Mathematics 5621-Financial Math II
Spring 2012
Final Examination Solutions
April 27 - April 30, 2012

This is an open book take-home exam. You may consult any books, notes, websites or other printed material that you wish. Having so consulted then submit your own answers as written by you.

Do NOT under any circumstances consult with any other person. Do NOT under any circumstances cut and paste any material from another source electronically into your answer. Do NOT under any circumstances electronically copy from a spreadsheet that was not created by you. Failure to follow these rules will be grounds for a failing grade for the course.

Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The grade will be based on your two best answers, equally weighted, out of the five questions below. If you like, you can just pick two questions to answer and you are done. Any more than two that you answer will just be buying you insurance in case you botched one and didn't know it. Exam grades and grades on your papers will be posted on the course website (by ID number) no later than 5 PM Saturday, May 4.

Submit your completed exams by 5 PM Monday April 30, at my office (under the door if closed), in my mailbox on the first floor, or by email.

1. Alpha Gaming has a current price of \$4 per share. You believe that the appropriate market capitalization rate for Alpha is 10%. Its annual sales are $\$2,400,000,000$. Total annual expenses including depreciation, amortization, interest, and taxes are $\$2,100,000,000$. On a book value basis debt is $\$720,000,000$. The payout ratio is 75%. The price/book ratio is 200%. There are 400 million shares outstanding. (a) What present value of growth opportunities (PVGO) is implied by Alpha's market valuation? (b) Grow or Die: what is the maximum possible growth rate Alpha Gaming can sustain without raising any new outside capital?

   Solution

   (a) From class notes

$$\begin{aligned} SharePrice &= \frac{EPS}{r} + \frac{PVGO}{shares} \text{ so} \\ PVGO &= shares\left(SharePrice - \frac{EPS}{r}\right) \\ &= 400\left(4 - \frac{\frac{2400-2100}{400}}{.10}\right) \\ &= -1,400 \text{ million or} \\ &= -3.50 \text{ per share} \end{aligned}$$

This means that the market is expecting either the company or its current profit margins, or both, to shrink in the future. "Grow or Die" is a pressing reality for this company if the market is correct.

(b) "Without raising any new outside capital" means that we are asking what growth rate can be attained without any new equity or debt: in the class notes this was the "internal growth rate."

$$\begin{aligned} g &= \frac{NetIncome \cdot PlowbackRatio}{NetAssets_{beginning}} \\ &= \frac{NetIncome \cdot PlowbackRatio}{E_{beg} + D_{beg}} \text{ note:} E \text{ and } D \text{ are book value basis} \\ &= \frac{NetIncome \cdot PlowbackRatio}{(E_{end} - NetIncome \cdot PlowbackRatio) + D_{beg}} \\ &= \frac{(2400 - 2100) \cdot (1 - .75)}{\left(\frac{4 \cdot 400}{2} - (2400 - 2100) \cdot (1 - .75)\right) + 720} \text{ note: no info about} \\ &\quad \text{any debt repayment so } D_{beg} \text{ is same as the given } D_{end} \\ &= .0519 \end{aligned}$$

Of course, the company can grow at more than 5.19% but it will need to raise more debt or equity or both to do so. A common but less accurate calculation would be

$$\begin{aligned} g &= \frac{NetIncome \cdot PlowbackRatio}{NetAssets} \\ &= \frac{NetIncome \cdot PlowbackRatio}{E + D} \\ &= \frac{(2400 - 2100) \cdot (1 - .75)}{\frac{4 \cdot 400}{2} + 720} \\ &= .0493 \end{aligned}$$

2. A company has net assets with a market value of $\$7,500,000$ and a financial structure involving 40% debt. The company believes that its current optimal financial structure should inolve 45% debt. The company is considering a new project that requires an investment of $\$2,375,000$. The company believes that afer taking on the project it will have an optimal capital structure requiring 50% debt. If the company's after tax $WACC$ is 15%, its marginal cost of new debt is 6% before tax, and its marginal tax rate is 40%, then what after tax rate of return does the project need to earn in order to to be acceptable, assuming that it will be financed optimally?

   Solution

   Use subscripts b for the company before the project, p for the project itself, and a for the company after the project.

To be acceptable the project needs to earn $WACC_p = \rho_p \left(1 - .40\frac{\Delta B}{\Delta(S+B)}\right)$ by equation (15.12) where ρ_p is the cost of capital for the project if it used no debt and $\frac{\Delta B}{\Delta(S+B)}$ is the proportion of debt in the optimal project financing. Since the optimal financing before the project was $45\%(7,500) = 3,375$ debt and after the project $50\%(7,500 + 2,375) = 4,937.5$ debt, then the optimal debt for the project must be $4,937.5 - 3,375 = 1,562.5$. Then $\frac{\Delta B}{\Delta(S+B)}$ for the project is $\frac{1,562.5}{2,375} = .6579$ so $WACC_p = \rho_p (1 - .40 \cdot .6579) = .7368\rho_p$. We still need to figure out ρ_p.

To do that we use the relationship $\rho_a = w_b\rho_b + w_p\rho_p$ where $w_b = \frac{7500}{7500+2375} = .7595$ and $w_p = \frac{2375}{7500+2375} = .2405$, and $\rho_b = \frac{WAAC_b}{(1-.40\cdot.45)} = \frac{.15}{.82} = .1829$ by (15.12) and the comments following it. The relationship comes from CAPM:

$$\begin{aligned}
\beta_a &= w_b\beta_b + w_p\beta_p \text{ by linearity of covariances so} \\
\rho_a &= r_f + \beta_a (r_M - r_f) \\
&= (w_b + w_p) r_f + \left(w_b\beta_b + w_p\beta_p\right)(r_M - r_f) \\
&= w_b (r_f + \beta_b (r_M - r_f)) + w_p \left(r_f + \beta_p (r_M - r_f)\right) \\
&= w_b\rho_b + w_p\rho_p \text{ so we now know that} \\
\rho_p &= \frac{\rho_a - w_b\rho_b}{w_p} \\
&= \frac{\rho_a - .7595 \cdot .1829}{.2405} \\
&= \frac{\rho_a}{.2405} - .5776 \text{ so} \\
WACC_p &= .7368\rho_p \\
&= 3.0636\rho_a - .4256
\end{aligned}$$

Now we only need to figure out what ρ_a is. For that use

$$\begin{aligned}
\rho_a (1 - .40 \cdot .50) &= WACC_a \\
&= .50k_a + .50 \cdot .60 \cdot .06 \text{ where } k_a \text{ is the} \\
&\quad \text{cost of equity after adding the project} \\
&= .50 (w_b k_b + w_p k_p) \ + .50 \cdot .60 \cdot .06 \\
.8\rho_a &= .50 \left(w_b \frac{WACC_b - .45 \cdot .60 \cdot .06}{.55} \right. \\
&\quad \left. + w_p \frac{WACC_p - .6579 \cdot .60 \cdot .06}{.3421} \right) + .018
\end{aligned}$$

$$
\begin{aligned}
\text{So } .8\rho_a &= .50\left(.7595\frac{.15-.0162}{.55}\right. \\
&\quad \left. +.2405\frac{3.0636\rho_a-.4256-.0237}{.3421}\right)+.018 \\
.8\rho_a &= .0924+1.0769\rho_a-.1579+.018 \\
\rho_a &= .1715 \text{ so} \\
WACC_p &= 3.0636\rho_a-.4256 \\
&= .0998 \text{ which is the answer to the question}
\end{aligned}
$$

3. Assume that the entire market contains only 3 risky assets $\tilde{r}_1$, $\tilde{r}_2$, and $\tilde{r}_3$ plus the risk free asset $\tilde{r}_f$. Assume that their expected returns are $r_1 = .10$, $r_2 = .20$, $r_3 = .01$, and $r_f = .03$ and that their variances of return are $\sigma_1 = .20$, $\sigma_2 = .25$, and $\sigma_3 = .35$. Finally, assume that the correlation coefficients among the returns are $\rho_{12} = .80$, $\rho_{13} = 0$, and $\rho_{23} = -.25$. (a) In the CAPM, what will be the relative weightings m_1, m_2, and m_3 (with $m_1+m_2+m_3 = 1$) of the three risky assets in the entire market (i.e. in the market portfolio $\tilde{r}_M$)? You can use formulas from your notes without actually deriving those formulas, but your answer must (b) give a reason why you used the formulas that you did.

Solution

The σs were meant to be standard deviations, not variances. If you used them as variances you got full credit; sorry for the badly stated question.

(a) From the notes:

$$
\underline{m} = \frac{\underline{\underline{\sigma}}^{-1}(\underline{r}-r_f\underline{1})}{\underline{1}^T\underline{\underline{\sigma}}^{-1}(\underline{r}-r_f\underline{1})} \quad \text{where}
$$

$$
\begin{aligned}
\underline{\underline{\sigma}} &= \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{31}\sigma_3\sigma_1 & \rho_{32}\sigma_3\sigma_2 & \sigma_3^2 \end{pmatrix} \\
&= \begin{pmatrix} .04 & .04 & 0 \\ .04 & .0625 & -.021875 \\ 0 & -.021875 & .1225 \end{pmatrix} \text{ so} \\
\underline{\underline{\sigma}}^{-1} &= \frac{1}{.000091109}\begin{pmatrix} .007177734 & -.0049 & -.000875 \\ -.0049 & .0049 & .000875 \\ -.000875 & .000875 & .0009 \end{pmatrix} \\
\underline{1}^T\underline{\underline{\sigma}}^{-1}(\underline{r}-r_f\underline{1}) &= 2.512829467 \\
m_1 &= -1.3674 \\
m_2 &= 2.0638 \\
m_3 &= .3036
\end{aligned}
$$

(b) We know that the market portfolio $\tilde{r}_M$ has a maximum Sharpe ratio. The formulas that were taken from the notes were the ones that gave weights which maximized the Sharpe ratio. The idea of a negative market weight for $\tilde{r}_1$ seems strange, but it could be interpretted to mean that a large number of investors have naked shorts in $\tilde{r}_1$. (Naked shorts are illegal in most large markets but are possible in theory. You could interpret $\tilde{r}_2$ to be government pension obligations held by current and retired workers around the world, especially government workers, and $\tilde{r}_1$ to be the net financial position – pv of future taxes minus pv of future pensions obligations – of governments around the world. In this model, $\tilde{r}_3$ would be the aggregate stock and bond markets of the world.)

4. Build a binomial pricing model using the following assumptions:$r_f = .03$, $\sigma = .25$, $T = 2$, $N = 4$, $S_0 = 50$, and $q_u = 1/2$. (Do NOT use any other choice for q_u. You will get NO credit at all for this question if you use any different choice for q_u). Use the model that you have built to price an American Call option on S with strike price 55 expiring at $T = 2$, assuming that S pays a dividend of 2 every 6 months. (a) What is the value of the call at time 0? (b) What is the value of the position held in S_0 at time 0 in the replicating portfolio? (c) How much larger is the value of the American Call at time 0 than the value of the corresponding European Call at time 0? Your boss remembers hearing in a class once that American Calls have the same value as the corresponding European Calls. (d) What would you tell your boss to explain why it makes sense for this American Call to be worth more than this European Call?

   Solution

   (a) 6.745892359 see solution spreadsheet (or 6.735020306 if you took r_f to be an effective annual rate, rather than an instantaneous annual rate)

   (b) 27.77489798 see solution speadsheet (or 27.74542369 if you took r_f to be an effective annual rate, rather than an instantaneous annual rate)

   (c) 0.255652323 see solution spreadsheet (or 0.259069537 if you took r_f to be an effective annual rate, rather than an instantaneous annual rate)

   (d) The European call gives you only the upside potential in the stock price with no chance to capture any dividend payments prior to expiry. The American call, on the other hand, gives you a chance to capture some dividend payments prior to expiry in addition to the upside potential in the stock price. But this occurs only in a few scenarios (those where most of the upside potential is actually realized early in the period) so the actual price difference between the two calls is not very large.

5. With a WAAC or Opportunity Cost of Capital of 17.5% (a) is a project with the following cash flows financially acceptable? (b) Is it acceptable to your boss who (irrationally) won't accept "any project with payouts that have less than a 20% return"? In justifying your answer, be sure to calculate (c) the Net Present Value (d) the IRR and (e) one other measure

of the rate of return that helps you to answer (a) and (b). Finally, (f) be sure to explain to your boss why your answer to (b) fits his rule about 20%

t	$\mathbf{CF}_t$
0	-2
1	0
2	1
3	-10
4	3
5	3
6	5
7	5
8	5
9	5
10	-7

Solution

(a) The project is financially acceptable because (c) it has a positive $NPV = 0.142128912$ (see solution speadsheet).

(b) The project is probably not acceptable to my irrational boss because (d) the $IRR = .181510005 < .20$ (use EXCEL Solver, an internet root-finder, or any other method you choose to see that this is the only positive solution to the IRR polynomial, but an accumulation of the cash flows at the IRR shows that the project mixes investment and financing so the IRR is an invalid tool in this case.)

But (e) even the $ModifiedIRR = 0.181133802 < .20$ (see solution spreadsheet ... modify the cash flows yourself, as shown in class, then the EXCEL IRR function succeeds, and an accumulation verifies that at the $ModifiedIRR$ the modified cash flows represent a pure investment project.)

Beware: the EXCEL MODIRR function does not give the correct $ModifiedIRR$ as defined in class! It gives a lazy analyst's answer that fails to look at the net project position (investment versus financing) at each point in time. This is a good lesson. As an analyst, *never* rely on a label in a piece of software! You are responsible to verify for yourself what the tools you choose to use are doing. In this case, an EXCEL Help query on its MODIRR function gives you the information that it is using a lazy shortcut.

(f) The spreadsheet shows an alternative method (but a mistaken one!) to get to a $ModifiedIRR = 0.205732743 > .20$ if I want to help my boss get to the correct decision despite his irrational fixation on .20. However, this is not correct because it discounts a net investment postion at .175 for some points in time. We can justify a modification as not violatiing the .20 only if it discounts only net financing positions at $.175 < .20$.

We might try one other way to convince our boss, in an honest way, to make the right decision. The spreadsheet shows a third modification

of cash flows using .175 when it's a financing position, switching to the boss's .20 when it is an investment position and then calculating a residual $ResidualIRR = 0.10128195$ for the investment positions prior to the big -10 investment. We say to the boss: Look this project is an experiment for the first three years. If we like the result, we'll invest the -10 at time 3 (along with the accumulated 1.57 from the experimental period) and get a 20% return until we've repaid the entire investment, at which point the remaining cash flows will provide .175 financing to the rest of the company until they need to repay us so we can meet the -7 at the end. (If we don't like the result of the first three years we will just forego the -10 investment and walk away.) Unfortunately, to reap this harvest, you'll need to settle for just a $ResidualIRR = 0.10128195$ for the fairly small investment involved in the three year experiment. What do you say?