Math 5621 Financial Math II
Spring 2016
Final Exam Soluitons
April 29 to May 2, 2016

This is an open book take-home exam. You may consult any books, notes, websites or other printed material that you wish. Having so consulted then submit your own answers as written by you.

Do NOT under any circumstances consult with any other person. Do NOT under any circumstances cut and paste any material from another source electronically into your answer. Do NOT under any circumstances electronically copy and paste from a spreadsheet that was not created entirely by you. Failure to follow these rules will be grounds for a failing grade for the course.

Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The eight questions will be equally weighted in the grading. Please return the completed exams by 12:30 PM Monday, May 2 to my mailbox in the department office, under my office door MSB408, or by email.

1. The Black-Scholes formula for the price of a call option is

$$p = S\Phi(d_1) - e^{-rT}K\Phi(d_2)$$

where d_1 and d_2 are expressions that you can evaluate. Once you know d_1 the value of $\Phi(d_1)$ can be obtained from a spreadsheet function of cumulative normal probability values (or a published table of them.) Presumably, then, $\Phi(d_1)$ must be the probability of some event. Explain exactly what that event is and why $\Phi(d_1)$ is its probability.

Solution

$\Phi(d_1)$ is not the probability of any event. It is the conditional expected value $\tilde{\mathbb{E}}\left[e^{-rT}\frac{S_T}{S}|S_T > K\right]$ mulitiplied by the probability $\tilde{\mathbb{P}}\left[S_T > K\right]$. It is just an accident of the mathematical form of the lognormal density function that this complicated expected value can be found in a table of probability values. (note: using concepts not covered in class, it is possible to identify $\Phi(d_1)$ as the probability that $S_T > K$ under an alternative make-believe risk-neutral probability measure that corresponds to using S_t rather than a risk-free investment account to discount future cash flows, also known as "using S_t as the numeraire" .)

2. Within the assumptions used to develop the CAPM, prove that the risk-free rate is unique. I.e. prove that if for two different assets $\tilde{r}_1$ and $\tilde{r}_2$ we have $\sigma_1 = \sigma_2 = 0$ then it must be the case that $r_1 = r_2$. Show all the steps in your reasoning (your proof) and say exactly what assumptions (axioms) you are using at each step.

Solution

Without loss of generality, assume that $r_1 > r_2$.

Let $\tilde{r}_{P2} = w_f \tilde{r}_2 + \sum_{i \neq 2} w_i \tilde{r}_i$ be any portfolio that contains $\tilde{r}_2$ and is owned by some investor (definition of portfolio.)

By axiom 7 (any investor can switch to any other portfolio freely with no cost to switch) construct a new portfolio

$\tilde{r}_{P1} = w_f \tilde{r}_1 + \sum_{i \neq 2} w_i \tilde{r}_i$ exactly the same as $\tilde{r}_{P2}$ except the new portfolio $\tilde{r}_{P1}$ has $\tilde{r}_1$ instead of the $\tilde{r}_2$ contained in $\tilde{r}_{P2}$.

$\sigma^2_{P1} = \sum_{i \neq 2} w_i^2 \sigma_i^2 + 2 \sum_{i,j \neq 2; i<j} w_i w_j \sigma_{ij}$ because $\sigma_1 = 0$ (in $\sum_{i \neq 2}$) and $\sigma_1 = 0$ implies $\sigma_{1j} = 0$ for all j (in $\sum_{i,j \neq 2; i<j}$).

$\sigma^2_{P2} = \sum_{i \neq 2} w_i^2 \sigma_i^2 + 2 \sum_{i,j \neq 2; i<j} w_i w_j \sigma_{ij}$ because $\sigma_2 = 0$ (in $\sum_{i \neq 2}$) and $\sigma_2 = 0$ implies $\sigma_{2j} = 0$ for all j (in $\sum_{i,j \neq 2; i<j}$).

So $\sigma^2_{P1} = \sigma^2_{P2}$.

But $\mathbb{E}[\tilde{r}_{P1}] = w_f r_1 + \sum_{i \neq 2} w_i r_i > w_f r_2 + \sum_{i \neq 2} w_i r_i = \mathbb{E}[\tilde{r}_{P2}]$ because we assumed $r_1 > r_2$.

Then every investor will prefer to own $\tilde{r}_{P1}$ rather than $\tilde{r}_{P2}$ by axiom 3 (with same σ every investor will prefer portfolio with higher $\mathbb{E}[\tilde{r}]$).

But $\tilde{r}_{P2}$ was any portfolio that owned $\tilde{r}_2$.

So $\tilde{r}_2$ does not exist, by axiom 2 (every investment must have at least one owner, must be in at least one portfolio.)

We have proved that a second risk free investment does not exist. $\tilde{r}_1$ is the unique risk-free investment.

3. A stock has a dividend yield of 2% and the company pays 7.5% interest on its long term debt. The ROE based on beginning of year equity is 16%. The are 10 million shares outstanding. The market to book ratio is 1.25 and the share price is $40. The interest payments on the long term debt amount to $2.00 per share. What is the maximum possible growth rate the company can finance without using any new external equity financing?

   Solution

   Since the problem excluded only equity financing, we are looking for the *sustainable* growth rate:

$$\begin{aligned}
g &= \frac{PB \cdot NI}{BV} \text{ using beginning of year } BV \\
&= PB \cdot ROE \;\; \text{using } ROE \text{ on begining of year } BV \\
&= \frac{NI - DIV}{NI} \cdot ROE \\
&= \left(ROE - \frac{DIV}{BV} \right) \\
&= \left(ROE - \frac{d \cdot MV}{BV} \right) \\
&= (.16 - (.02) \cdot (1.25)) \\
&= .135
\end{aligned}$$

4. This problem involves a European put option expiring in T years with strike price K on an asset whose value today is S_0. The risk free rate (continuously compounded) is r.

   (a) Write down the Black-Scholes formula for V_0 the value today of the European put option

   Solution

   By put-call parity

$$\begin{aligned}
V_0 + S_0 &= C_0 + e^{-rT} K \\
\text{where } S_0 &= \text{value of underlying and} \\
C_0 &= S_0 \Phi \left(\frac{\ln \frac{S_0}{K} + rT}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \right) - e^{-rT} K \Phi \left(\frac{\ln \frac{S_0}{K} + rT}{\sigma \sqrt{T}} - \frac{1}{2} \sigma \sqrt{T} \right) \\
& \quad \text{is the value of the corresponding call option} \\
\text{so } V_0 &= e^{-rT} K \left[1 - \Phi \left(\frac{\ln \frac{S_0}{K} + rT}{\sigma \sqrt{T}} - \frac{1}{2} \sigma \sqrt{T} \right) \right] \\
& \quad - S_0 \left[1 - \Phi \left(\frac{\ln \frac{S_0}{K} + rT}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \right) \right]
\end{aligned}$$

   (b) Write down the N-stage binomial tree formula for V_0 the value today of the European put option. (Use the $\sum$ symbol, don't actually write out a tree.)

   Solution

   By the definition of a european put option, the value is the sum of the present values of each possible $(K - S_T) > 0$ multiplied by the probability of that value occurring at time T in the tree:

$$\begin{aligned}
V_0 &= \sum_{j=0}^{j_K-1} e^{-rT}\left(K - u^j d^{N-j} S_0\right)\left(\frac{1}{2}\right)^N \frac{N!}{j!\,(N-j)!} \\
\text{where } j_K &\geq 1 \text{ is the first such value with } u^{j_K} d^{N-j_K} S_0 > K \\
u &= e^{\frac{T}{N}\left(r-\frac{1}{2}\sigma^2\right)+\sigma\sqrt{\frac{T}{N}}};\ d = e^{\frac{T}{N}\left(r-\frac{1}{2}\sigma^2\right)-\sigma\sqrt{\frac{T}{N}}} \\
\text{and } \left(\frac{1}{2}\right)^N \frac{N!}{j!\,(N-j)!} &= \text{binomial probability of } j \text{ up } N-j \text{ down}
\end{aligned}$$

$$\text{so } V_0 = e^{-rT} K \sum_{j=0}^{j_K-1}\left(\frac{1}{2}\right)^N \frac{N!}{j!\,(N-j)!} - S_0 \sum_{j=0}^{j_K-1} e^{-rT} u^j d^{N-j}\left(\frac{1}{2}\right)^N \frac{N!}{j!\,(N-j)!}$$

(c) Explain why the term involving S_0 in the Black-Scholes formula for the value today of the European put option is NOT multiplied by e^{-rT}.

Solution

Approximately

$$\left[1-\Phi\left(\frac{\ln\frac{S_0}{K}+rT}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T}\right)\right] \approx \sum_{j=0}^{j_K-1} e^{-rT} u^j d^{N-j}\left(\frac{1}{2}\right)^N \frac{N!}{j!\,(N-j)!}$$

or exactly, using the lognormal and after integrating and simplifying as in the class notes,

$$\begin{aligned}
\left[1-\Phi\left(\frac{\ln\frac{S_0}{K}+rT}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T}\right)\right] &= \int_{-\infty}^{\frac{K}{S_0}} e^{-rT} e^x \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-T\left(r-\frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}}\right)^2} dx \\
&= \mathbb{E}\left[e^{-rT}\frac{S_T}{S_0}|S_T<K\right]\mathbb{P}\left[S_T<K\right]
\end{aligned}$$

So the term involving S_0 actually does have a factor e^{-rT} contained within it. The factor e^{-rT} is disguised within the integration that gives rise to the coefficient $\left[1-\Phi\left(\frac{\ln\frac{S_0}{K}+rT}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T}\right)\right]$

5. Assume your company has three classes of securities in its financing structure: \$500 million (market value) of senior perpetual debt with a market yield of 5%; \$5 billion (market value) of junior high yield (junk) perpetual debt with a market yield of 15%; and \$250 million (market value) of common equity with a market capitalization rate of 40%. Assume a corporate tax rate of 35% and also assume that, because of the high proportion of junk financing, the tax authorities grant tax deductibility to only 1/3 of the interest on the high yield financing.

(a) What is the firm's weighted average cost of capital (WACC)?

Solution

Since the given facts included a market capitalization rate we can compute the $WACC$ *directly from (15.19) in the text (generalized to include the junk debt) as* $WACC = \frac{250}{5750}.40 + \frac{500}{5750}(1-.35).05 + \frac{5000}{5750}(1-\frac{1}{3}(.35)).15 = .13543$ *or about* 13.54%

(b) What can you conclude (if anything) about the cost of capital for an all-equity firm with the same operating risks? If you answer "nothing" give reasons.

Solution

This is like exercise 15.1 but with junk debt instead of the preferred stock in the exercise, i.e. it is like equations (15.1) through (15.11) in the text, adjusted for the presence of the junk bonds. Let J *stand for the market value of the junk bonds and* X the portion of its interest that is deductible *and then, following the solution manual for 15.1, the value of the levered firm is*

$$
\begin{aligned}
V_L &= V_U + \tau_c B + X\tau_c J \\
\textit{where } V_U &= \textit{value of the unlevered (all equity) firm,} \\
\textit{so } V_U &= V_L - \tau_c B - X\tau_c J \\
\textit{But } V_U &= \frac{\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c)}{\rho} \\
\textit{where } EBIT(1-T) &= \textit{cash flow from operations (perpetual)} \\
\textit{and } \rho &= \textit{the cost of capital for the unlevered firm.} \\
\textit{Thus, } \rho &= \frac{\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c)}{V_L - \tau_c B - X\tau_c J} \\
\textit{But } V_L &= \frac{\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c)}{WAAC} \textit{ so} \\
\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c) &= WAAC \cdot V_L \textit{ and so} \\
\rho &= \frac{WAAC \cdot V_L}{V_L - \tau_c B - X\tau_c J} \\
&= \frac{WACC}{1 - \tau_c \frac{B}{V_L} - X\tau_c \frac{J}{V_L}} \\
&= \frac{.13543}{1 - .35\frac{500}{5750} - \frac{1}{3}(.35)\frac{5000}{5750}} \\
&= .15600 \textit{ or about } 15.6\%
\end{aligned}
$$

This entire analysis uses Modigliani-Miller style assumptions except for the taxes. Thus if considerations involving (1) expected value of future loss of deductions on debt (beyond what's already assumed) (2) expected value of future financial distress or (3) expected value of financial flexibility are important (as they are in fact likely to be for such a highly levered firm), then we have overstated the cost of *capital for an all equity firm. Nothing in the given facts allows us to estimate the amount of this overstatement.*

6. With the following expected returns and covariance matrix what are the weights w_1,w_2, and w_3 of each of the three assets in the optimal portfolio assuming the risk free rate is .0005? You don't have to prove your answer but you do have to show how you calculated it.

	$\mathbf{j}=$	**1**	**2**	**3**
	$\mathbf{r}_j=$	.0076	.0673	.1480
	$\boldsymbol{\sigma}_{i,j}=$			
$\mathbf{i}=$	**1**	.01	−.009	0
	2	−.009	.03	.02
	3	0	.02	.06

Solution

see class notes on CAPM

The weight vector will be $\mathbf{w} = \frac{\boldsymbol{\sigma}^{-1}(\mathbf{r}-r_f\mathbf{1})}{\mathbf{1}^T\boldsymbol{\sigma}^{-1}(\mathbf{r}-r_f\mathbf{1})}$

$$\boldsymbol{\sigma}^{-1} = \left\langle \begin{array}{ccc} .01 & -.009 & 0 \\ -.009 & .03 & .02 \\ 0 & .02 & .06 \end{array} \right\rangle^{-1} = \left\langle \begin{array}{ccc} 153.173 & 59.0810 & -19.6937 \\ 59.0810 & 65.6455 & -21.8818 \\ -19.6937 & -21.8818 & 23.9606 \end{array} \right\rangle$$

$$\boldsymbol{\sigma}^{-1}(\mathbf{r}-r_f\mathbf{1}) = \left\langle \begin{array}{ccc} 153.173 & 59.0810 & -19.6937 \\ 59.0810 & 65.6455 & -21.8818 \\ -19.6937 & -21.8818 & 23.9606 \end{array} \right\rangle \left\langle \begin{array}{c} .0071 \\ .0668 \\ .1475 \end{array} \right\rangle =$$

$$\left\langle \begin{array}{c} 2.1293 \\ 1.5770 \\ 1.9327 \end{array} \right\rangle$$

So the weights are $\left\langle \begin{array}{c} 2.1293 \\ 1.5770 \\ 1.9327 \end{array} \right\rangle \div (2.1293 + 1.5770 + 1.9327) = \left\langle \begin{array}{c} .3776 \\ .2797 \\ .3427 \end{array} \right\rangle$

7. Your nuclear research department just discovered a way to turn lead into gold. With the price of gold at \$1300 per ounce this week you are quite excited and are making plans. You've already learned, for example, that you'll need to plan on annual spending of 1% of the value of any gold you produce just to store it safely and insure it. It's going to take you 15 years and a lot of money to implement the nuclear technology before you get your first output of gold, however, so you need to make an assumption about the price of gold 15 years from now in order to evaluate whether to go ahead with the investment today. The best experts that you can find tell you that in their opinion the price of gold has a beta of 0, will be flat for the next two years while the market digests the Fed's tapering plans, but then it will advance 10% a year for 3 years reflecting the inflation of the dollar that must come sooner or later, followed by a steady 5% annual increase thereafter. The annual risk free rate for a 15 year horizon is 3%.

What is the present value today of an ounce of gold produced 15 years from now?

Solution

Always trust the market price more than any expert's opinion, unless you are in the business of speculating (outguessing the market). Here your business is gold production, not speculation, so trust the market price of gold. With storage and insurance costs of 1% of the value of the gold per year the market is telling you that one ounce of gold fifteen years from now can be produced without fail by putting $\$(.99)^{-15}\,Price-per-ounce-today$ worth of gold into insured storage today. It is a replicating portfolio guaranteed to pay off for one ounce of gold in fifteen years. So the present value today of an ounce of gold produced fifteen years from now is $\$(.99)^{-15}\,Price-per-ounce-today = \$1.1627118 \times 1300 \ = \$1511.53$.

8. For many years, a company has plowed back 60% of earnings while making a 20% return on equity and maintaining a 2% dividend yield. The debt ratio has remained constant. The market has priced the shares as if the growth rate corresponding to this financial performance could continue forever. By what % and in what direction will the share price change if the company suddenly announces, in a complete surprise to the market, that is has no further opportunities for profitable growth beyond its current scale of operations, it now plans no further growth at all, and will begin to pay out all of its earnings as dividends every year?

   Solution

   Under the scenario described, all of the current $PVGO$, present value of growth opportunities per-share, will disappear from the stock price at the time of the surprise announcement. So we get a decline in price:

$$
\begin{aligned}
-\frac{PVGO}{P} &= -\frac{1}{P}\left(P-\frac{eps}{k_S}\right) \\
&= -\frac{1}{P}\left(P-\frac{\frac{eps(1-PB)}{1-PB}}{d+g}\right) \\
&= -\frac{1}{P}\left(P-\frac{\frac{\text{div}}{1-PB}}{d+g}\right) \\
&= -\frac{1}{P}\left(P-\frac{\text{div}}{(1-PB)\,(d+PB\cdot ROE)}\right) \\
&= -\left(1-\frac{d}{(1-PB)\,(d+PB\cdot ROE)}\right) \\
&= -\left(1-\frac{.02}{(1-.60)\,(.02+.60\,(.20))}\right) \\
&= -.6429 \\
&= 64.29\%\ \text{price decline}
\end{aligned}
$$