Math 5660 Advanced Financial Math
Spring 2010
Final Exam
April 30 to May 5, 2010

This is an open book take-home exam. You may use any books, notes, websites or other printed material that you wish but do not consult with any other person. Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The six questions will be equally weighted in the grading. Return the completed exams by 5 PM Wednesday May 5, to my mailbox in the department office, under my office door MSB408, or by email.

1. Let $\{\mathcal{F}(t)\}$ be the filtration generated by a Brownian motion $W(t)$ and assume the existence of a risk-neutral measure $\tilde{\mathbb{P}}$ and a risk-free discount process $D(t)$ and interest rate process $R(t)$ with $dD(t) = -R(t)D(t)dt$. Let V be an $\mathcal{F}(T)$-measurable random variable with $V > 0$ almost surely. Show that there is a generalized geometric Brownian motion $V(t)$ adapted to $\{\mathcal{F}(t)\}$ with $V(T) = V$. In other words, all strictly positive payoffs come from generalized geometric Brownian motions.

   Solution 1 *By the definition of a risk-neutral measure,* $\tilde{\mathbb{E}}[D(T)V \mid \mathcal{F}(t)]$ *is a* $\tilde{\mathbb{P}}$ *martingale. Define* $V(t)$ *by* $V(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R(u)du} V \mid \mathcal{F}(t)\right]$. *Notice that according to this definition* $V(t) > 0$ *almost surely and* $V(T)$ *is defined by* $V(T) = \tilde{\mathbb{E}}[V \mid \mathcal{F}(T)] = V\tilde{\mathbb{E}}[1 \mid \mathcal{F}(T)] = V$ *(taking out what's known.)* $D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T) \mid \mathcal{F}(t)]$ *is now another expression for the* $\tilde{\mathbb{P}}$ *martingale. By the martingale representation theorem there is an adapted process* $\tilde{\Gamma}(t)$ *where* $d(D(t)V(t)) = \tilde{\Gamma}(t)\, d\tilde{W}(t)$ *for* $\tilde{W}(t)$ *Brownian under* $\tilde{\mathbb{P}}$. *But Itô's product formula says that* $d(D(t)V(t)) = dD(t)V(t) + D(t)dV(t) + dD(t)dV(t)$. *Equating the two expressions for* $d(D(t)V(t))$

gives

$$\begin{aligned}
dD(t)V(t)+D(t)dV(t)+dD(t)dV(t) &= \tilde{\Gamma}(t)d\tilde{W}(t)\\
dV(t) &= -\frac{dD(t)}{D(t)}\left(V(t)+dV(t)\right)+\frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t)\\
&= R(t)\left(V(t)+dV(t)\right)dt+\frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t)\\
& \quad \textit{a solution for which is}\\
dV(t) &= R(t)V(t)dt+\frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t)\\
\textit{because } dtdt &= 0 \textit{ and } dtd\tilde{W}(t)=0\\
\textit{This can be rewritten as } dV(t) &= R(t)V(t)dt+\frac{\tilde{\Gamma}(t)}{D(t)V(t)}V(t)d\tilde{W}(t)\\
\textit{because } V(t) &> 0 \textit{ almost surely.}
\end{aligned}$$

Therefore, $V(t)$ is a generalized $\tilde{\mathbb{P}}$ Brownian motion with drift process $R(t)$ and volatility process $\frac{\tilde{\Gamma}(t)}{D(t)V(t)}$. From the Girsanov Theorem, $d\tilde{W}(t) = \Theta(t)dt + dW(t)$ for a market price of risk process $\Theta(t)$ so $dV(t) = (R(t) + \Theta(t))V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)V(t)}V(t)dW(t)$, which shows that $V(t)$ is a generalized $\mathbb{P}$ Brownian motion with drift process $R(t)+\Theta(t)$ and volatility process $\frac{\tilde{\Gamma}(t)}{D(t)V(t)}$.

2. In the situation of question 1, assume that the risk-neutral measure $\tilde{\mathbb{P}}$ is equal to $\tilde{\mathbb{P}}_{S_1}$, the risk-neutral measure derived from some generalized geometric Brownian motion $S_1(t)$ such that $D(t)S_1(t)$ is a $\tilde{\mathbb{P}}_{S_1}$-martingale. Let $\tilde{\mathbb{P}}_{S_2}$ be the risk-neutral measure derived from some other generalized geometric Brownian motion $S_2(t)$ so that $D(t)S_2(t)$ is a $\tilde{\mathbb{P}}_{S_2}$-martingale. Show that $\tilde{\mathbb{P}}_{S_1}(A) = \tilde{\mathbb{P}}_{S_2}(A)$ for all sets $A \,\epsilon\, \mathcal{F}$. In other words, the risk-neutral measure is unique.

   Solution 2 *Let $dS_1(t) = \alpha_1(t)S_1(t)dt + \sigma_1(t)S_1(t)dW(t)$ and $dS_2(t) = \alpha_2(t)S_2(t)dt + \sigma_2(t)S_2(t)dW(t)$ be the two generalized geometric Brownian motions. Since $\mathcal{F} = \bigcup\limits_t \mathcal{F}(t)$, for any $A\epsilon\mathcal{F}$, there is a t such that*

$A \epsilon \mathcal{F}(t) \subseteq \mathcal{F}$. The Girsanov Theorem says that

$$\begin{aligned}
\tilde{\mathbb{P}}_{S_1}[A] &= \int_A e^{-\int_0^t \Theta_1(u)dW(u) - \frac{1}{2}\int_0^t \Theta_1^2(u)du} d\mathbb{P}[\omega] \\
\textit{where } \Theta_1(t) &= \frac{\alpha_1(t) - R(t)}{\sigma_1(t)} \textit{ and that} \\
\tilde{\mathbb{P}}_{S_2}[A] &= \int_A e^{-\int_0^t \Theta_2(u)dW(u) - \frac{1}{2}\int_0^t \Theta_2^2(u)du} d\mathbb{P}[\omega] \\
\textit{where } \Theta_2(t) &= \frac{\alpha_2(t) - R(t)}{\sigma_2(t)}.
\end{aligned}$$

In question 1 let $V = S_2(T)$ for some $T > t$. Then, $X(t) = \Delta(t)S_1(t) + \frac{S_2(t) - \Delta(t)S_1(t)}{B(t)}B(t)$ with $dX(t) = \Delta(t)dS_1(t) + R(t)\left(S_2(t) - \Delta(t)S_1(t)\right)dt$ defines a portfolio process with $X(t) = S_2(t) = V(t)$. This gives

$$\begin{aligned}
dS_2(t) &= \Delta(t)dS_1(t) + R(t)\left(S_2(t) - \Delta(t)S_1(t)\right)dt \\
\alpha_2(t)S_2(t)dt + \sigma_2(t)S_2(t)dW(t) &= \left(\Delta(t)\alpha_1(t)S_1(t) + R(t)\left(S_2(t) - \Delta(t)S_1(t)\right)\right)dt \\
&\quad + \Delta(t)\sigma_1(t)S_1(t)dW(t) \textit{ so} \\
\alpha_2(t)S_2(t) &= \Delta(t)\alpha_1(t)S_1(t) + R(t)\left(S_2(t) - \Delta(t)S_1(t)\right) \\
\alpha_2(t) &= \Delta(t)\left(\alpha_1(t) - R(t)\right)\frac{S_1(t)}{S_2(t)} + R(t) \textit{ and} \\
\sigma_2(t)S_2(t) &= \Delta(t)\sigma_1(t)S_1(t) \\
\sigma_2(t) &= \Delta(t)\sigma_1(t)\frac{S_1(t)}{S_2(t)}
\end{aligned}$$

Plug these expressions for $\alpha_2(t)$ and $\sigma_2(t)$ into the definition of $\Theta_2(t)$:

$$\begin{aligned}
\Theta_2(t) &= \frac{\alpha_2(t) - R(t)}{\sigma_2(t)} \\
&= \frac{\Delta(t)\left(\alpha_1(t) - R(t)\right)\frac{S_1(t)}{S_2(t)} + R(t) - R(t)}{\Delta(t)\sigma_1(t)\frac{S_1(t)}{S_2(t)}} \\
&= \frac{\alpha_1(t) - R(t)}{\sigma_1(t)} \\
&= \Theta_1(t)
\end{aligned}$$

This means that $\tilde{\mathbb{P}}_{S_2}[A] = \tilde{\mathbb{P}}_{S_1}[A]$.

3. If $S(t) = S(0)e^{\int_0^t \nu(u)du + \int_0^t \sigma(u)dW(u)}$ is a martingale, what is the relationship between $\nu(t)$ and $\sigma(t)$? Prove it.

Solution 3 *Let* $X(t) = \int_0^t \nu(u)du + \int_0^t \sigma(u)dW(u)$*, so* $S(t) = S(0)e^{X(t)}$*.*
By the Itô Lemma

$$\begin{aligned} dS(t) &= S(0)e^{X(t)}dX(t) + \frac{1}{2}S(0)e^{X(t)}dX(t)dX(t) \\ &= S(t)\left\{\nu(t)dt + \sigma(t)dW(t) \right. \\ &\quad \left. + \frac{1}{2}\left(\nu(t)dt + \sigma(t)dW(t)\right)\left(\nu(t)dt + \sigma(t)dW(t)\right)\right\} \\ &= S(t)\left\{\left(\nu(t) + \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dW(t)\right\} \end{aligned}$$

So $S(t)$ *is a martingale only if* $\nu(t) = -\frac{1}{2}\sigma^2(t)$*, making the* dt *term equal to* 0*.*

4. If $M(t) = \int_0^t h(u)dN(u)$ where $N(t)$ is the semi-martingale with $dN(t) = \alpha(t)dt + \beta(t)dW(t)$, $g(t)$ is an adapted process, and $[M, M](t)$ is the quadratic variatiion of $M(t)$, what is the simplest expression for $\int_0^t g(u)d[M, M](u)$ in terms of $g(t)$, $h(t)$, $\alpha(t)$, $\beta(t)$, and $W(t)$?

Solution 4

$$\begin{aligned} \int_0^t g(u)d[M, M](u) &= \int_0^t g(u)dM(u)dM(u) \\ &= \int_0^t g(u)h(u)dN(u)h(u)dN(u) \\ &= \int_0^t g(u)h^2(u)\left(\alpha(u)du + \beta(u)dW(u)\right)\left(\alpha(u)du + \beta(u)dW(u)\right) \\ &= \int_0^t g(u)h^2(u)\beta^2(u)dW(u)W(u) \\ &= \int_0^t g(u)h^2(u)\beta^2(u)du \end{aligned}$$

5. Use stochastic calculus to derive a formula for $\mathbb{E}\left[W^6(t)\right]$ assuming only that you know that $dW(t)dW(t) = dt$ and the usual rules of stochastic calculus.

Solution 5

$$
\begin{aligned}
dW^2t) &= 2W(t)dW(t) + dW(t)dW(t) \\
W^2(t) &= 2\int_0^t W(u)dW(u) + \int_0^t du \\
\mathbb{E}\left[W^2(t)\right] &= t \textit{ because the first integral is a martingale.} \\
dW^4(t) &= 4W^3(t)dW(t) + 6W^2(t)dW(t)dW(t) \\
W^4(t) &= 4\int_0^t W^3(u)dW(u) + 6\int_0^t W^2(u)du \\
\mathbb{E}\left[W^4(t)\right] &= 6\int_0^t \mathbb{E}\left[W^2(u)\right] du \textit{ because the first integral is a martingale.} \\
&= 6\int_0^t udu = 3t^2 \\
dW^6(t) &= 6W^5(t)dW(t) + 15W^4(t)dW(t)dW(t) \\
W^6(t) &= 6\int_0^t W^5(u)dW(u) + 15\int_0^t W^4(u)du \\
\mathbb{E}\left[W^6(t)\right] &= 15\int_0^t \mathbb{E}\left[W^4(u)\right] du \textit{ because the first integral is a martingale.} \\
&= 15\int_0^t 3u^2du = 15t^3
\end{aligned}
$$

6. If $B_1(t)$ and $B_2(t)$ are Brownian motions with $dB_1(t)dB_2(t) = \rho(t)dt$ for an adapted process -1< $\rho(t) < 1$ find an adapted process $W(t)$ that is a Brownian motion independent of $B_1(t)$.

Solution 6 *Define* $W(t)$ *by*

$$
\begin{aligned}
W(0) &= 0 \\
dW(t) &= \frac{dB_2(t) - \rho(t)dB_1(t)}{\sqrt{1-\rho^2(t)}}. \quad \textit{Then} \\
dW(t)dW(t) &= \frac{(dB_2(t) - \rho(t)dB_1(t))\,(dB_2(t) - \rho(t)dB_1(t))}{1-\rho^2(t)} \\
&= \frac{dB_2(t)dB_2(t) - 2\rho(t)dB_1(t)dB_2(t) + \rho^2(t)dB_1(t)dB_1(t)}{1-\rho^2(t)} \\
&= \frac{dt - 2\rho(t)\rho(t)dt + \rho^2(t)dt}{1-\rho^2(t)} \\
&= \frac{1-\rho^2(t)}{1-\rho^2(t)}dt = dt
\end{aligned}
$$

and by Lévy's Theorem $W(t)$ *is Brownian.*

$$\begin{aligned} dW(t)dB_1(t) &= \frac{dB_2(t) - \rho(t)dB_1(t)}{\sqrt{1-\rho^2(t)}}dB_1(t) \\ &= \frac{dB_2(t)dB_1(t) - \rho(t)dB_1(t)dB_1(t)}{\sqrt{1-\rho^2(t)}} \\ &= \frac{\rho(t)dt - \rho(t)dt}{\sqrt{1-\rho^2(t)}} = 0 \end{aligned}$$

so $W(t)$ *is independent of* $B_1(t)$