Math 5660 Advanced Financial Math
Spring 2011
Final Exam
April 29 to May 2, 2011

This is an open book take-home exam. You may use any books, notes, websites or other printed material that you wish but do not consult with any other person. Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The five questions will be equally weighted in the grading. Please return the completed exams by 5 PM Monday, May 2 to my mailbox in the department office, under my office door MSB408, or by email.

1. Let $W(u)$ be Brownian motion relative to a filtration $\mathcal{F}(u)$ and let $a(u)$, $b(u)$, $\gamma(u)$, and $\sigma(u)$ be processes adapted to the filtration. Fix a time t and define

$$
\begin{aligned}
Z(u) &= e^{\left(\int_t^u \sigma(v)dW(v) + \int_t^u \left(b(v) - \frac{1}{2}\sigma^2(v)\right)dv\right)} \\
Y(u) &= x + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)} dv \ + \int_t^u \frac{\gamma(v)}{Z(v)} dW(v)
\end{aligned}
$$

Show that $X(u) = Y(u)Z(u)$ solves the stochastic differential equation

$$dX(u) = (a(u) + b(u)X(u))du + (\gamma(u) + \sigma(u)X(u))dW(u)$$

with initial condition $X(t) = x$

SOLUTION

$$
\begin{aligned}
dX(u) &= d(Y(u)Z(u)) \\
&= dY(u)Z(u) + Y(u)dZ(u) + dY(u)dZ(u) \\
&= \left(\frac{a(u) - \sigma(u)\gamma(u)}{Z(u)} du + \frac{\gamma(u)}{Z(u)} dW(u)\right) Z(u) \\
&\quad + Y(u)Z(u) \left[\left(\sigma(u)dW(u) + \left(b(u) - \frac{1}{2}\sigma^2(u)\right) du\right)\right. \\
&\quad \left. + \frac{1}{2}\left(\sigma(u)dW(u) + \left(b(u) - \frac{1}{2}\sigma^2(u)\right) du\right)^2\right] \\
&\quad + \left(\frac{a(u) - \sigma(u)\gamma(u)}{Z(u)} du + \frac{\gamma(u)}{Z(u)} dW(u)\right) \cdot \\
&\quad Z(u) \left[\sigma(u)dW(u) + \left(b(u) - \frac{1}{2}\sigma^2(u)\right) du\right. \\
&\quad \left. + \frac{1}{2}\left(\sigma(u)dW(u) + \left(b(u) - \frac{1}{2}\sigma^2(u)\right) du\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
dX(u) &= (a(u) - \sigma(u)\gamma(u))\,du + \gamma(u)dW(u) \\
&\quad + Y(u)Z(u)\left[\left(\sigma(u)dW(u) + \left(b(u) - \frac{1}{2}\sigma^2(u)\right)du\right)\right. \\
&\quad \left. + \frac{1}{2}\sigma^2(u)du\right] \\
&\quad + [(a(u) - \sigma(u)\gamma(u))\,du + \gamma(u)dW(u)] \cdot \\
&\quad \left[\sigma(u)dW(u) + \left(b(u) - \frac{1}{2}\sigma^2(u)\right)du + \frac{1}{2}\sigma^2(u)du\right] \\
&= (a(u) + b(u)Y(u)Z(u))\,du + (\gamma(u) + \sigma(u)Y(u)Z(u))\,dW(u) \\
&= (a(u) + b(u)X(u))du + (\gamma(u) + \sigma(u)X(u))dW(u)
\end{aligned}
$$

2. Let $S(t)$ be the price of a security that follows the dynamics

$$dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW(t)$$

where $r(t)$ and $\sigma(t)$ are ordinary, non-random functions of t. Show that

$$S(t) = e^X$$

where X is a normal random variable and give the mean and variance of X.

SOLUTION

$$
\begin{aligned}
d\ln(S(t)) &= \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S^2(t)}d(S(t))d(S(t)) \\
&= (r(t)dt + \sigma(t)dW(t)) - \frac{1}{2}\sigma^2(t)dt \\
&= \left(r(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dW(t) \\
\ln(S(t) &= \ln(S(0)) + \int_0^t \left(r(u) - \frac{1}{2}\sigma^2(u)\right)du + \int_0^t \sigma(u)dW(u) \\
S(t) &= e^{\ln(S(0) + \int_0^t \left(r(u) - \frac{1}{2}\sigma^2(u)\right)du + \int_0^t \sigma(u)dW(u)} \\
X &= \ln(S(0)) + \int_0^t \left(r(u) - \frac{1}{2}\sigma^2(u)\right)du + \int_0^t \sigma(u)dW(u)
\end{aligned}
$$

is normal with mean $\ln(S(0)) + \int_0^t \left(r(u) - \frac{1}{2}\sigma^2(u)\right)du$ and variance $\int_0^t \sigma^2(u)du$ because $\ln(S(0)) + \int_0^t \left(r(u) - \frac{1}{2}\sigma^2(u)\right)du$ is non-random and $\int_0^t \sigma(u)dW(u)$ is normal (because $\sigma(u)$ is non-random) with mean 0 and variance equal to the quadratic variation $\sigma(u)dW(u)\sigma(u)dW(u) = \sigma^2(u)du$ accumulated from 0 to t.

3. Let $S(t)$ be the price of a security that follows the dynamics

$$dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW(t)$$

where $r(t)$ and $\sigma(t)$ are random processes adapted to the filtration $\mathcal{F}(t)$ corresponding to the Brownian motion $W(t)$. Given a non-random value $S(0)$ what is the formula for the solution $S(t)$ to the dynamics? Is $S(t)$ a lognormal random variable? Why or why not?

SOLUTION

By exactly the same derivation as for #2,

$$S(t) = S(0)e^{\int_0^t \left(r(u) - \frac{1}{2}\sigma^2(u)\right)du + \int_0^t \sigma(u)dW(u)}$$

This is not necessarily lognormal because $r(t)$ and $\sigma(t)$ are random processes that are not necessarily normal and, even they were normal, there is no guarantee that $\int_0^t \sigma(u)dW(u)$is normal.

4. If $S(t) = S(0)e^{\left(\int_0^t \nu(u)du + \int_0^t \sigma(u)dW(u)\right)}$ is a martingale where $\nu(t)$ and $\sigma(t)$ are random processes, what is the relationship between $\nu(t)$ and $\sigma(t)$? Prove it.

SOLUTION

$$\begin{aligned} dS(t) &= S(0)e^{\left(\int_0^t \nu(u)du + \int_0^t \sigma(u)dW(u)\right)} \{\nu(t)dt + \sigma(t)dW(t) \\ &\quad + \frac{1}{2}[\nu(t)dt + \sigma(t)dW(t)][\nu(t)dt + \sigma(t)dW(t)]\Big\} \\ &= S(t)\left[\left(\nu(t) + \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dW(t)\right] \end{aligned}$$

which is a martingale if and only if the dt term is 0, namely,

$$\nu(t) = -\frac{1}{2}\sigma^2(t)$$

5. A random interest rate process $R(t)$ follows

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$$

for constants α, β, and σ. Solve for $R(t)$.

SOLUTION

You could do this one of two ways. First, you might notice that this is a special case of #1 with $a(u) = \alpha$, $b(u) = -\beta$, $\gamma(u) = \sigma$, and $\sigma(u) = 0$. So for $u > t$ define

$$
\begin{aligned}
Z(u) &= e^{\left(\int_t^u -\beta dv\right)} \\
&= e^{-\beta(u-t)} \\
Y(u) &= R(t) + \int_t^u \frac{\alpha}{Z(v)} dv \ + \int_t^u \frac{\sigma}{Z(v)} dW(v) \\
&= R(t) + \int_t^u \alpha e^{\beta(v-t)} dv + \int_t^u \sigma e^{\beta(v-t)} dW(v) \\
&= R(t) + \frac{\alpha}{\beta}\left(e^{\beta(u-t)} - 1\right) + \sigma \int_t^u e^{\beta(v-t)} dW(v) \\
R(u) &= Y(u)Z(u) \\
&= R(t)e^{-\beta(u-t)} + \frac{\alpha}{\beta}\left(1 - e^{-\beta(u-t)}\right) + \sigma \int_t^u e^{-\beta(u-v)} dW(v)
\end{aligned}
$$

where $\int_t^u e^{-\beta(u-v)} dW(v)$ is normal mean 0 and variance $\sigma^2 \int_t^u e^{-2\beta(u-v)} dW(v) dW(v) = \sigma^2 \int_t^u e^{-2\beta(u-v)} dv = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta(u-t)}\right)$

Alternatively, you can guess from the fact that $-\beta R(t)dt$ appears that there might be an appearance of $e^{\beta t} R(t)$ involved. Try taking its stochastic differential:

$$
\begin{aligned}
d(e^{\beta t} R(t)) &= \beta e^{\beta t} R(t) dt + e^{\beta t} dR(t) \\
&= \beta e^{\beta t} R(t) dt + e^{\beta t}\left[(\alpha - \beta R(t))dt + \sigma dW(t)\right] \\
&= e^{\beta t}\left[\alpha dt + \sigma dW(t)\right] \text{ and integrate for} \\
e^{\beta u} R(u) &= e^{\beta t} R(t) + \alpha \int_t^u e^{\beta v} dv + \sigma \int_t^u e^{\beta v} dW(v) \\
R(u) &= e^{-\beta(u-t)} R(t) + \frac{\alpha}{\beta}\left(1 - e^{-\beta(u-t)}\right) + \sigma \int_t^u e^{-\beta(u-v)} dW(v)
\end{aligned}
$$

same as the first way gave.