Faa's Formula

$$\left[f(g(x))\right]^{(n)} = \sum_{\substack{\{j_k\} \\ \text{s/t } \sum_k k\cdot j_k = n}} \frac{n!}{\prod_k j_k!\, k!^{j_k}} f^{\left(\sum_k j_k\right)}(g(x)) \prod_k \left(g^{(k)}(x)\right)^{j_k}$$

(for one of your projects, prove this using Taylor's series)

The coefficient is "n-choose j_k indistinguishable groups of k for each k."

Examples (each row inside the table is a set $\{j_k\}$. The sum is over all such sets.)

$n=1$

$\sum_k j_k$	k: 1
1	1

$$[f(g)]^{(1)} = f^{(1)}(g)\cdot g^{(1)} = f^{(1)} g^{(1)}$$

$n=2$

$\sum_k j_k$	k: 2	1
1	1	0
2	0	2

$$[f(g)]^{(2)} = \frac{2!}{1!\,2!^1} f^{(1)} g^{(2)} + \frac{2!}{2!\,1!^2} f^{(2)} g^{(1)2} = f^{(1)} g^{(2)} + f^{(2)} g^{(1)2}$$

$n=3$

$\sum_k j_k$	k: 3	2	1
1	1	0	0
2	0	1	1
3	0	0	3

$$[f(g)]^{(3)} = \frac{3!}{1!\,3!^1} f^{(1)} g^{(3)} + \frac{3!}{1!\,1!^1\,1!\,2!^1} f^{(2)} g^{(1)} g^{(2)} + \frac{3!}{3!\,1!^3} f^{(3)} g^{(1)3}$$

$$= f^{(1)} g^{(3)} + 3 f^{(2)} g^{(1)} g^{(2)} + f^{(3)} g^{(1)3}$$

$n=4$

$\sum_k j_k$	k: 4	3	2	1
1	1	0	0	0
2	0	1	0	1
2	0	0	2	0
3	0	0	1	2
4	0	0	0	4

$$[f(g)]^{(4)} = \frac{4!}{1!\,4!^1} f^{(1)} g^{(4)} + \frac{4!}{1!\,1!^1\,1!\,3!^1} f^{(2)} g^{(1)} g^{(3)} + \frac{4!}{2!\,2!^2} f^{(2)} g^{(2)2}$$

$$+ \frac{4!}{2!\,1!^2\,1!\,2!^1} f^{(3)} g^{(1)2} g^{(2)} + \frac{4!}{4!\,1!^4} f^{(4)} g^{(1)4}$$

$$= f^{(1)} g^{(4)} + 4 f^{(2)} g^{(1)} g^{(3)} + 3 f^{(2)} g^{(2)2} + 6 f^{(3)} g^{(1)2} g^{(2)} + f^{(4)} g^{(1)4}$$

See: Flanders, American Mathematical Monthly, v. 108 #6, June-July 2001
Johnson, " " ", v. 109 #3, March 2002
Craik, " " ", v. 112 #2, February 2005