Math 5660 Advanced Financial Math
Spring 2015
Final Exam
May 1 to May 6, 2015

This is an open book take-home exam. You may use any books, notes, websites or other printed material that you wish but do not consult with any other person (doing so will be grounds for failure of the course). Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The eight questions will be equally weighted in the grading. Please return the completed exams by 5 PM Wednesday, May 6 to my mailbox in the department office, under my office door MSB408, or by email.

1. Let $\frac{d\mathbb{Q}}{d\mathbb{P}}(t)$ be the Radon-Nikodym derivative process that defines a martingale measure $\mathbb{Q}$ according to the formula

$$\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}(t)\,\mathbf{1}_A\right]$$

for any set $A \in \mathcal{F}_t$ of outcomes (paths) in the time t level $\mathcal{F}_t \subseteq \mathcal{F}$ of the filtration of a sigma-algebra $\mathcal{F}$ of sets of outcomes (paths) in our sample space Ω, where $\mathbf{1}_A$ is the indicator random variable for an outcome (path) $\omega \in \Omega$ to be in the set A (i.e. $\mathbf{1}_A(\omega) = 1$ when $\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ when $\omega \notin A$), $\mathbb{P}$ is our original probability measure, and the dynamics of our model is

$$dS(u) = \alpha(u, S(u))\,du + \sigma(u, S(u))\,dW_{\mathbb{P}}(u)$$

where $S(t)$ is the risky asset and $W_{\mathbb{P}}(t)$ is Brownian motion in the original probability measure $\mathbb{P}$. Prove that $\frac{d\mathbb{Q}}{d\mathbb{P}}(t)$ is a martingale in the original $\mathbb{P}$ probability measure.

2. For each n define

$$f_n(x) = \sqrt{\frac{n}{2\pi}}e^{-\frac{nx^2}{2}}.$$

Show that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx \neq \int_{-\infty}^{\infty}\left(\lim_{n\to\infty} f_n(x)\right)dx$$

and explain why this does not violate the Dominated Covergence Theorem and the Monotone Convergence Theorem.

3. Given a probability space with a filtratration, a Brownian motion on that filtration and a unique risk neutral measure, assume some asset has a random value $A(T)$ at a time T such that $A(T) > 0$ almost surely. Prove that for all $t < T$ the value of that asset at t must follow a geometric Brownian motion.

4. Let $G(t)$ be a $\mathbb{Q}$-martingale for a risk-neutral measure $\mathbb{Q}$. Use the martingale representation theorem for the original measure $\mathbb{P}$ and some stochastic calculus to come up with a stochastic process $\Lambda(t)$ that is adapted to the filtration $\mathcal{F}_t$ generated by the original Brownian motion $W_{\mathbb{P}}(t)$ and that satisfies
$$G(t) = G(0) + \int_0^t \Lambda(u)\, dW_{\mathbb{Q}}(u)$$
where $W_{\mathbb{Q}}(u)$ is Brownian motion with respect to the risk-neutral measure $\mathbb{Q}$.

5. Let $Y(t)$ be a stochastic process with respect to a filtration $\mathcal{F}_t$. Fix some $s < t$ and let $E(t) = Y(t) - \mathbb{E}[Y(t)|\mathcal{F}_s]$. What is $\mathbb{E}[E(t)]$? Let $X(t)$ be any other stochastic process with respect to the filtration $\mathcal{F}_t$. Prove that
$$\mathbb{V}[E(t)] \leq \mathbb{V}[Y(t) - X(s)]$$
where $\mathbb{V}[\,]$ stands for the variance of the random variable inside the $[\ ]$.

6. Use stochastic calculus to find the sixth-central-moment of a normal random variable with variance T.

7. Derive the general solution to the stochastic differential equation
$$dR(u) = p(u)\, R(u)\, du + q(u)\, R(u)\, dW(u)$$

8. Derive a solution to the stochastic differential equation
$$dS(u) = (p(u) + q(u)\, S(u))\, du + (a(u) + b(u)\, S(u))\, dW(u)$$
subject to an initial condition $S(0) = s$.