Math 5621 Financial Math II
Spring 2011
Final Exam
April 29 to May 2, 2011

This is an open book take-home exam. You may use any books, notes, websites or other printed material that you wish but do not consult with any other person. Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The four questions will be equally weighted in the grading. Please return the completed exams by 5 PM Monday, May 2 to my mailbox in the department office, under my office door MSB408, or by email.

1. A binomial pricing model is built using the following assumptions: $r_f = .02$, $\sigma = .20$, $T = 2$, $N = 4$, $S_0 = 20$

				35.213
			30.569	
		26.538		26.538
	23.038		23.038	
20.000		20.000		20.000
	17.362		17.362	
		15.073		15.073
			13.085	
				11.359

There is something unique about this particular binomial pricing model. What is unique about it?

The model is used to price an American Put option on S with strike price 25 expiring at $T = 2$, producing a value of 5.288 for the put option, as follows, where **bold** entries indicate early exercise:

				0
			0	
		1.225		0
	3.024		2.475	
5.288		4.883		5
	7.331		7.389	
	7.638	9.679	**7.638**	9.927
		9.927	11.667	
			11.915	13.641

There is an error in the calculation. Where is the error and what is the correct value for the put option?

SOLUTION

The unique aspect of this model is that it features BOTH $p_u = p_d = \frac{1}{2}$ and $ud = 1$. Usually, you have to choose which of these desireable conditions you want and live without the other. But in this case $r_f - \frac{1}{2}\sigma^2 = 0$ so we can have our cake and eat it, too: for $p_u = p_d = \frac{1}{2}$, $u = e^{\frac{T}{N}(r_f - \frac{1}{2}\sigma^2) + \sqrt{\frac{T}{N}}\sigma} = e^{+\sqrt{\frac{T}{N}}\sigma} = \left(e^{-\sqrt{\frac{T}{N}}\sigma}\right)^{-1} = \left(e^{\frac{T}{N}(r_f - \frac{1}{2}\sigma^2) - \sqrt{\frac{T}{N}}\sigma}\right)^{-1} = d^{-1}$.

The error in the calculation is the number 4.883. It should be $e^{-\frac{1}{2}(.02)}\frac{7.638+2.475}{2} = 5.006$ NOT $e^{-\frac{1}{2}(.02)}\frac{7.389+2.475}{2} = 4.883$. (Also, NOT early exercise 5.000).

		3.0845
This makes the rest of the table	5.308	
		7.392
		7.638

2. An investor's entire portfolio consists of 25% in the risk free investment, with a return of .03, and the balance invested in two stocks S_1 and S_2 where $r_1 = .08$, $r_2 = .12$, $\sigma_1^2 = .04$, $\sigma_2^2 = .09$, and $\rho_{12} = .6$. What is the expected return r_P and the variance of return σ_P^2 on the investor's entire portfolio?

   Solution

   Since it is the investor's entire portfolio, it needs to have a maximum Sharpe ratio. Since the Sharpe ratio is the slope of the line from risk-free to the portfolio, the portfolio consisting of just the two stocks has the same Sharpe ratio, the maximum one. The portfolio with maximum Sharpe ratio consisting of just the two stocks can be found by setting equal to zero the derivative of the Sharpe ratio with respect to the weight w_1 in S_1, since $w_2 = 1 - w_1$ so $r = w_1 r_1 + (1 - w_1) r_2$ and $\sigma = \left(w_1^2\sigma_1^2 + (1 - w_1)^2\sigma_2^2 + 2w_1(1 - w_1)\rho_{12}\sigma_1\sigma_2\right)^{\frac{1}{2}}$:

$$
\begin{aligned}
0 &= \frac{d}{dw_1}\left(\frac{r - .03}{\sigma}\right) \\
&= \frac{\frac{d}{dw_1}(r - .03)\sigma - (r - .03)\frac{d}{dw_1}\sigma}{\sigma^2} \\
&= \frac{(r_1 - r_2)\sigma - [(r_1 - r_2)w_1 + r_2 - .03]\,\frac{1}{2}\sigma^{-1} \cdot \begin{bmatrix} 2w_1\sigma_1^2 - 2(1 - w_1)\sigma_2^2 + \\ 2(1 - w_1 - w_1)\rho_{12}\sigma_1\sigma_2 \end{bmatrix}}{\sigma^2}
\end{aligned}
$$

so multiplying both sides by σ^3

$$
\begin{aligned}
0 &= (r_1 - r_2)\,\sigma^2 - [(r_1 - r_2)w_1 + r_2 - .03]\left[w_1\sigma_1^2 - (1 - w_1)\,\sigma_2^2 + (1 - w_1 - w_1)\,\rho_{12}\sigma_1\sigma_2\right] \\
0 &= (r_1 - r_2)\left(w_1^2\sigma_1^2 + (1 - w_1)^2\,\sigma_2^2 + 2w_1\,(1 - w_1)\,\rho_{12}\sigma_1\sigma_2\right) \\
&\quad - [(r_1 - r_2)w_1 + r_2 - .03]\left[w_1\sigma_1^2 - (1 - w_1)\,\sigma_2^2 + (1 - w_1 - w_1)\,\rho_{12}\sigma_1\sigma_2\right] \\
w_1 &= \frac{(r_1 - r_2)\,\sigma_2^2 + (r_2 - .03)\left(\sigma_2^2 - \rho_{12}\sigma_1\sigma_2\right)}{(r_1 - r_2)\left(\sigma_2^2 - \rho_{12}\sigma_1\sigma_2\right) + (r_2 - .03)\left(\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2\right)} \\
&= .411765 \\
w_2 &= 1 - w_1 = .588235 \\
r_P &= .25(.03) + .75\,[.411765(.08) + .588235(.12)] = .085147 \\
\sigma_P^2 &= .75^2\left[.411765^2(.04) + .588235^2(.09) + 2\,(.411765)\,(.588235)\,(.6)\,(.2)\,(.3)\right] = .031142
\end{aligned}
$$

3. Assume that markets are as efficient as they can be in the practical world. Describe the expected abnormal return that can be earned by arbitrageurs, such as some hedge funds, who seek to profit by finding and exploiting inefficiencies in the market.

   SOLUTION

   (See exercise 10.5) In the practical world, it is the action of arbitrageurs in markets that create market efficiency by finding and exploiting opportunities for abnormal returns until those abnormal returns are reduced to the practical minimum. The practical minimum for those abnormal returns is the point at which it would no longer be worthwhile financially (i.e. no longer value producing) for competing arbitrageurs to exploit the remaining abnormal return. This point is reached when the remaining abnormal return, that which the arbitrageur can expect to earn, is equal to the cost of information to find the opportunity for abnormal return, plus the trading costs to exploit it, plus the cost of capital to support the operation, including human, physical and financial capital.

4. A company has net assets with a market value of $\$5,000,000$ and a financial structure involving 50% debt. The company believes that it has too much debt relative to the risks in its operations. The company is considering a new project that requires an investment of $\$1,250,000$. Taking on the project will leave the company's overall operating risk exactly where it is today. The company plans to finance the project 100% with new equity, believing that this new equity will leave it with a financial structure exactly compatible with its operating risks. If the company's after tax $WACC$ is 15%, its marginal cost of new debt is 6% before tax, and its marginal tax rate is 40%, then what after tax rate of return does the project need to earn in order to to be acceptable?

   Solution

(See exercise 15.7)

$$
\begin{aligned}
WACC &= \rho\left(1-\tau_c\frac{B}{B+S}\right)\\
\rho &= .\frac{WACC}{\left(1-\tau_c\frac{B}{B+S}\right)}\\
&= \frac{.15}{1-.4\,(.5)}\\
&= .1875
\end{aligned}
$$

In this case the correct capital structure to assume for the project is the capital structure for the company after taking on the project, because (a) it is believed that the project has the same risk characteristics (i.e. can support the same debt) as the company and (b) it is believed that the capital structure after taking on the project is the correct capital structure for the company. So the required return is the $WACC$ in the new capital structure.

$$
\begin{aligned}
WACC &= \rho\left(1-\tau_c\frac{B}{B+S}\right)\\
&= .1875\left(1-.4\frac{2,500,000}{6,250,000}\right)\\
&= .1575
\end{aligned}
$$