

Math 5660 Advanced Financial Math
Spring 2011
Final Exam
April 29 to May 2, 2011

This is an open book take-home exam. You may use any books, notes, websites or other printed material that you wish but do not consult with any other person. Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The five questions will be equally weighted in the grading. Please return the completed exams by 5 PM Monday, May 2 to my mailbox in the department office, under my office door MSB408, or by email.

1. Let $W(u)$ be Brownian motion relative to a filtration $\mathcal{F}(u)$ and let $a(u)$, $b(u)$, $\gamma(u)$, and $\sigma(u)$ be processes adapted to the filtration. Fix a time t and define

$$\begin{aligned} Z(u) &= e^{\left(\int_t^u \sigma(v) dW(v) + \int_t^u (b(v) - \frac{1}{2}\sigma^2(v)) dv \right)} \\ Y(u) &= x + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)} dv + \int_t^u \frac{\gamma(v)}{Z(v)} dW(v) \end{aligned}$$

Show that $X(u) = Y(u)Z(u)$ solves the stochastic differential equation

$$dX(u) = (a(u) + b(u)X(u))du + (\gamma(u) + \sigma(u)X(u))dW(u)$$

with initial condition $X(t) = x$

2. Let $S(t)$ be the price of a security that follows the dynamics

$$dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW(t)$$

where $r(t)$ and $\sigma(t)$ are ordinary, non-random functions of t . Show that

$$S(t) = e^X$$

where X is a normal random variable and give the mean and variance of X .

3. Let $S(t)$ be the price of a security that follows the dynamics

$$dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW(t)$$

where $r(t)$ and $\sigma(t)$ are random processes adapted to the filtration $\mathcal{F}(t)$ corresponding to the Brownian motion $W(t)$. Given a non-random value $S(0)$ what is the formula for the solution $S(t)$ to the dynamics? Is $S(t)$ a lognormal random variable? Why or why not?

4. If $S(t) = S(0)e^{\left(\int_0^t \nu(u)du + \int_0^t \sigma(u)dW(u)\right)}$ is a martingale where $\nu(t)$ and $\sigma(t)$ are random processes, what is the relationship between $\nu(t)$ and $\sigma(t)$? Prove it.

5. A random interest rate process $R(t)$ follows

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$$

for constants α , β , and σ . Solve for $R(t)$.