Math 5660 Advanced Financial Math
Spring 2015
Final Exam Solutions
May 1 to May 6, 2015

This is an open book take-home exam. You may use any books, notes, websites or other printed material that you wish but do not consult with any other person (doing so will be grounds for failure of the course). Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The eight questions will be equally weighted in the grading. Please return the completed exams by 5 PM Wednesday, May 6 to my mailbox in the department office, under my office door MSB408, or by email.

1. Let $\frac{d\mathbb{Q}}{d\mathbb{P}}(t)$ be the Radon-Nikodym derivative process that defines a martingale measure $\mathbb{Q}$ according to the formula

$$\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}(t)\,\mathbf{1}_A\right]$$

for any set $A \in \mathcal{F}_t$ of outcomes (paths) in the time t level $\mathcal{F}_t \subseteq \mathcal{F}$ of the filtration of a sigma-algebra $\mathcal{F}$ of sets of outcomes (paths) in our sample space Ω, where $\mathbf{1}_A$ is the indicator random variable for an outcome (path) $\omega \in \Omega$ to be in the set A (i.e. $\mathbf{1}_A(\omega) = 1$ when $\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ when $\omega \notin A$), $\mathbb{P}$ is our original probability measure, and the dynamics of our model is

$$dS(u) = \alpha(u, S(u))\,du + \sigma(u, S(u))\,dW_{\mathbb{P}}(u)$$

where $S(t)$ is the risky asset and $W_{\mathbb{P}}(t)$ is Brownian motion in the original probability measure $\mathbb{P}$. Prove that $\frac{d\mathbb{Q}}{d\mathbb{P}}(t)$ is a martingale in the original $\mathbb{P}$ probability measure.

Solution

From Theorem 5.2.3 and formula (5.2.21) of the textbook, or from page

12 of the outline

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(t) = e^{-\int_0^t \frac{1}{2}\left(\frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))}\right)^2 du - \int_0^t \frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))} dW_{\mathbb{P}}(u)} \quad \text{and now use Itô}$$

$$\begin{aligned}
d\frac{d\mathbb{Q}}{d\mathbb{P}}(t) &= e^{-\int_0^t \frac{1}{2}\left(\frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))}\right)^2 du - \int_0^t \frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))} dW_{\mathbb{P}}(u)} \\
&\quad \left\{ \begin{array}{c} d\left[-\int_0^t \frac{1}{2}\left(\frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))}\right)^2 du - \int_0^t \frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))} dW_{\mathbb{P}}(u)\right] \\ +\frac{1}{2} d\left[-\int_0^t \frac{1}{2}\left(\frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))}\right)^2 du - \int_0^t \frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))} dW_{\mathbb{P}}(u)\right] \\ d\left[-\int_0^t \frac{1}{2}\left(\frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))}\right)^2 du - \int_0^t \frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))} dW_{\mathbb{P}}(u)\right] \end{array} \right\} \\
&= e^{-\int_0^t \frac{1}{2}\left(\frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))}\right)^2 du - \int_0^t \frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))} dW_{\mathbb{P}}(u)} \\
&\quad \left\{ \begin{array}{c} -\frac{1}{2}\left(\frac{\alpha(t,S(t))-r(t)S(t)}{\sigma(t,S(t))}\right)^2 dt - \frac{\alpha(t,S(t))-r(t)S(t)}{\sigma(t,S(t))} dW_{\mathbb{P}}(t) \\ +\frac{1}{s}\left(\frac{\alpha(t,S(t))-r(t)S(t)}{\sigma(t,S(t))}\right)^2 dW_{\mathbb{P}}(t)\, dW_{\mathbb{P}}(t) \end{array} \right\} \\
&= e^{-\int_0^t \frac{1}{2}\left(\frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))}\right)^2 du - \int_0^t \frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))} dW_{\mathbb{P}}(u)} \left\{-\frac{\alpha(t,S(t))-r(t)S(t)}{\sigma(t,S(t))} dW_{\mathbb{P}}(t)\right\}
\end{aligned}$$

because $dW_{\mathbb{P}}(t)\, dW_{\mathbb{P}}(t)$ is dt, and so $d\frac{d\mathbb{Q}}{d\mathbb{P}}(t)$ has only a $\mathbb{P}$-Brownian component in its dynamics and is therefore a $\mathbb{P}$-martingale.

2. For each n define

$$f_n(x) = \sqrt{\frac{n}{2\pi}} e^{-\frac{nx^2}{2}}.$$

Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx \neq \int_{-\infty}^{\infty} \left(\lim_{n\to\infty} f_n(x)\right) dx$$

and explain why this does not violate the Dominated Covergence Theorem and the Monotone Convergence Theorem.

Solution

$$\begin{aligned}
\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx &= \lim_{n\to\infty} 1 = 1 \\
\int_{-\infty}^{\infty} \left(\lim_{n\to\infty} f_n(x)\right) dx &= \int_{-\infty}^{0} 0 dx + \int_{0}^{\infty} 0 dx = 0
\end{aligned}$$

For $m < n$, for large x, $f_n(x) < f_m(x)$ but $f_n(0) > f_m(0)$ so Monotone Convergence cannot apply. For any non-negative integrable function $g(x)$

on $\mathbb{R}$, let $L = \displaystyle\int_{-\infty}^{\infty} g(x)dx$. Then

$$\begin{aligned}
\lim_{n\to\infty}\left[L - \int_{-\infty}^{-\sqrt{\frac{1}{n}}} g(x)dx - \int_{\sqrt{\frac{1}{n}}}^{\infty} g(x)dx\right] - 1 &= -1\text{, so for large enough } n, \\
\left(L - \int_{-\infty}^{-\sqrt{\frac{1}{n}}} g(x)dx - \int_{\sqrt{\frac{1}{n}}}^{\infty} g(x)dx\right) - ((1-\Phi(-1)) - \Phi(-1)) &< 0 \text{ for } \Phi \text{ std norm cdf,} \\
\int_{-\sqrt{\frac{1}{n}}}^{\sqrt{\frac{1}{n}}} g(x)dx - \int_{-\sqrt{\frac{1}{n}}}^{\sqrt{\frac{1}{n}}} f_n(x)\,dx &< 0 \\
\int_{-\sqrt{\frac{1}{n}}}^{\sqrt{\frac{1}{n}}} (g(x) - f_n(x))\,dx &< 0
\end{aligned}$$

which means that $g(x) > f_n(x)$ fails for a set of x of measure greater than 0 and so Dominated Convergence cannot apply.

3. Given a probability space with a filtratration, a Brownian motion on that filtration and a unique risk neutral measure, assume some asset has a random value $A(T)$ at a time T such that $A(T) > 0$ almost surely. Prove that for all $t < T$ the value of that asset at t must follow a geometric Brownian motion.

   Solution

   Since there is a unique risk-neutral measure, the Second Fundamental Theorem guarantees that the market is complete. In that case, there is a replicating portfolio for the payoff $A(T)$ and the present value of the payoff is the present value of the replicating portfolio. The risk-neutral conditional expected value $\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T R(u)du} A(T) \mid \mathcal{F}_t\right]$ of the present value of any portfolio is a $\mathbb{Q}$-martingale, so $\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T R(u)du} A(T) \mid \mathcal{F}_t\right]$ is a $\mathbb{Q}$-martingale. Define $A(t)$ by $A(t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T R(u)du} A(T) \mid \mathcal{F}_t\right]$. Notice that according to this definition $A(t) > 0$ almost surely and $e^{-\int_0^t R(u)du} A(t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T R(u)du} A(T) \mid \mathcal{F}_t\right]$ is now another expression for the $\mathbb{Q}$-martingale.

By the martingale representation theorem there is an adapted process $\Gamma_{\mathbb{Q}}(t)$ where $d(e^{-\int_0^t R(u)du}A(t)) = \Gamma_{\mathbb{Q}}(t)dW_{\mathbb{Q}}(t)$ for $W_{\mathbb{Q}}(t)$ Brownian under $\mathbb{Q}$. But Itô's product formula says that $d(e^{-\int_0^t R(u)du}A(t)) = de^{-\int_0^t R(u)du}A(t) + e^{-\int_0^t R(u)du}dA(t) + de^{-\int_0^t R(u)du}dA(t)$. Equating the two expressions for $d(e^{-\int_0^t R(u)du}A(t))$ gives

$$\begin{aligned}
\Gamma_{\mathbb{Q}}(t)dW_{\mathbb{Q}}(t) &= de^{-\int_0^t R(u)du}A(t) + e^{-\int_0^t R(u)du}dA(t) + de^{-\int_0^t R(u)du}dA(t) \\
dA(t) &= e^{\int_0^t R(u)du}\left\{-de^{-\int_0^t R(u)du}[A(t)+dA(t)] + \Gamma_{\mathbb{Q}}(t)dW_{\mathbb{Q}}(t)\right\} \\
&= R(t)\left(A(t)+dA(t)\right)dt + e^{\int_0^t R(u)du}\Gamma_{\mathbb{Q}}(t)dW_{\mathbb{Q}}(t) \\
& \text{a solution for which is} \\
dA(t) &= R(t)A(t)dt + e^{\int_0^t R(u)du}\Gamma_{\mathbb{Q}}(t)dW_{\mathbb{Q}}(t) \\
\text{because } dtdt &= 0 \text{ and } dtdW_{\mathbb{Q}}(t) = 0 \\
\text{This can be rewritten as } dA(t) &= R(t)A(t)dt + e^{\int_0^t R(u)du}\frac{\Gamma_{\mathbb{Q}}(t)}{A(t)}A(t)dW_{\mathbb{Q}}(t) \\
\text{because } A(t) &> 0 \text{ almost surely.}
\end{aligned}$$

Therefore, $A(t)$ is a generalized $\mathbb{Q}$-Brownian motion with drift process $R(t)$ and volatility process $e^{\int_0^t R(u)du}\frac{\Gamma_{\mathbb{Q}}(t)}{A(t)}$. From the Girsanov Theorem, $dW_{\mathbb{Q}}(t) = \Theta(t)dt + dW(t)$ for some market price of risk process $\Theta(t)$ so $dA(t) = (R(t)+\Theta(t))A(t)dt + e^{\int_0^t R(u)du}\frac{\Gamma_{\mathbb{Q}}(t)}{A(t)}A(t)dW(t)$, which shows that $A(t)$ is a generalized Brownian motion with drift process $R(t)+\Theta(t)$ and volatility process $e^{\int_0^t R(u)du}\frac{\Gamma_{\mathbb{Q}}(t)}{A(t)}$.

4. Let $G(t)$ be a $\mathbb{Q}$-martingale for a risk-neutral measure $\mathbb{Q}$. Use the martingale representation theorem for the original measure $\mathbb{P}$ and some stochastic calculus to come up with a stochastic process $\Lambda(t)$ that is adapted to the filtration $\mathcal{F}_t$ generated by the original Brownian motion $W_{\mathbb{P}}(t)$ and that

satisfies

$$G(t) = G(0) + \int_0^t \Lambda(u)\, dW_{\mathbb{Q}}(u)$$

where $W_{\mathbb{Q}}(u)$ is Brownian motion with respect to the risk-neutral measure $\mathbb{Q}$.

Solution

If $\Theta(t)$ is the market price of risk process that defines the risk-neutral measure $\mathbb{Q}$, so that $dW_{\mathbb{Q}}(u) = dW_{\mathbb{P}}(u) + \Theta(u)du$, for any r.v. X, $\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}[XZ(T)]$ where the exponential martingale $Z(t) = e^{-\int_0^t \Theta(u)dW_{\mathbb{P}}(u) - \frac{1}{2}\int_0^t \Theta^2(u)du}$ is the Radon-Nikodym derivative process, and, for any s, $\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_s] = \mathbb{E}\left[X\frac{Z(T)}{Z(s)}|\mathcal{F}_s\right] = \frac{1}{Z(s)}\mathbb{E}[XZ(T)|\mathcal{F}_s]$, then $M(t) = G(t)Z(t)$ is a $\mathbb{P}$-martingale because for any $s < t$

$$\begin{aligned}
\mathbb{E}[M(t)|\mathcal{F}_s] &= \mathbb{E}[G(t)Z(t)|\mathcal{F}_s] \\
&= \mathbb{E}[G(t)\mathbb{E}[Z(T)|\mathcal{F}_t]|\mathcal{F}_s] \\
&= \mathbb{E}[\mathbb{E}[G(t)Z(T)|\mathcal{F}_t]|\mathcal{F}_s] \\
&= \mathbb{E}[G(t)Z(T)|\mathcal{F}_s] \\
&= Z(s)\mathbb{E}_{\mathbb{Q}}[G(t)|\mathcal{F}_s] \\
&= Z(s)G(s) \\
&= M(s)
\end{aligned}$$

By the Martingale Representation Theorem in $\mathbb{P}$, then, there is a process $\Gamma(t)$ adapted to $\mathcal{F}_t$ with $dM(t) = \Gamma(t)dW_{\mathbb{P}}(t)$. Use that in what follows, along with

$$\begin{aligned}
d\frac{1}{Z(t)} &= \frac{1}{Z(t)}\left[\Theta(t)dW_{\mathbb{P}}(t) + \frac{1}{2}\Theta^2(t)dt + \frac{1}{2}\Theta^2(t)dt\right] \\
&= \frac{1}{Z(t)}\left[\Theta(t)W_{\mathbb{P}}(t) + \Theta^2(t)dt\right] \\
&= \frac{\Theta(t)}{Z(t)}[dW_{\mathbb{P}}(t) + \Theta(t)dt] \\
&= \frac{\Theta(t)}{Z(t)}dW_{\mathbb{Q}}(t), \text{ so that}
\end{aligned}$$

$$\begin{aligned}
dG\left(t\right) &= d\left(M(t)\frac{1}{Z(t)}\right) \\
&= dM(t)\frac{1}{Z(t)} + M(t)d\frac{1}{Z(t)} + dM(t)d\frac{1}{Z(t)} \\
&= \Gamma(t)dW_{\mathbb{P}}\left(t\right)\frac{1}{Z(t)} + M(t)\frac{\Theta(t)}{Z(t)}dW_{\mathbb{Q}}\left(t\right) + \Gamma(t)dW\left(t\right)\frac{\Theta(t)}{Z(t)}dW_{\mathbb{Q}}\left(t\right) \\
&= \frac{\Gamma(t)}{Z(t)}\left(dW_{\mathbb{Q}}\left(t\right) - \Theta(t)dt\right) + M(t)\frac{\Theta(t)}{Z(t)}dW_{\mathbb{Q}}\left(t\right) + \left(dW_{\mathbb{Q}}\left(t\right) - \Theta(t)dt\right)\frac{\Gamma(t)\Theta(t)}{Z(t)}dW_{\mathbb{Q}}\left(t\right) \\
&= \frac{\Gamma(t)}{Z(t)}\left(-\Theta(t) + \Theta(t)\right)dt + \frac{\Gamma(t) + M(t)\Theta(t)}{Z(t)}dW_{\mathbb{Q}}\left(t\right) \\
&= \frac{\Gamma(t) + M(t)\Theta(t)}{Z(t)}dW_{\mathbb{Q}}\left(t\right), \text{ so } \Lambda\left(t\right) = \frac{\Gamma(t) + M(t)\Theta(t)}{Z(t)}
\end{aligned}$$

5. Let $Y\left(t\right)$ be a stochastic process with respect to a filtration $\mathcal{F}_t$. Fix some $s < t$ and let $E\left(t\right) = Y\left(t\right) - \mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right]$. What is $\mathbb{E}\left[E\left(t\right)\right]$? Let $X\left(t\right)$ be any other stochastic process with respect to the filtration $\mathcal{F}_t$. Prove that

$$\mathbb{V}\left[E\left(t\right)\right] \leq \mathbb{V}\left[Y\left(t\right) - X\left(s\right)\right]$$

where $\mathbb{V}[\ ]$ stands for the variance of the random variable inside the $[\ ]$.

Solution

$\mathbb{E}\left[E\left(t\right)\right] = \mathbb{E}\left[Y\left(t\right) - \mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right]\right] = \mathbb{E}\left[Y\left(t\right)\right] - \mathbb{E}\left[\mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right]\right] = 0$, and obviously $\mathbb{E}\left[E(t)|\mathcal{F}_s\right] = 0$, so

$$\begin{aligned}
\mathbb{V}\left[Y\left(t\right) - X\left(s\right)\right] &= \mathbb{E}\left[\left(Y\left(t\right) - X\left(s\right) - \mathbb{E}\left[Y\left(t\right) - X\left(s\right)\right]\right)^2\right] \\
&= \mathbb{E}\left[\left(E(t) + \mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right] - X\left(s\right) - \mathbb{E}\left[Y\left(t\right) - X\left(s\right)\right]\right)^2\right] \\
&= \mathbb{E}\left[\begin{array}{c} E^2(t) + 2E(t)\left(\mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right] - X\left(s\right) - \mathbb{E}\left[Y\left(t\right) - X\left(s\right)\right]\right) \\ +\left(\mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right] - X\left(s\right) - \mathbb{E}\left[Y\left(t\right) - X\left(s\right)\right]\right)^2 \end{array}\right] \\
&= \mathbb{E}\left[E^2(t)\right] + 2\mathbb{E}\left[E(t)\left(\mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right] - X\left(s\right) - \mathbb{E}\left[Y\left(t\right) - X\left(s\right)\right]\right)\right] \\
&\quad +\mathbb{E}\left[\left(\mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right] - X\left(s\right) - \mathbb{E}\left[Y\left(t\right) - X\left(s\right)\right]\right)^2\right] \\
&\geq \mathbb{V}\left[E\left(t\right)\right] + 2\mathbb{E}\left[\mathbb{E}\left[E(t)\left(\mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right] - X\left(s\right) - \mathbb{E}\left[Y\left(t\right) - X\left(s\right)\right]\right)|\mathcal{F}_s\right]\right] \\
&= \mathbb{V}\left[E\left(t\right)\right] + 2\mathbb{E}\left[\mathbb{E}\left[E(t)|\mathcal{F}_s\right]\left(\mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right] - X\left(s\right) - \mathbb{E}\left[Y\left(t\right) - X\left(s\right)\right]\right)\right] \\
&= \mathbb{V}\left[E\left(t\right)\right]
\end{aligned}$$

6. Use stochastic calculus to find the sixth-central-moment of a normal random variable with variance T.

Solution

$W\left(T\right)$ is normal with mean zero and variance T, so the sixth central moment of a normal random variable with variance T will be the sixth moment of $W(T)$

$$\begin{aligned}
dW^6(t) &= 6W^5(t)\,dW(t) + \frac{1}{2}30W^4(t)\,dt \\
W^6(T) &= \int_0^T 6W^5(t)\,dW(t) + \int_0^T 15W^4(t)\,dt \\
\mathbb{E}\left[W^6(T)\right] &= \int_0^T 15\mathbb{E}\left[W^4(t)\right]dt \\
dW^4(t) &= 4W^3(t)\,dW(t) + \frac{1}{2}12W^2(t)\,dt \\
W^4(T) &= \int_0^T 4W^3(t)\,dW(t) + \int_0^T 6W^2(t)\,dt \\
\mathbb{E}\left[W^4(T)\right] &= \int_0^T 6\mathbb{E}\left[W^2(t)\right]dt \\
&= \int_0^T 6t dt = 3T^2 \quad \text{so} \\
\mathbb{E}\left[W^6(T)\right] &= \int_0^T 15{\cdot}3t^2 dt = 15T^3
\end{aligned}$$

7. Derive the general solution to the stochastic differential equation

$$dR(u) = p(u)\,R(u)\,du + q(u)\,R(u)\,dW(u)$$

Solution

$$\begin{aligned}
d\ln R(u) &= \frac{1}{R(u)}dR(u) - \frac{1}{2}\frac{1}{R^2(u)}dR(u)dR(u) \\
&= p(u)\,du + q(u)\,dW(u) - \frac{1}{2}q^2(u)\,du \\
&= \left(p(u) - \frac{1}{2}q^2(u)\right)du + q(u)\,dW(u) \\
\ln R(t) &= \ln R(0) + \int_0^t \left(p(u) - \frac{1}{2}q^2(u)\right)du + \int_0^t q(u)\,dW(u) \\
R(t) &= e^{\ln R(0) + \int_0^t \left(p(u)-\frac{1}{2}q^2(u)\right)du + \int_0^t q(u)dW(u)} = R(0)e^{\int_0^t \left(p(u)-\frac{1}{2}q^2(u)\right)du + \int_0^t q(u)dW(u)}
\end{aligned}$$

8. Derive a solution to the stochastic differential equation

$$dS(u) = (p(u) + q(u)\,S(u))\,du + (a(u) + b(u)\,S(u))\,dW(u)$$

subject to an initial condition $S(0) = s$.

Solution

Define

$$
\begin{aligned}
Z(t) &= e^{\int_0^t b(u)dW(u) + \int_0^t \left(q(u) - \frac{1}{2}b^2(u)\right)du} \\
s(t) &= s + \int_0^t \frac{a(u)}{Z(u)} dW(u) + \int_0^t \frac{p(u) - b(u)a(u)}{Z(u)} du \\
S(t) &= s(t)Z(t), \text{ then} \\
S(0) &= se^0 = s \text{ and} \\
dS(t) &= ds(t)Z(t) + s(t)dZ(t) + ds(t)dZ(t) \\
&= \left(\frac{a(t)}{Z(t)} dW(t) + \frac{p(t) - b(t)a(t)}{Z(t)} dt\right) Z(t) + s(t)Z(t)\left(b(t)dW(t) + q(t)dt\right) \\
&\quad + \left(\frac{a(t)}{Z(t)} dW(t) + \frac{p(t) - b(t)a(t)}{Z(t)} dt\right) Z(t)\left(b(t)dW(t) + q(t)dt\right) \\
&= a(t)dW(t) + \left(p(t) - b(t)a(t)\right)dt + S(t)\left(b(t)dW(t) + q(t)dt\right) + a(t)b(t)dt \\
&= \left(a(t) + b(t)S(t)\right)dW(t) + \left(p(t) + q(t)S(t)\right)dt
\end{aligned}
$$