Math 5637
Risk Theory
Fall 2015
Final Examination Solutions
December 11 - 16, 2015

Due back to me by 5 PM on Wednesday, December 16, in my mailbox, under my door, or by email. You may consult with any written source, including textbooks, solution manuals, notes, websites, or anything else in writing. Remember to use Appendix A if you can! Do NOT consult with any other person. Doing so will be grounds for failing the course. The four questions will be equally weighted in the grading.

1. Individual loss amounts (ground up) this year follow a two parameter Pareto distribution with $\alpha = 3$ and expected value 1000. Next year you confidently expect loss amounts to inflate by 5% uniformly across all losses. What will be the standard deviation next year for loss payments that are limited to $2,000$ per payment with the original loss amount first subjected to a 200 deductible per loss before any payment? (HINT: use the fact that $\beta(3,1;u) = u^3$ to help your calculations.)

Solution

If X represents ground up losses this year then losses next year subject to deductible and limits will be

$$\begin{aligned} Y^L &= (1.05X - 200)_+ \wedge 2000 \\ &= \left[1.05(X - \frac{200}{1.05})_+\right] \wedge 2000 \\ &= 1.05\left[(X - \frac{200}{1.05})_+ \wedge \frac{2000}{1.05}\right] \\ &= 1.05\left[X \wedge \left(\frac{2000}{1.05} + \frac{200}{1.05}\right) - \frac{200}{1.05}\right]_+ \end{aligned}$$

Using the surface interpretation and translating to payments, i.e. $Y^P = Y^L | \left(Y^L > 0\right)$

$$\begin{aligned} \mathbb{E}\left[Y^L\right] &= 1.05\left\{\mathbb{E}\left[X \wedge \left(\frac{2000}{1.05} + \frac{200}{1.05}\right)\right] - \mathbb{E}\left[X \wedge \frac{200}{1.05}\right]\right\} \\ \mathbb{E}\left[Y^P\right] &= \frac{1}{S_X\left(\frac{200}{1.05}\right)}\mathbb{E}\left[Y^L\right] \end{aligned}$$

From Appendix A, with $\alpha = 3$, $\mathbb{E}[X] = 1000$ means $\theta = 2000$ in the

2-parameter Pareto. Using the formulae in Appendix A

$$\begin{aligned}
\mathbb{E}[Y] &= 1.05\left\{\frac{2000}{2}\left[1-\left(\frac{2000}{\frac{2000}{1.05}+\frac{200}{1.05}+2000}\right)^2\right]-\frac{2000}{2}\left[1-\left(\frac{2000}{\frac{200}{1.05}+2000}\right)^2\right]\right\}\\
&= 624.90\\
S_X\left(\frac{200}{1.05}\right) &= \left(\frac{2000}{\frac{200}{1.05}+2000}\right)^3\\
&= .761157\\
\mathbb{E}\left[Y^P\right] &= \frac{1}{.761157}624.90 = 820.99
\end{aligned}$$

Using the surface interpretation, and translatiing to payments

$$\begin{aligned}
\mathbb{E}\left[\left(Y^L\right)^2\right] &= (1.05)^2\left\{\mathbb{E}\left[\left(X\wedge\left(\frac{2000}{1.05}+\frac{200}{1.05}\right)\right)^2\right]-\mathbb{E}\left[\left(X\wedge\frac{200}{1.05}\right)^2\right]\right.\\
&\quad\left.-2(\frac{200}{1.05})\left(\mathbb{E}\left[X\wedge\left(\frac{2000}{1.05}+\frac{200}{1.05}\right)\right]-\mathbb{E}\left[X\wedge\frac{200}{1.05}\right]\right)\right\}\\
\mathbb{E}\left[\left(Y^P\right)^2\right] &= \frac{1}{S_X\left(\frac{200}{1.05}\right)}\mathbb{E}\left[\left(Y^L\right)^2\right]
\end{aligned}$$

Using the formulae in Appendix A, and the hint that $\beta(3,1;u)=u^3$,

$$\begin{aligned}
\mathbb{E}\left[\left(Y^L\right)^2\right] &= (1.05)^2\left\{\frac{2000^2(2)}{2}\left(\frac{\frac{2000}{1.05}+\frac{200}{1.05}}{\frac{2000}{1.05}+\frac{200}{1.05}+2000}\right)^3+\left(\frac{2000}{1.05}+\frac{200}{1.05}\right)^2\left(\frac{2000}{\frac{2000}{1.05}+\frac{200}{1.05}+2000}\right)^3\right.\\
&\quad-\left[\frac{2000^2(2)}{2}\left(\frac{\frac{200}{1.05}}{\frac{200}{1.05}+2000}\right)^3+\left(\frac{200}{1.05}\right)^2\left(\frac{2000}{\frac{200}{1.05}+2000}\right)^3\right]\\
&\quad\left.-2(\frac{200}{1.05})\left(\frac{2000}{2}\left[1-\left(\frac{2000}{\frac{2000}{1.05}+\frac{200}{1.05}+2000}\right)^2\right]-\frac{2000}{2}\left[1-\left(\frac{2000}{\frac{200}{1.05}+2000}\right)^2\right]\right)\right\}\\
&= 871,070\\
\mathbb{E}\left[\left(Y^P\right)^2\right] &= \frac{1}{.761157}\,871,070 = 1,144,403
\end{aligned}$$

So

$$\sigma = \left(1,144,403\ - 820.99^2\right)^{\frac{1}{2}} = 470,378^{\frac{1}{2}} = 685.84$$

2. Write down formulas for the first five moments of the Inverse Gaussian random variable in terms of its parameters μ and θ.

Solution

According to Appendix A the moment generating function of the inverse Gaussian random variable is

$$M(z) = e^{\frac{\theta}{\mu}\left(1-\left(1-\frac{2\mu^2}{\theta}z\right)^{\frac{1}{2}}\right)}$$

So the cumulant generating function is

$$C(z) = \frac{\theta}{\mu}\left(1 - \left(1 - \frac{2\mu^2}{\theta} z\right)^{\frac{1}{2}}\right)$$

and first five cumulants are

$$\begin{aligned}
\kappa_1 &= C^{(1)}(0) = \mu\left(1 - \frac{2\mu^2}{\theta} z\right)^{-\frac{1}{2}} |_{z=0} = \mu \\
\kappa_2 &= C^{(2)}(0) = \frac{\mu^3}{\theta}\left(1 - \frac{2\mu^2}{\theta} z\right)^{-\frac{3}{2}} |_{z=0} = \frac{\mu^3}{\theta} \\
\kappa_3 &= \frac{3\mu^5}{\theta^2}\left(1 - \frac{2\mu^2}{\theta} z\right)^{-\frac{5}{2}} |_{z=0} = \frac{3\mu^5}{\theta^2} \\
\kappa_4 &= \frac{15\mu^7}{\theta^3}\left(1 - \frac{2\mu^2}{\theta} z\right)^{-\frac{7}{2}} |_{z=0} = \frac{15\mu^7}{\theta^3} \\
\kappa_5 &= \frac{105\mu^9}{\theta^4}\left(1 - \frac{2\mu^2}{\theta} z\right)^{-\frac{9}{2}} |_{z=0} = \frac{105\mu^9}{\theta^4}
\end{aligned}$$

Now write the moment generating function in terms of the cumulant generating function

$$M(z) = e^{C(z)}$$

and use Faá to get moments

$$\begin{aligned}
\mu &= M^{(1)}(z)|_{z=0} = e^{C(0)} C^{(1)}(0) = \kappa_1 = \mu \\
\mu_2' &= M^{(2)}(z)|_{z=0} = e^{C(0)}\left[C^{(2)}(0) + C^{(1)}(0)^2\right] = \kappa_2 + \kappa_1^2 = \frac{\mu^3}{\theta} + \mu^2 \\
\mu_3' &= M^{(3)}(z)|_{z=0} = e^{C(0)}\left[\kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3\right] = \frac{3\mu^5}{\theta^2} + \frac{3\mu^4}{\theta} + \mu^3 \\
\mu_4' &= M^{(4)}(z)|_{z=0} = e^{C(0)}\left[\kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4\right] = \frac{15\mu^7}{\theta^3} + \frac{15\mu^6}{\theta^2} + \frac{6\mu^5}{\theta} + \mu^4 \\
\mu_5' &= M^{(5)}(z)|_{z=0} = e^{C(0)}\left[\kappa_5 + 5\kappa_4\kappa_1 + 10\kappa_3\kappa_2 + 10\kappa_3\kappa_1^2 + 15\kappa_2^2\kappa_1 + 10\kappa_2\kappa_1^3 + \kappa_1^5\right] \\
&= \frac{105\mu^9}{\theta^4} + \frac{105\mu^8}{\theta^3} + \frac{45\mu^7}{\theta^2} + \frac{10\mu^6}{\theta} + \mu^5
\end{aligned}$$

3. Among all continuous non-negative random variables with mean μ

   (a) Show which one has maximum entropy and tell me its name

Solution

Entropy is

$$-\int_0^\infty f(x) \ln f(x)\, dx$$

Constraints are

$$\begin{aligned} \int_0^\infty f(x)\,dx &= 1 \\ \int_0^\infty \frac{x}{\mu} f(x)\,dx &= 1 \end{aligned}$$

Euler-Lagrange equations to maximize entropy subject to these constraints are

$$\frac{d}{dx}\left[\frac{\partial}{\partial f'}\left(-f(x)\ln f(x) + \lambda_1 f(x) + \lambda_2 \frac{x}{\mu} f(x)\right)\right] - \frac{\partial}{\partial f}\left(-f(x)\ln f(x) + \lambda_1 f(x) + \lambda_2 \frac{x}{\mu} f(x)\right) = 0$$

$$\begin{aligned} 0 + \ln f(x) + \frac{f(x)}{f(x)} - \lambda_1 - \lambda_2 \frac{x}{\mu} &= 0 \\ \ln f(x) &= -1 + \lambda_1 + \lambda_2 \frac{x}{\mu} \\ f(x) &= e^{-1+\lambda_1+\lambda_2 \frac{x}{\mu}} \end{aligned}$$

The constraints then imply (just integrate)

$$\begin{aligned} \lambda_2 &= -1 \\ e^{-1+\lambda_1} &= \frac{1}{\mu} \\ f(x) &= \frac{1}{\mu} e^{-\frac{x}{\mu}} \end{aligned}$$

This is the density for the exponential random variable with mean μ

(b) Write down the density for the CTM-τ transformation of the random variable in 3.a. and tell me its name

Solution

The CTM transformation with parameter τ for a random variable X with density $f_X(x)$ was defined in your class notes as the random variable Y with density

$$f_Y(y) = \frac{y^\tau}{\mathbb{E}[X^\tau]} f_X(y)$$

Clearly this is the density of some random variable (it integrates to 1) and we called that the CTM transformation of X with parameter τ. In this case

$$\begin{aligned} f_X(y) &= \frac{1}{\mu} e^{-\frac{y}{\mu}} \\ \mathbb{E}[X^\tau] &= \mu^\tau \Gamma(\tau+1) \text{ from App.A} \\ f_Y(y) &= \frac{y^\tau}{\mu^{\tau+1}\Gamma(\tau+1)} e^{-\frac{y}{\mu}} \end{aligned}$$

and in App. A this is the density for a Gamma random variable with parameters $\tau+1$ and μ

4. If S is a compound Negative Binomial - Negative Binomial random variable with parameters $\beta_N = 2$ and $r_N = .5$ for the primary variable and $\beta_M = .5$ and $r_M = 2$ for the secondary variable then calculate numerical values for the first 5 probabilities $\mathbb{P}[S=0]$, $\mathbb{P}[S=1]$, ... ,$\mathbb{P}[S=4]$.

Solution

From the definition of a compound random variable and from the App B facts about Negative Binomial:

$$
\begin{aligned}
S &= M_1 + ... + M_N \\
g_n &= \mathbb{P}[S=n] \\
f_n &= \mathbb{P}[M=n] \\
p_n &= \mathbb{P}[N=n] \\
P_N(z) &= [1-2(z-1)]^{-.5} \\
a_N &= \frac{2}{1+2} = \frac{2}{3} \\
b_N &= \frac{(.5-1)\,2}{1+2} = -\frac{1}{3} \\
p_0 &= (1+2)^{-.5} = \frac{1}{\sqrt{3}} \\
P_M(z) &= [1-.5(z-1)]^{-2} \\
a_M &= \frac{.5}{1+.5} = \frac{1}{3} \\
b_M &= \frac{(2-1)\,.5}{1+.5} = \frac{1}{3} \\
f_0 &= (1+.5)^{-2} = \frac{4}{9} = .444444
\end{aligned}
$$

Now set up the Panjer recursion with

$$g_0 = P_N(f_0) = \left[1-2\left(\frac{4}{9}-1\right)\right]^{-.5} = \frac{3}{\sqrt{19}} = .688247$$

$$f_n = f_{n-1}\left(\frac{1}{3}+\frac{1}{3n}\right)$$

and

$$
\begin{aligned}
g_n &= \frac{1}{1-\frac{2}{3}(.444444)} \sum_{k=1}^{n} \left(\frac{2}{3}-\frac{k}{3n}\right) f_k g_{n-k} \\
&= \frac{1}{1-\frac{2}{3}(.444444)} \left[\frac{2}{3}\sum_{k=1}^{n} f_k g_{n-k} - \frac{1}{3n}\sum_{k=1}^{n} k f_k g_{n-k}\right]
\end{aligned}
$$

$\mathbf{n}$	$\mathbf{f}_n$	$\sum_{k=1}^{\mathbf{n}} \mathbf{f}_k \mathbf{g}_{n-k}$	$\sum_{k=1}^{\mathbf{n}} k \mathbf{f}_k \mathbf{g}_{n-k}$	$\mathbf{g}_n$
0	.444444			.688247
1	.296296	.20392483	.20392483	.096596
2	.148148	.13058342	.23254584	.068634
3	.065844	.07996342	.18490779	.046559
4	.027435	.04920556	.12874125	.031370