We had reached the equation:

$\psi(u)= \psi(0)[\int_{[0,u]}f_{L1}(y)\ \psi(u-y)dy+\int_{[u,\infty]} f_{L1}(y)dy]$ and interpreted it in probability terms.

Substitute $\psi(u-y)=1- \varphi(u-y)$ and use the fact that $\int_{[0,\infty]} f_{L1}(y)dy = 1$ (it's a probability density):

$\psi(u)= \psi(0)[1- \int_{[0,u]}f_{L1}(y)\ \varphi\ (u-y)dy] = \psi(0)[1- \int_{[0,\infty]}f_{L1}(y)\ \varphi\ (u-y)dy]$ since $\varphi\ (u-y)=0$ for $y>u$, so

$\psi(u)= \psi(0)[1- (f_{L1}*\ \varphi)(u)]$ using the analytic definition of convolution (*).

Now use $1- \psi(u)= \varphi(u)$ and $1- \psi(0)= \varphi(0)$ to get

$\varphi(u)= \varphi(0)+ \psi(0)\ (f_{L1}*\ \varphi)(u)$ which you can read as prob surv from u =

=(prob always stay above u) + (prob drop below u) X $\int$[(prob drop by y, u>y>0)(prob surv from u-y)]dy

This is called the *renewal equation* for $\varphi(u)$.

A solution for this equation is (plug it in, it works … perhaps you can see why it was a good guess?):

$\varphi(u)= \varphi(0)\sum_{k=0,\infty} [\psi(0)]^k F_{L1}^{*k}(u)$ where $F_{L1}^{*k}(u)$ is the cum prob dist for $L_1 +\ldots+L_1$, k times, where the L_1's are independent. Remember from several weeks ago that in general $F_X^{*k}(u)= (f_X* F_X^{*(k-1)})(u)$ for any random variable X. Seeing the solution for the renewal equation so quickly is the payoff for using the convolution notation.

$\sum_{k=0,\infty} [\psi(0)]^k = 1/[1- \psi(0)]=1/\varphi(0)=(1+\theta)/\theta$ (remember $\psi(0)=1/(1+\theta)$ from the other day so $\varphi(0)= \theta/(1+\theta)$).

Thus, the solution for $\varphi(u)$ just given is the cumulative probability distribution for a random variable we'll call L that is a compound Geometric-L_1 distribution, i.e. $L = L_1+\ldots+L_1$, K times, where K has a geometric distribution, i.e. a negative binomial with $\beta=1/\theta$, r=1 or $a=1/(1+\theta)$, b=0.

L is the random variable representing the maximum aggregate loss, the maximum amount by which u(t) ever drops below its starting point u(0) on a random path. K is the number of time u(t) drops below its prior low point. (Each such drop has probability $\psi(0)=1/(1+\theta)$ so the total number of such drops is the geometric K just described.) Each random L_1 is how far below the prior low point the new drop reached. So the compound variable L is the total of how far below the original starting point all of the drops below prior lows bring you, the maximum aggregate loss. Obviously, you survive forever from a starting point u only if $L\leq u$, so it makes sense that $\varphi(u)$= the cumulative probability distribution for L, the compound Geometric-L_1 as just described.

This formula for $\varphi(u)$ gives us a lot of alternatives to learn about probabilities of survival $\varphi(u)$ or of ruin

$\psi(u)=1-\varphi(u)=S_L(u)$.

I. We can use Panjer recursion to calculate numerical values for $\varphi(u)$ (and $\psi(u)$),

II. We can use Faa's formula and our knowledge about moments of L_1 from previous class to determine the moments of the maximum aggregate loss variable L.

III. If the single loss variable X has nice properties we can write down an exact analytic formula for $\psi(u)$, for example if X is an exponential or a gamma random variable.

I. Panjer Recursion

To use Panjer recursion on the formula for $\varphi(u)$ we need to discretize the distribution for L_1. Remember that $f_{L1}(y)=S_X(y)/\mu_X$ so discretizing L_1 just means to pick up values of $S_X(y)$ at points half-way between your discrete values of y, i.e. $S_X(y+d/2)d/\mu_X$ where d is the discrete interval.

(see Example of Compound Geometric and Panjer Recursion For Ruin Probabilities on the course website. There is an issue of whether to use the true value of μ_X and treat the approximation just as an approximate $f_{L1}(y)$, or view the discrete values as coming from an approximation discretizing $f_X(y)$ itself, in which case you have to pick up any change in μ_X (and later μ'_{X2}) that result from the approximation. I prefer the true value approach, which is "alternative #2" on the website example. It means that you need to calculate each $f_{L1}(y)$ based on the surface interpretation. Subtract successive values of $\sum_{z>y}S_X(z+d/2))d$ from the true value of μ_X to get successive values of the approximation for $S_{L1}(y)\ \mu_X$, and then take differences of the resulting $S_{L1}(y)$ values to get $f_{L1}(y)$ values.)

The Panjer recursion formula here will be:

$f_L(u)=[1/(1-af_{L1}(0))]\ \sum_{y=1,u}(a+b(y/u))f_{L1}(y)f_L(u-y)$ where $a=1/(1+\theta)$, $b=0$, $f_{L1}(y)=S_X(y+d/2)d/\mu_X$ (by whichever of the two methods discussed above that you choose), and the starting value is $f_L(0)=P_K[f_{L1}(0)]=[1-(1/\theta)\{f_{L1}(0)-1\}]$^(-1). You can just go ahead and program your spreadsheet with this and get values for $f_L(u)$ which will sum to values for $F_L(u)=\varphi(u)$ and then $\psi(u)=1-\varphi(u)$ gives you the ruin probabilities.

Although you don't need it for the spreadsheet calculations, you can do a little algebra on the values just given and get $f_L(u)=[1/\{(1+\theta)\ \mu_X - S_X(d/2))d\}]\ \sum_{y=1,u}S_X(y+d/2))d\ f_L(u-y)$, where once again you need to choose which alternative you'll use to come up with μ_X and $S_X(z+d/2))d$ in the approximation. This version of the formula shows directly how the tail probabilities $S_X(y)$ of the original single loss variable X compound themselves to determine the probability density $f_L(u)$ for the maximum aggregate loss to be u.

II. Moments of the Maximum Aggregate Loss Variable L

Since it is a compound Geometric-L_1 variable, we can write down the moment generating and cumulant generating functions for L:

$M_L(r) = P_K(M_{L1}(r)) = [1 – (1/\theta)\{M_{L1}(r)-1\}]^{\wedge}(-1)$

We can get raw moments of L by using Faa's formula on this, remembering what we already know about the raw moments of L_1. Namely, $\mu'_{L1\ k} = (1/(k+1))\ \mu'_{X\ (k+1)} / \mu_X$ follows directly from surface interpretation and $f_{L1}(y)=S_X(y)/\mu_X$.

Taking logarithms, $C_L(r) = -\ln[1 – (1/\theta)\{M_{L1}(r)-1\}]$ so once again Faa's formula and knowledge of the raw moments of L_1 will let us calculate cumulants of the aggregate loss variable L.

III. Exact Analytic Expressions for ψ(u)

$\psi(u)=S_L(u)$ where L is the compound Geometric-L_1 as shown above.

If X happens to be exponential, then by the memory-less property L_1 with $f_{L1}(y)=S_X(y)/\mu_X$ is also exponential (with same mean). We showed some weeks ago that for a compound distribution L with exponential secondary and primary denoted by K we could rewrite the decumulative distribution as $S_L(x)=e^{-x/\mu} \sum_{j=0,\infty}[(x/\mu)^j / j!]S_K(j)$ where in this case K is geometric with $a=1/(1+\theta)$ and μ is μ_X, the mean of the exponential. Putting in this value for a, summing the geometric series that appears in $S_K(j)$, and doing some algebra will give $\psi(u)=S_L(u)=(1/(1+\theta))e^{-(\theta/(1+\theta))\ u/\ \mu}$ so the ruin probability is exactly a constant times an exponential function of starting surplus, with the parameters shown.

A similar, but more complex analysis can be done starting from the result we had that $S_S(x)=e^{-x/\mu} \sum_{j=0,\infty}[(x/\mu)^j / j!]S_K([j/\alpha])$ for S a compound distribution with gamma secondary.