over time** *(Δ * P)* in the mature portfolio into closed forms, combine VI.3 and VI.4 to give:

$$(\Delta * P)(w) \approx {}_{\mathrm{B}}^{\mathrm{g}}s\left\{\left[I\cdot\left(\frac{{}_{B}^{g}\mu^{\infty}}{{}_{A}^{g}\mu^{\infty}}\,{}_{A}^{g}\mu - {}_{B}^{g}\mu\right) - \left(\frac{1}{2}\right)\left(\frac{{}_{B}^{g}\mu^{\infty}}{{}_{A}^{g}\mu^{\infty}}\,{}_{A}^{g}m_2 - {}_{B}^{g}m_2\right)\right] * \left({}^{g}\Delta\rho'\right)\right\}(w) +$$

$$+\frac{1}{2}\,{}_{\mathrm{B}}^{\mathrm{g}}s\left(\frac{{}_{B}^{g}\mu^{\infty}}{{}_{A}^{g}\mu^{\infty}}\,{}_{\mathrm{A}}^{\mathrm{g}}m_2^{\infty} - {}_{\mathrm{B}}^{\mathrm{g}}m_2^{\infty}\right)\left[\rho(y') - \rho(0)\right] +$$

$$+\,{}_{\mathrm{B}}^{\mathrm{g}}s\cdot{}_{A}\mu^{\infty}\left\{\Delta * \left(\left[\left(\sum a^{**}\right) * \left(\frac{{}_{B}^{g}\mu^{\infty}}{{}_{A}^{g}\mu^{\infty}}\,{}_{A}^{g}\mu - {}_{B}^{g}\mu\right)\right]\cdot\rho'\right)\right\}(y')$$

for large *w* and some *y´* chosen large enough. **(VI.5)**

VII. Simplest Applications

In the mature portfolio, VI.2 demonstrates that the contribution to the **interest rate spread** *P(z)* at time *z* from interest rates having moved by an amount *ρ´(y)dy* at some time $y \le z$ is given by a fairly simple expression involving partial mean functions:

$$P(z) \approx {}_{\mathrm{B}}^{\mathrm{g}}s[({}_{\mathrm{B}}^{\mathrm{g}}\mu^{\infty}/{}_{\mathrm{A}}^{\mathrm{g}}\mu^{\infty})\,{}_{\mathrm{A}}^{\mathrm{g}}\mu(z-y) - {}_{\mathrm{B}}^{\mathrm{g}}\mu(z-y)](1+g)^{y}\,\rho'(y)dy \qquad \textbf{(VII.1)}$$

Two questions suggest themselves immediately as applications of (VII.1):

(1) Given a **stochastic model** for *ρ´(y)*, the rate of change in interest rates over time, what results as a stochastic model for the current interest rate spreads *P(z)* in the portfolio?

(2) From a specific time *y*, what total shock to future interest spreads *P(z)* in the portfolio over all future time $z \ge y$ emanates from a single **isolated pulse** *ρ´(y)dy* of change to the market interest rate at time *y*? In other words, how vulnerable is the portfolio at any given time to a single jump in interest rates, measuring vulnerability by total subsequent interest rate spreads over time, rather than by any particular present value theory?

THE STOCHASTIC MODEL

For each *y* let *ρ´(y)* be a random variable (not necessarily independent for different *y*'s) distributed identically to the distribution of a random variable *ρ´,* so that the means $\mu_{\rho'(y)} = \mu_{\rho'}$ for all *y*. (To interpret this, it's easy to show that with this assumption the annual change in interest rates *[ρ(y + 1) - ρ(y)]* is a random variable with mean $\mu_{\rho'}$.)

By VI.2, this implies that *P(z)* at any time *z* in the mature portfolio (*z* large) is a random variable with mean

$$\mu_{P(z)} \approx {}_{\mathrm{B}}^{\mathrm{g}}s\{[({}_{\mathrm{B}}^{\mathrm{g}}\mu^{\infty}/{}_{\mathrm{A}}^{\mathrm{g}}\mu^{\infty})\,{}_{\mathrm{A}}^{\mathrm{g}}\mu - {}_{\mathrm{B}}^{\mathrm{g}}\mu] * {}^{\mathrm{g}}\Delta\}(z)\,\mu_{\rho'}$$

Using II.20, this gives

$$\mu_{P(z)} \approx {}_B{}^g s\{{}^g\Delta\, [({}_B{}^g\mu^\infty / {}_A{}^g\mu^\infty)\, ((\Delta * (\Delta / {}^g\Delta))_A{}^g\mu - (1/2)\, {}_{gA}{}^g m_2\ ) -$$

$$- ((\Delta * (\Delta / {}^g\Delta))_B{}^g\mu - (1/2)\, {}_{gB}{}^g m_2\ )]\}(z)\ \mu_{\rho'}$$

which as $z \to \infty$ is equivalent to

$$\mu_{P(z)} \approx -(1/2)\, {}_B{}^g s\ {}^g\Delta(z)[({}_B{}^g\mu^\infty / {}_A{}^g\mu^\infty)\, {}_{gA}{}^g m_2{}^\infty - {}_{gB}{}^g m_2{}^\infty]\ \mu_{\rho'} \qquad \textbf{(VII.2)}$$

by the definitions of ${}_A{}^g\mu^\infty$, ${}_B{}^g\mu^\infty$, ${}_{gA}{}^g m_2{}^\infty$, and ${}_{gB}{}^g m_2{}^\infty$. Finally, according to III.5, the size of the entire mature portfolio at time z is $[(\Delta - B) * {}^g\Delta](z) \approx {}_B{}^g s\ {}_B{}^g\mu^\infty\ {}^g\Delta(z)$

(VII.3)

So, in relation to the portfolio size at time z, the **expected value** of the stochastic variable $P(z)$ representing the **interest rate spread** in the mature portfolio is given by dividing VII.2 by VII.3.

$$\{\mu_{P(z)}\ / [(\Delta - B) * {}^g\Delta](z)\} \approx -(1/2)[({}_{gA}{}^g m_2{}^\infty / {}_A{}^g\mu^\infty) - ({}_{gB}{}^g m_2{}^\infty / {}_B{}^g\mu^\infty)]\ \mu_{\rho'} \qquad \textbf{(VII.4)}$$

Thus, the expected value of the interest rate spread at any point in time, in proportion to the size of the portfolio at that time, is given by the expected value of the change in interest rates over time (the secular trend in interest rates) modulated by a simple factor involving the difference between the **ratio of a generalized second moment to a generalized first moment** for the asset maturity schedule in each generation of assets versus the same ratio for the liability maturity schedule in each generation of liabilities. As respects expected values of interest rate spreads, the entire story of asset/liability mismatch is told by these ratios of generalized second to first moments (given the highly idealized model constructed here.)

VII.4 was first derived in 1977 (unpublished) by Edmund F. Kelly, FSA, for a restricted case ($\rho'(y)$ constant, $g = 0$, and simple maturity schedules for A and B) in response to simulation studies by the author that surprisingly (at the time) displayed simple linear dependence on second moments. We called the ratio of second to first moments the "**maturity index**."

VARIANCE

Now assume further that the random variables $\rho'(y)$ are **independent** and each has variance $(\sigma_{\rho'(y)})^2 = (\sigma_{\rho'})^2/dy$ where $(\sigma_{\rho'})^2$ is the variance of ρ'. (To interpret this, it's easy to show that with this assumption the annual change in interest rates $[\rho(y + 1) - \rho(y)]$ is a random variable with variance $(\sigma_{\rho'})^2$ so ρ' can be identified with the annualized evolution of $\rho'(y)$. This seemingly cavalier assumption of infinite variance for the instantaneous random variable can be fully justified within the framework of ch. III and ch. V. of Robinson.) By VI.2, the random variable $P(z)$ at any time z in the mature (z large) portfolio has **variance**

$$(\sigma_{P(z)})^2 \approx ({}_B{}^g s)^2\{[({}_B{}^g\mu^\infty/{}_A{}^g\mu^\infty)\ {}_A{}^g\mu\ -\ {}_B{}^g\mu]^2 * {}^g\Delta^2\}(z)(\sigma_{\rho'})^2 \qquad \textbf{(VII.5)}$$

This paper offers no closed form simplification for VII.5, nor for the interesting related question of the covariance among the random variables $P(z)$ for different values of z. These gaps need to be filled, as VII.5 captures the volatility of the interest rate spread in response to random volatility (as opposed to secular trend) in market interest rates. And the covariance among the $P(z)$ for different values of z governs the persistency over time of random volatile perturbations to the interest rate spread.

To a certain extent VII.8 below indirectly addresses the covariance issue by aggregating annual interest rate spreads over time. Section VIII. will present an entirely different approach, Fourier analysis, to the volatility issue.

EFFECT OF AN INTEREST RATE JUMP

Suppose the change in market interest rates $\rho'(y)dy = 0$ for all y except **a single value** $\rho'(y)dy \neq 0$, and further suppose that the time y of the jump satisfies $y>y'$ where y' is the value in VI.4 and VI.5 beyond which the permanent "residual" in $(\Delta * P)$ generated by $\rho'(y)dy$ is sufficiently small to ignore. That is, we are looking at a jump in market interest rates where the jump itself occurs in the mature portfolio. Then taking the limit as $w \to \infty$ in VI.3 (or VI.4 or VI.5 to properly verify the smallness of the "residual") shows that the total future interest rate spread emanating from that single jump $\rho'(y)dy$ is:

$$(\Delta * P)(\infty) \approx -(1/2){}_B{}^g s[({}_B{}^g\mu^\infty/{}_A{}^g\mu^\infty){}_A{}^g m_2{}^\infty - {}_B{}^g m_2{}^\infty]\ {}^g\Delta(y)\ \rho'(y)dy \qquad \textbf{(VII.6)}$$

Finally, according to III.5, the size of the entire mature portfolio at time y is

$$[(\Delta - B) * {}^g\Delta](y) \approx {}_B{}^g s\ {}_B{}^g\mu^\infty\ {}^g\Delta(y) \qquad \textbf{(VII.7)}$$

So, in relation to the portfolio size at time y, the **total future interest rate spread** emanating from that single jump $\rho'(y)dy$ is given by dividing VII.6 by VII.7:

$$\{(\Delta * P)(\infty) / [(\Delta - B) * {}^g\Delta](y)\} \approx -(1/2)[({}_A{}^g m_2{}^\infty / {}_A{}^g\mu^\infty) - ({}_B{}^g m_2{}^\infty / {}_B{}^g\mu^\infty)]\ \rho'(y)dy \qquad \textbf{(VII.8)}$$

which (just as VII.4) modulates the interest rate change by a simple factor involving the difference between **ratios of second to first moments**. (Aficionados of stationary population problems will not be surprised!) Because of the growth factor g in the model, different generalizations of the second moment appear in VII.4 and VII.8. If $g = 0$, the two expressions are identical.

Referring to VII.1 and VI.2, recall that **all** of the effects that changing interest rates have on **interest rate spreads** can be expressed as linear combinations of the effects of single jumps such as $\rho'(y)dy$. This makes VII.8 a very strong result for immunization considerations. Of particular note, the accuracy of approximation in VII.8 depends only upon the simplifying assumption of a mature stationary (i.e. stably growing) model. Within that range of approximation, VII.8 relates the **actual values** of its component terms, not just expected values in a stochastic sense.

This fact makes the appearance of **different second moment concepts** in VII.4 compared with VII.8 troubling upon first notice. The conflict resolves itself, however, upon consideration that VII.4 reflects response to a secular trend of market interest rates persisting right up to the moment *z*. Although the effects of the changes $\rho'(y)dy$ to market interest rates reflected in VII.8 have been fully realized by time *z* for any *y* far prior to *z*, the full response guaranteed by VII.8 for the more recent $\rho'(y)dy$ still lies largely in times *w* beyond *z*.

Perhaps this contains a hint that any strategy to attain true economic immunization of total interest rate spreads over time against volatility in market interest rates must inevitably tolerate at least the degree of period to period volatility in interest rate spreads demanded by the difference between VII.4 and VII.8. An opening toward a rigorous quantitative "**uncertainty principle**" corresponding to the complementary relationship between balance sheet and income statement may lurk near here.

PRACTICAL NOTE

In applying VII.4 or VII.8 the use of II.11, II.15, and II.19 provides the most practical basis for quick calculation of the generalized second moments and means. The same results can be derived, however, from more elaborate calculations based on creating tables of generalized partial mean and partial second moment functions, which then also can be used in calculations involving some of the formulas less compact than VII.4 or VII.8.

VIII. Fourier Analysis

The most visible property of actual interest rates is **cyclicity**. After they go up, interest rates always come down. After they come down, they eventually go back up. It seems perverse then that both classical immunization theory and the simplest applications of our model deal only with the response to one-time jumps in interest rates (classical immunization and our VII.8) or to the expected secular trend in interest rates (our VII.4). Jumping somewhere and staying there forever, or moving forever around one expected secular direction (whether deterministically or on average), are two things that actual interest rates just don't do.

However, before retreating into the miasma of simulation trials with which actuaries have become so familiar over the past twenty five years, the model of interest rate spreads expressed in VI.1 and VI.2 creates the opportunity at least to sketch an analytic description of how the interest rate spread in the model portfolio responds over time to external market interest rate cyclicity. The **Fourier transform** provides the preeminent mathematical tool to describe and analyze cyclic phenomena. It yields up interesting information when applied to VI.2. We will not pursue here the grail of the Fourier spectrum for interest rates themselves. Rather, the goal is to understand precisely and in full generality, given our highly simplified model, what response in the interest rate spread over time is stimulated by each of the possible cyclic components of the external interest rate signal, whatever they may be.

If one happens to have a conviction about which cyclic components dominate the actual external interest rate signal, this approach has the potential to offer specific insight into what would be required to minimize (conceivably eliminate) unwanted volatility in the resulting interest rate spread output signal. But it is wholly consistent with Redington's original motives to take the point of view that to immunize means to avoid exposing the outcome to anyone's incorrect conviction about the structure of the external interest rate spectrum; that to immunize means to create asset/liability relationships that will dampen the interest rate spread response uniformly (i.e. within some uniform reasonable upper limit) across the entire spectrum of possible cyclic components in the external interest rate signal.

To a function *h(x)* there corresponds another function *FT(h)(f)* called the **Fourier transform** of *h* that encodes the cyclic components of *h* in the sense that $h(x) = \int_{-\infty}^{\infty} FT(h)(f)\, e^{2\pi i x f}\, df$, which expresses *h(x)* as a phased and weighted sum of cyclic components of all possible frequencies, *f*. The phase and weight attaching to the cyclic component $e^{2\pi i x f}$ of frequency *f* is given by the value *FT(h)(f)* of the Fourier transform of *h* evaluated at the frequency *f*. (The Fourier transform *FT(h)(f)* is a complex-number valued function of *f*, allowing it to carry both phase and weight information.) The **value** of the Fourier transform at each frequency *f* is given by

$$FT(h)(f) = \int_{-\infty}^{\infty} h(x)\, e^{-2\pi i f x}\, dx$$

Properties of the Fourier transform relevant to our goal here include:

$FT(h + k) = FT(h) + FT(k)$	$FT(ch) = cFT(h)$ for c a constant
$FT(h \cdot k) = FT(h) * FT(k)$	$FT(h * k) = FT(h) \cdot FT(k)$
$FT[FT(h)](x) = h(-x)$	$FT(h')(f) = 2\pi i f FT(h)(f)$
$FT(2\pi i x h(x))(f) = - FT(h)'(f)$	$FT(c) = c \cdot \delta$ for c a constant
$FT(\delta)(f) = 1$ for all f	$FT(\Delta)(f) = (1/2)\delta(f) + 1/(2\pi i f)$
$FT(1/{}^{g}\Delta)(f) = 1/(\ln(1+g) + 2\pi i f)$	**(VIII.1)**

All follow more or less directly from the definition of the Fourier transform. Brigham and James each provide good introductions to the Fourier transform and its properties. Brigham is especially good for visualization of the concepts. Ch. 2 of Brigham illustrates, and his sec. 3-11 demonstrates, the important point that for real-valued functions h all frequencies $f \neq 0$ in the Fourier transform must come in $\pm f$ pairs with complex conjugate transform values. Ch. 9 of Rudin presents a rigorous development of the Fourier transform. There is no general agreement in the literature, however, on the notation to use for the Fourier transform nor on a convention for selecting constants and signs in the definition. It is necessary to verify in each source which notation and definitional conventions apply.

Now the expression in VI.2 for the interest rate spread $P(z)$ **fails** to have a Fourier transform (unless $g = 0$) because it contains an expression ${}^{g}\Delta$ which does not have a finite Fourier transform for $g > 0$. But $1/{}^{g}\Delta$ does have a Fourier transform so we **can** investigate the cyclic components (the Fourier transform) of

$$\{P(z)/[(\Delta - B) * {}^{g}\Delta](z)\} \approx P(z)/({}_{B}{}^{g}s\ {}_{B}{}^{g}\mu^{\infty}\ {}^{g}\Delta(z)) \qquad \textbf{(VIII.2)}$$

based upon the approximation III.5, $[(\Delta - B) * {}^{g}\Delta](z) \approx {}_{B}{}^{g}s\ {}_{B}{}^{g}\mu^{\infty}\ {}^{g}\Delta(z)$, for the size of the mature portfolio in our model at time z. VIII.2 is the expression for the **interest rate spread** in the portfolio at time z expressed **in relation to the portfolio size** at time z (in basis points, rather than dollars, if you will). First, we prepare for the most useful form of its Fourier transform.

Substituting the expression for $P(z)$ from VI.2 into VIII.2, and using II.1, we can conclude that in the mature portfolio

$$\{P/[(\Delta - B) * {}^{g}\Delta]\} \approx [(1/{}^{g}\Delta)({}_{A}{}^{g}\mu / {}_{A}{}^{g}\mu^{\infty} - {}_{B}{}^{g}\mu / {}_{B}{}^{g}\mu^{\infty})] * (\Delta \cdot \rho')$$

$$\approx [(1/{}^{g}\Delta)({}_{A}{}^{g}\mu / {}_{A}{}^{g}\mu^{\infty} - {}_{B}{}^{g}\mu / {}_{B}{}^{g}\mu^{\infty})] * \rho' \qquad \textbf{(VIII.3)}$$

where the last step is justified by noting that the $[\ ](z - x)$ term $\rightarrow 0$ for z large when $x < 0$, by the definitions of ${}_{A}{}^{g}\mu^{\infty}$ and ${}_{B}{}^{g}\mu^{\infty}$, making the Δ multiplier on ρ' superfluous in the mature portfolio.

By definition $\quad {}_{A}{}^{g}\mu = \Delta * ({}^{g}\Delta \cdot (\Delta - {}_{g}A))$

$$= {}^{g}\Delta \cdot [(1/{}^{g}\Delta) * (\Delta - {}_{g}A)], \text{ by II.1.}$$

Substituting this, and its analogue for ${}_{B}{}^{g}\mu$, into VIII.3 gives

$$\{P/[(\Delta - B) * {}^{g}\Delta]\} \approx (1/{}^{g}\Delta) * [((\Delta - {}_{g}A) / {}_{A}{}^{g}\mu^{\infty}) - ((\Delta - {}_{g}B) / {}_{B}{}^{g}\mu^{\infty})] * \rho'$$

$$\approx (1/{}^{g}\Delta) * \Delta * [((\delta - {}_{g}a) / {}_{A}{}^{g}\mu^{\infty}) - ((\delta - {}_{g}b) / {}_{B}{}^{g}\mu^{\infty})] * \rho'$$

Finally, we can take advantage of the basic properties VIII.1 of Fourier transforms, beginning with the fact that convolution transforms into multiplication:

$$FT\{\ P/[(\Delta - B) *^{g}\Delta]\}(f) \approx$$

$$\approx FT(1/^{g}\Delta)(f)\cdot FT(\Delta)(f)\cdot FT[((\delta - {}_{g}a) / {}_{A}{}^{g}\mu^{\infty}) - ((\delta - {}_{g}b) / {}_{B}{}^{g}\mu^{\infty})](f)\cdot FT(\rho')(f)$$

$$\approx [1/(\ln(1+g) + 2\pi if)]\cdot[(1/2)\delta(f) + (1/(2\pi if))]\cdot$$

$$\cdot[((1 - FT({}_{g}a)(f)) / {}_{A}{}^{g}\mu^{\infty}) - ((1 - FT({}_{g}b)(f)) / {}_{B}{}^{g}\mu^{\infty})]\ \cdot FT(\rho')(f)$$

Now $\delta(f) = 0$ for $f \neq 0$. But for $f = 0$, both $(1 - FT({}_{g}a)(0)) = 0$ and $(1 - FT({}_{g}b)(0)) = 0$, because ${}_{g}a$ and ${}_{g}b$ are density functions and $FT(h)(0) = \int_{-\infty}^{\infty} h(x)dx$ for any function h (by definition of the Fourier Transform.) So the $\delta(f)$ term disappears entirely and the **Fourier transform** of the **interest rate spread** over time in the mature portfolio expressed **in relation to the portfolio size** at each time is

$$FT\{\ P/[(\Delta - B) *^{g}\Delta]\}(f) \approx$$

$$\approx [1/(\ln(1+g)+2\pi if)]\cdot[1/(2\pi if)]\cdot[((1-FT({}_{g}a)(f))/{}_{A}{}^{g}\mu^{\infty})-((1-FT({}_{g}b)(f))/{}_{B}{}^{g}\mu^{\infty})]\cdot FT(\rho')(f)$$

(VIII.4)

This is a wonderfully rich encapsulation of information about the interest rate spread in our model portfolio of assets and liabilities in its mature state. A number of observations follow immediately, working from right to left in VIII.4:

1. The phase and weight *FT{ }(f)* of **each cyclic component of the interest rate spread** in the mature portfolio is just **some multiple of** the phase and weight $FT(\rho')(f)$ of exactly **the same cyclic component** of the external signal ρ' **of changes in the market interest rate**. These are **the only** interest rate spread output **responses stimulated** in the mature portfolio by the external signal of changes in the market interest rate. As a "tuner" the mature insurance company functions purely by amplitude modulation (AM) and phase modulation, with no frequency modulation (FM) at all.

2. The **cyclic component** of frequency f, with phase and weight $FT(\rho')(f)$, in the external signal ρ' of changes over time in the market interest rate **is modulated** into the frequency f component of the interest rate spread output signal principally by the factor $[((1-FT({}_{g}a)(f))/{}_{A}{}^{g}\mu^{\infty})-((1-FT({}_{g}b)(f))/{}_{B}{}^{g}\mu^{\infty})]$. That is, the relative importance of the frequency f input contribution of phase and weight $FT(\rho')(f)$ to the interest rate spread output signal can be tuned only **by tuning** the relationship between the phases and weights (i.e. Fourier transform values at f)

$FT(_g\,a)(f)$ and $FT(_g\,b)(f)$ of the **frequency f components of the generational asset and liability maturity schedules** $_g a$ and $_g b$ (adjusted for the growth rate g and scaled to their respective generalized means $_A{}^g\mu^\infty$ and $_B{}^g\mu^\infty$.) Relative immunization of the interest rate spread across the whole spectrum of possible cyclic components in the change over time in market interest rates depends upon tuning the relationship between the adjusted and scaled asset and liability maturity schedules relatively across the whole spectrum of frequencies, as measured by their Fourier transforms.

3. The factors $[1/(ln(1+g)+2\pi if)]$ and $[1/(2\pi if)]$ **at lower frequency** ($/f/$ smaller, cycle longer) draw **heavier relative contributions** into the interest rate spread output signal from cyclic components of frequency f in the external signal of changes over time in the market interest rate, as modulated by observation 2. To achieve uniform immunization across the entire spectrum of possible cyclic components in the changes over time in market interest rates, it is relatively more important to align the lower frequency components of the asset and liability maturity schedule Fourier transforms than the higher frequency components.

4. The presence of $ln(1+g)$ in the denominator of one of the factors indicates that a **higher rate of stable growth**, all other things being equal, **tends to moderate** the relative overweighting of the **lower frequency contributions** into the interest rate spread output signal, as described in observation 3.

5. As the frequency f approaches zero (i.e. for **a secular trend** $FT(\rho')(0)$ within the cyclic evolution of ρ) one last application of L'Hôspital's rule (to VIII.4) yields the result expressed in VII.4, with $FT(\rho')(0)$ in place of $\mu_{\rho'}$. (The details require the relationships $FT(2\pi ixh(x))(f) = -FT(h)'(f)$; $FT(h)(0) = \int_{-\infty}^{\infty} h(x)dx$; plus II.11 and II.19.) The work of section VII. stands revealed as but one point of information within the full spectrum encoded in VIII.4. Even generalized moments are just derivatives of Fourier transforms evaluated at $f = 0$.

6. The qualitative aspects of observations 1. through 4., which would not necessarily have required Fourier analysis in order to be observed, each find precise quantitative expression in VIII.4.

Together, observations 3. and 5. above might seem to suggest a conclusion that VII.4 (i.e. evaluation of VIII.4 at only the single frequency $f = 0$) is in some sense an optimal simplified approach to immunization. On empirical grounds, however, the weight $/FT(\rho')(f)/$ of the cyclic component of frequency f in the external signal ρ' of changes in market interest rates is unlikely to be anywhere near a maximum at $f = 0$. That is, some pair of cyclic components of ρ' with frequency $/f/ \neq 0$ likely carries materially more weight in the spectrum of ρ' than does the secular trend. For example, work of Becker that found the log-log plot for the Hurst coefficient to have an elbow at approximately 5 years seems to suggest that for U. S. government interest rates, at least, $f \approx \pm 1/5$ per year may be the lowest frequency with clear importance in the spectrum of ρ'.

In the context of VIII.4 this suggests that single frequency immunization at the value $f = 0$ may not be even the most important single reference frequency for immunization. Instead, the factor $[((1-FT({}_g\, a)(f))/{}_A{}^g\mu^\infty)-((1-FT({}_g\, b)(f))/{}_B{}^g\mu^\infty)]$ of VIII.4 evaluated at some single "critical frequency" pair somewhere between the values $f = 0$ and $f = \pm 1/5$, reflecting the effect of observation 3. above, should provide the optimal single frequency immunization if the full spectrum encoded in VIII.4 is too overwhelming to work with. Use of an appropriately selected such critical frequency pair $/f/ \neq 0$ should yield a better uniform limit on interest rate spread volatility than would, for example, an approximation series using higher moments at $f = 0$. This is reminiscent of the Esscher approximation in risk theory (see Wooddy), which actually amounts to a Taylor's series expansion of a certain Fourier transform about an arguably better point than 0, about which the Edgeworth approximation expands (author, unpublished manuscript.)

As a final technical point, the Fourier transforms of the growth adjusted maturity schedules ${}_g a$ and ${}_g b$ required for the use of VIII.4 can be expressed in terms of the growth rate g and the basic generational maturity schedules a and b. The following comes directly from applying parts of VIII.1 to the definition of ${}_g a$:

$$FT({}_g a)(f_0) = \{(FT(a)(f)/{}_A{}^g s) * [1/(\ln(1+g)+2\pi i f)]\}(f_0), \text{ for any value } f = f_0 \quad \textbf{(VIII.5)}$$

As $g \rightarrow 0$ it can be shown that the $*\ [\ ]$ term in VIII.5 becomes an identity operator owing to the fact that $a(x) = 0$ for $x < 0$. The analogue of VIII.5 also holds for ${}_g b$.

IX. Conclusions

Stationary immunization shows promise as an alternative viewpoint to classical immunization theory. This does not mean replacement, but a formally complementary relationship between the two. Radically different assumptions underlie the two models, but highly idealized and unrealistic constructions characterize both. Together, they ought to enfold more of the rich economics of the insurance enterprise than either alone. Further development seems to be warranted.

Stationary immunization features a buy-and-hold model of the investment process contrasted to the dynamic rebalancing demands of the classical model. It seems (so far) to reduce to manageable complexity levels only in the mature phase, whereas the classical present value model can be calculated for most any state of the enterprise. The classical model seems to promise full immunization given the right duration relationships, but then equivocates with convexity conditions, rebalancing assumptions, and (as a practical matter) cash flow testing requirements. Stationary immunization builds cash flow and interest rate spreads into its very foundations. Its results assert explicitly that full immunization is a chimera, and focus attention directly on the practical task: how to modulate the inevitable volatility. The price, sofar, is a woefully oversimplified model that nevertheless suffers from woeful complexity.

It seems likely that a more robust foundation could arise from further development, teasing more simplified core results out of the thorny base developed here. In that regard, VII.1, VII.4, VII.8 and especially VIII.4 (the Fourier transform) show great promise. After all, to focus exclusively on interest rate spreads represents a very foreign starting point for actuarial practice. No surprise, then, that it might take a while to develop a workable level of balance between theoretical and practical comparable to what the present-value starting point offers today, after more than a century of actuarial elaboration.

Once accomplished, however, it could point the way to the principles for a model of the insurance enterprise rigorously complementary to the balance sheet present-value account of the enterprise. The latter suffers more and more from an unfortunate conflation of the concepts of present value and market value, a conflation encouraged by classical immunization theory. With market-value concepts uncritically linked to present-value calculations holding sway over fair-value-of-liability discussions, never has there been more need to have as rigorous an intellectual foundation for an amortized-cost balance sheet as for a market-value balance sheet, allowing each a full and unique measure of importance in describing and controlling the insurance enterprise, recognizing that neither alone captures reality.

Classical immunization theory provides tools relevant for a market-value balance sheet model of the enterprise. Stationary immunization theory suggests a direction to find tools that would clarify the role of an interest-rate-spread income statement model of the enterprise, complementary to the balance sheet model. Based on the success (so far as it goes) of the over-simplified model displayed in this paper, the complementary model that emerges looks a lot like amortized-cost accounting. If concepts developed here can be pushed far enough, that model can be made every bit as rigorous and scientifically productive within a volatile world as the more fashionable market-value account of things.

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