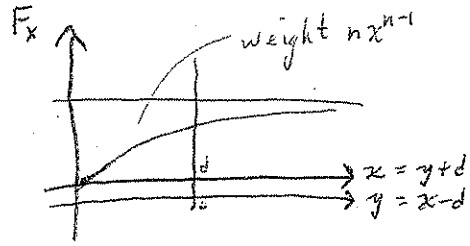
ERLANG DISTRIBUTION

(GAMMA DISTRIBUTION WITH α = integer = n)

Let X be an exponential random variable with mean $\theta = 1$

Let $Y = (X - d)\,|_{X \geq d}$, also exponential with mean $\theta = 1$ (memoryless)



$$\Gamma(n; d) = \frac{1}{\Gamma(n)} \int_0^d x^{n-1} e^{-x} dx$$ is the definition of the gamma distribution function (incomplete gamma function)

$$= \frac{1}{\Gamma(n)n} \int_0^d n x^{n-1} e^{-x} dx$$

$$= \frac{1}{n!} \mathbb{E}\left[(X \wedge d)^n\right]$$ by the surface interpretation for $\mathbb{E}\left[(X \wedge d)^n\right]$

$$= \frac{1}{n!} \left\{\mathbb{E}\left[X^n\right] - \text{weight } n x^{n-1} \text{on } S_X(x) \text{ for } x \epsilon\ [d, \infty)\right\}$$ by the surface interpretation for $\mathbb{E}\left[X^n\right]$

$$= \frac{1}{n!} \left\{n! - \text{weight } n x^{n-1} \text{on } e^{-x} \text{ for } x \epsilon\ [d, \infty)\ \right\}$$ since X is exponential

$$= \frac{1}{n!} \left\{n! - \text{weight } n\,(y+d)^{n-1} \text{ on } e^{-(y+d)} \text{ for } y \epsilon\ [0, \infty)\right\}$$ since $x = y + d$

$$= \frac{1}{n!} \left\{n! - \text{weight } n \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} d^k y^{n-1-k} \text{ on } e^{-(y+d)} \text{ for } y \epsilon\ [0, \infty)\right\}$$

$$= \frac{1}{n!} \left\{n! - \text{weight } n \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!(n-k)} d^k (n-k) y^{n-1-k} \text{ on } e^{-d} e^{-y} \text{ for } y \epsilon\ [0, \infty)\right\}$$

$$= \frac{1}{n!} \left\{n! - e^{-d} \sum_{k=0}^{n-1} \frac{n!}{k!\,(n-k)!} d^k \text{ weight } (n-k) y^{n-1-k} \text{ on } S_Y(y) \text{ for } y \epsilon\ [0, \infty)\right\}$$ since Y is exponential (by memoryless property)

$$= \frac{1}{n!} \left\{n! - e^{-d} \sum_{k=0}^{n-1} \frac{n!}{k!\,(n-k)!} d^k\, \mathbb{E}\left[Y^{n-k}\right]\right\}$$ by surface interpretation of $\mathbb{E}\left[Y^{n-k}\right]$

$$= \frac{1}{n!} \left\{n! - e^{-d} \sum_{k=0}^{n-1} \frac{n!}{k!\,(n-k)!} d^k (n-k)!\right\}$$ since Y is exponential

$$= 1 - e^{-d} \sum_{k=0}^{n-1} \frac{1}{k!} d^k$$