Math 5660 Advanced Financial Math
Spring 2014
Final Exam Solutions
May 2 to May 7, 2014

This is an open book take-home exam. You may use any books, notes, websites or other printed material that you wish but do not consult with any other person (doing so will be grounds for failure of the course). Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The nine questions will be equally weighted in the grading. Please return the completed exams by 5 PM Wednesday, May 7 to my mailbox in the department office, under my office door MSB408, or by email.

1. Use stochastic calculus to find the sixth-moment of a normal random variable with mean zero and variance T.

   Solution

   W_T is normal with mean zero and variance T.

$$
\begin{aligned}
dW_t^6 &= 6W_t^5 dW_t + \frac{1}{2}30W_t^4 dt \\
W_T^6 &= \int_0^T 6W_t^5 dW_t + \int_0^T 15W_t^4 dt \\
\mathbb{E}\left[W_T^6\right] &= \int_0^T 15\mathbb{E}\left[W_t^4\right] dt \\
dW_t^4 &= 4W_t^3 dW_t + \frac{1}{2}12W_t^2 dt \\
W_T^4 &= \int_0^T 4W_t^3 dW_t + \int_0^T 6W_t^2 dt \\
\mathbb{E}\left[W_T^4\right] &= \int_0^T 6\mathbb{E}\left[W_t^2\right] dt \\
&= \int_0^T 6t dt = 3T^2 \quad \text{so} \\
\mathbb{E}\left[W_T^6\right] &= \int_0^T 15 \cdot 3t^2 dt = 15T^3
\end{aligned}
$$

2. Derive a solution to the stochastic differential equation

$$dR(t) = (a(t) + b(t) R(t)) dt + (\gamma(t) + \sigma(t) R(t)) dW_t$$

   subject to an initial condition $R(0) = r$.

   Solution

Define

$$
\begin{aligned}
Z(t) &= e^{\int_0^t \sigma(u)dW_u + \int_0^t \left(b(u)-\frac{1}{2}\sigma^2(u)\right)du} \\
r(t) &= r + \int_0^t \frac{\gamma(u)}{Z(u)}dW_u + \int_0^t \frac{a(u)-\sigma(u)\gamma(u)}{Z(u)}du \\
R(t) &= r(t)Z(t), \text{ then} \\
R(0) &= re^0 = r \text{ and} \\
dR(t) &= dr(t)Z(t) + r(t)dZ(t) + dr(t)dZ(t) \\
&= \left(\frac{\gamma(t)}{Z(t)}dW_t + \frac{a(t)-\sigma(t)\gamma(t)}{Z(t)}dt\right) Z(t) + r(t)Z(t)\left(\sigma(t)dW_t + b(t)dt\right) \\
&\quad + \left(\frac{\gamma(t)}{Z(t)}dW_t + \frac{a(t)-\sigma(t)\gamma(t)}{Z(t)}dt\right) Z(t)\left(\sigma(t)dW_t + b(t)dt\right) \\
&= \gamma(t)dW_t + \left(a(t)-\sigma(t)\gamma(t)\right)dt + R(t)\left(\sigma(t)dW_t + b(t)dt\right) + \gamma(t)\sigma(t)dt \\
&= \left(\gamma(t) + \sigma(t)R(t)\right)dW_t + \left(a(t) + b(t)R(t)\right)dt
\end{aligned}
$$

3. Derive the general solution to the stochastic differential equation

$$dS\left(t\right) = \alpha\left(t\right)S\left(t\right)dt + \sigma\left(t\right)S\left(t\right)dW_t$$

Solution

$$
\begin{aligned}
d\ln S(t) &= \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S^2(t)}dS(t)dS(t) \\
&= \alpha\left(t\right)dt + \sigma\left(t\right)dW_t - \frac{1}{2}\sigma^2\left(t\right)dt \\
&= \left(\alpha\left(t\right) - \frac{1}{2}\sigma^2\left(t\right)\right)dt + \sigma\left(t\right)dW_t \\
\ln S(t) &= \ln S(0) + \int_0^t \left(\alpha\left(t\right) - \frac{1}{2}\sigma^2\left(t\right)\right)dt + \int_0^t \sigma\left(t\right)dW_t \\
S(t) &= e^{\ln S(0) + \int_0^t \left(\alpha(t)-\frac{1}{2}\sigma^2(t)\right)dt + \int_0^t \sigma(t)dW_t} = S(0)e^{\int_0^t \left(\alpha(t)-\frac{1}{2}\sigma^2(t)\right)dt + \int_0^t \sigma(t)dW_t}
\end{aligned}
$$

4. Let $Y\left(t\right)$ be a stochastic process with respect to a filtration $\mathcal{F}_t$. Fix some $s < t$ and let $E\left(t\right) = Y\left(t\right) - \mathbb{E}\left[Y\left(t\right)|\mathcal{F}_s\right]$. What is $\mathbb{E}\left[E\left(t\right)\right]$? Let $X\left(t\right)$ be any other stochastic process with respect to the filtration $\mathcal{F}_t$. Prove that

$$\mathbb{V}\left[E\left(t\right)\right] \leq \mathbb{V}\left[Y\left(t\right) - X\left(s\right)\right]$$

where $\mathbb{V}\left[\,\right]$ stands for the variance of the random variable inside the $[\ \ ]$.

Solution

$\mathbb{E}[E(t)] = \mathbb{E}[Y(t) - \mathbb{E}[Y(t)|\mathcal{F}_s]] = \mathbb{E}[Y(t)] - \mathbb{E}[\mathbb{E}[Y(t)|\mathcal{F}_s]] = 0$, and obviously $\mathbb{E}[E(t)|\mathcal{F}_s] = 0$, so

$$\begin{aligned}
\mathbb{V}[Y(t) - X(s)] &= \mathbb{E}\left[(Y(t) - X(s) - \mathbb{E}[Y(t) - X(s)])^2\right] \\
&= \mathbb{E}\left[(E(t) + \mathbb{E}[Y(t)|\mathcal{F}_s] - X(s) - \mathbb{E}[Y(t) - X(s)])^2\right] \\
&= \mathbb{E}\left[\begin{array}{c} E^2(t) + 2E(t)(\mathbb{E}[Y(t)|\mathcal{F}_s] - X(s) - \mathbb{E}[Y(t) - X(s)]) \\ +(\mathbb{E}[Y(t)|\mathcal{F}_s] - X(s) - \mathbb{E}[Y(t) - X(s)])^2 \end{array}\right] \\
&= \mathbb{E}\left[E^2(t)\right] + 2\mathbb{E}[E(t)(\mathbb{E}[Y(t)|\mathcal{F}_s] - X(s) - \mathbb{E}[Y(t) - X(s)])] \\
&\quad +\mathbb{E}\left[(\mathbb{E}[Y(t)|\mathcal{F}_s] - X(s) - \mathbb{E}[Y(t) - X(s)])^2\right] \\
&\geq \mathbb{V}[E(t)] + 2\mathbb{E}[\mathbb{E}[E(t)(\mathbb{E}[Y(t)|\mathcal{F}_s] - X(s) - \mathbb{E}[Y(t) - X(s)])|\mathcal{F}_s]] \\
&= \mathbb{V}[E(t)] + 2\mathbb{E}[\mathbb{E}[E(t)|\mathcal{F}_s](\mathbb{E}[Y(t)|\mathcal{F}_s] - X(s) - \mathbb{E}[Y(t) - X(s)])] \\
&= \mathbb{V}[E(t)]
\end{aligned}$$

5. Give an example (with proof) of two standard normal random variables that are uncorrelated but not independent.

   Solution

   Let X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{2|x|+y}{\sqrt{2\pi}} e^{-\frac{(2|x|+y)^2}{2}} & \text{when } y \geq -|x| \\ 0 & \text{when } y < -|x| \end{cases}$$

   $\int_{-|x|}^{\infty} \frac{2|x|+y}{\sqrt{2\pi}} e^{-\frac{(2|x|+y)^2}{2}} dy = \int_{|x|}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \left[-\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right]_{|x|}^{\infty} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, so for fixed x, the marginal density is standard normal.

   $2\int_{\max(0,-y)}^{\infty} \frac{2|x|+y}{\sqrt{2\pi}} e^{-\frac{(2|x|+y)^2}{2}} dx = 2\int_{\pm y}^{\infty} \frac{1}{2} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$, so for fixed y, the marginal density is standard normal.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-|x|}^{\infty} xy\frac{2|x|+y}{\sqrt{2\pi}} e^{-\frac{(2|x|+y)^2}{2}} dydx &= \int_{-\infty}^{0}\int_{-|x|}^{\infty} xy\frac{2|x|+y}{\sqrt{2\pi}} e^{-\frac{(2|x|+y)^2}{2}} dydx \\
&\quad +\int_{0}^{\infty}\int_{-|x|}^{\infty} xy\frac{2|x|+y}{\sqrt{2\pi}} e^{-\frac{(2|x|+y)^2}{2}} dydx \\
&= \int_{0}^{\infty}\int_{-|x|}^{\infty} (-z)\, y\frac{2|z|+y}{\sqrt{2\pi}} e^{-\frac{(2|z|+y)^2}{2}} dydz \\
&\quad +\int_{0}^{\infty}\int_{-|x|}^{\infty} xy\frac{2|x|+y}{\sqrt{2\pi}} e^{-\frac{(2|x|+y)^2}{2}} dydx \\
&= 0, \text{ so } \mathbb{E}[XY] = 0.
\end{aligned}$$

But $\mathbb{P}[Y \leq 0 \& X \leq -1] = \int_{-\infty}^{-1} \int_{-|x|}^{0} \frac{2|x|+y}{\sqrt{2\pi}} e^{-\frac{(2|x|+y)^2}{2}} dy dx = \int_{-\infty}^{-1} \int_{|x|}^{2|x|} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz dx =$ $\int_{-\infty}^{-1} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{4x^2}{2}} \right) = \Phi(-1) - \frac{1}{2}\Phi(-2) > \frac{1}{2}\Phi(-1)$, while $\mathbb{P}[Y \leq 0] \mathbb{P}[X \leq -1] =$ $\frac{1}{2}\Phi(-1)$ so X and Y are not independent.

6. Let $\widetilde{M}(t)$ be a $\widetilde{\mathbb{P}}$-martingale for a risk-neutral measure $\widetilde{\mathbb{P}}$. Use the martingale representation theorem for the original measure $\mathbb{P}$ and some stochastic calculus to come up with a stochastic process $\widetilde{\Gamma}(t)$ that is adapted to the filtration $\mathcal{F}_t$ generated by the original Brownian motion W_t and that satisfies

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u) \, d\widetilde{W}_u$$

where $\widetilde{W}_u$ is Brownian motion with respect to the risk-neutral measure $\widetilde{\mathbb{P}}$.

Solution

If $\Theta(t)$ is the market price of risk process that defines the risk-neutral measure $\widetilde{\mathbb{P}}$, so that $d\widetilde{W}_t = dW_t + \Theta(t)dt$, for any r.v. X, $\widetilde{\mathbb{E}}[X] = \mathbb{E}[XZ(T)]$ where the exponential martingale $Z(t) = e^{-\int_0^t \Theta(u)dW_u - \frac{1}{2}\int_0^t \Theta^2(u)du}$ is the Radon-Nikodym derivative process, and, for any s, $\widetilde{\mathbb{E}}[X|\mathcal{F}_s] = \mathbb{E}\left[X\frac{Z(T)}{Z(s)}|\mathcal{F}_s\right] =$ $\frac{1}{Z(s)}\mathbb{E}[XZ(T)|\mathcal{F}_s]$, then $M(t) = \widetilde{M}(t)Z(t)$ is a $\mathbb{P}$-martingale because for any $s < t$

$$\begin{aligned}
\mathbb{E}[M(t)|\mathcal{F}_s] &= \mathbb{E}\left[\widetilde{M}(t)Z(t)|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\widetilde{M}(t)\mathbb{E}[Z(T)|\mathcal{F}_t]|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\widetilde{M}(t)Z(T)|\mathcal{F}_t\right]|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\widetilde{M}(t)Z(T)|\mathcal{F}_s\right] \\
&= Z(s)\widetilde{\mathbb{E}}\left[\widetilde{M}(t)|\mathcal{F}_s\right] \\
&= Z(s)\widetilde{M}(s) \\
&= M(s)
\end{aligned}$$

By the Martingale Representation Theorem in $\mathbb{P}$, then, there is a process $\Gamma(t)$ adapted to $\mathcal{F}_t$ with $dM(t) = \Gamma(t)dW_t$. Use that in what follows, along with

$$\begin{aligned}
d\frac{1}{Z(t)} &= \frac{1}{Z(t)}\left[\Theta(t)dW_t + \frac{1}{2}\Theta^2(t)dt + \frac{1}{2}\Theta^2(t)dt\right] \\
&= \frac{1}{Z(t)}\left[\Theta(t)dW_t + \Theta^2(t)dt\right] \\
&= \frac{\Theta(t)}{Z(t)}\left[dW_t + \Theta(t)dt\right] \\
&= \frac{\Theta(t)}{Z(t)}d\widetilde{W}_t, \text{ so that}
\end{aligned}$$

$$\begin{aligned}
d\widetilde{M}(t) &= d\left(M(t)\frac{1}{Z(t)}\right) \\
&= dM(t)\frac{1}{Z(t)} + M(t)d\frac{1}{Z(t)} + dM(t)d\frac{1}{Z(t)} \\
&= \Gamma(t)dW_t\frac{1}{Z(t)} + M(t)\frac{\Theta(t)}{Z(t)}d\widetilde{W}_t + \Gamma(t)dW_t\frac{\Theta(t)}{Z(t)}d\widetilde{W}_t \\
&= \frac{\Gamma(t)}{Z(t)}\left(d\widetilde{W}_t - \Theta(t)dt\right) + M(t)\frac{\Theta(t)}{Z(t)}d\widetilde{W}_t + \left(d\widetilde{W}_t - \Theta(t)dt\right)\frac{\Gamma(t)\Theta(t)}{Z(t)}d\widetilde{W}_t \\
&= \frac{\Gamma(t)}{Z(t)}\left(-\Theta(t) + \Theta(t)\right)dt + \frac{\Gamma(t) + M(t)\Theta(t)}{Z(t)}d\widetilde{W}_t \\
&= \frac{\Gamma(t) + M(t)\Theta(t)}{Z(t)}d\widetilde{W}_t, \text{ so } \widetilde{\Gamma}(t) = \frac{\Gamma(t) + M(t)\Theta(t)}{Z(t)}
\end{aligned}$$

7. Given a probability space with a filtratration, a Brownian motion on that filtration and a unique risk neutral measure, assume some asset has a random value $V(T)$ at a time T such that $V(T) > 0$ almost surely. Prove that for all $t < T$ the value of that asset at t must follow a geometric Brownian motion.

   Solution

   By the second fundamental theorem of asset pricing this market is complete, so there is a portfolio with payoff equal to $V(T)$ at T. The value of the portfolio at each time is the value of the asset at that time. Define $V(t) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T r(u)du} V(T)|\mathcal{F}_t\right]$. Clearly this equals $V(T)$ when $t = T$, so it is the value at t of the portfolio and of the asset. By the definition of risk-neutral measure, the present value at 0 of the portfolio, $D(t)V(t)$, is a $\widetilde{\mathbb{P}}$-martingale where $D(t) = e^{-\int_0^t r(u)du}$. The result of problem #6 then provides an adapted process $\widetilde{\Gamma}(t)$ for which.

$$\begin{aligned} d(D(t)V(t)) &= \widetilde{\Gamma}(t)d\widetilde{W}_t \text{ so} \\ -r(t)D(t)V(t)dt + D(t)dV(t) &= \widetilde{\Gamma}(t)d\widetilde{W}_t \\ dV(t) &= r(t)V(t)dt + \frac{\widetilde{\Gamma}(t)}{D(t)}d\widetilde{W}_t \\ dV(t) &= r(t)V(t)dt + \frac{\widetilde{\Gamma}(t)}{D(t)V(t)}V(t)d\widetilde{W}_t \end{aligned}$$

so $V(t)$ follows a geometric Brownian motion with $\alpha(t) = r(t)$ and $\sigma(t) = \frac{\widetilde{\Gamma}(t)}{D(t)V(t)}$. (Note that, for all t, $V(t) > 0$ almost surely because the same is true for $V(T)$).

8. Give an example (with proof) of an uncountably infinite set with probability zero.

   Solution

   In the probability space consisting of all possible infinite sequences $\omega_1\omega_2\omega_3...$of independent coin-tosses with probability $0 < p < 1$ of heads on each toss, let A be the subset consisting of all sequences where $\omega_1 = \omega_2, \omega_3 = \omega_4$, and so on forever. First, the set A is uncountable, because if you had a listing $(\omega_1^{(1)}\omega_1^{(1)}\omega_3^{(1)}\omega_3^{(1)}...), (\omega_1^{(2)}\omega_1^{(2)}\omega_3^{(2)}\omega_3^{(2)}...), ...$ of them, then it would have to omit the sequence $(\overline{\omega}_1^{(1)}\overline{\omega}_1^{(1)}\overline{\omega}_3^{(2)}\overline{\omega}_3^{(2)}...)$, where $\overline{\omega}$ is the opposite toss of ω, contradicting countability. But the set A has probability 0 because the probability that $\omega_2 = \omega_1$ is $0 < 1 - 2p(1-p) < 1$ (because it is $p^2 + (1-p)^2$). The probability that both $\omega_2 = \omega_1$ and $\omega_4 = \omega_3$ is then $(1 - 2p(1-p))^2$ and so on, so that the probability that a sequence is in A is $\lim_{n\to\infty} (1 - 2p(1-p))^n = 0$.

9. For each n define
$$f_n(x) = \frac{1}{\sqrt{2\pi n}}e^{-\frac{x^2}{2n}}.$$
   Show that
$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx \neq \int_{-\infty}^{\infty}\left(\lim_{n\to\infty} f_n(x)\right)dx$$
   and explain why this does not violate the Dominated Covergence Theorem and the Monotone Convergence Theorem.

   Solution

$$\begin{aligned} \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx &= \lim_{n\to\infty} 1 = 1 \\ \int_{-\infty}^{\infty}\left(\lim_{n\to\infty} f_n(x)\right)dx &= \int_{-\infty}^{\infty} 0dx = 0 \end{aligned}$$

For $m < n$, for large x, $f_n(x) > f_m(x)$ but $f_n(0) < f_m(0)$ so Monotone Convergence cannot apply. For any non-negative integrable function $g(x)$ on $\mathbb{R}$, let $L = \int_{-\infty}^{\infty} g(x)dx$. Then

$$\begin{aligned}
\lim_{n\to\infty}\left[L - \int_{-\sqrt{n}}^{\sqrt{n}} g(x)dx\right] - 1 &= -1, \text{ so for large enough } n, \\
\left(L - \int_{-\sqrt{n}}^{\sqrt{n}} g(x)dx\right) - (\Phi(-1) + (1 - \Phi(-1))) &< 0 \text{ for } \Phi \text{ std norm cdf}, \\
\left(\int_{-\infty}^{-\sqrt{n}} g(x)dx + \int_{\sqrt{n}}^{\infty} g(x)dx\right) - \left(\int_{-\infty}^{-\sqrt{n}} f_n(x)\,dx + \int_{\sqrt{n}}^{\infty} f_n(x)\,dx\right) &< 0 \\
\int_{-\infty}^{-\sqrt{n}} (g(x) - f_n(x))\,dx + \int_{\sqrt{n}}^{\infty} (g(x) - f_n(x))\,dx &< 0
\end{aligned}$$

which means that $g(x) > f_n(x)$ fails for a set of x of measure greater than 0 and so Dominated Convergence cannot apply.