Math 5621 Financial Math II
Spring 2010
Final Examination Solutions
April 30 to May 5

This is an open book take home examination. You may use any books, notes, websites or other printed material but you may not consult with any other person. Please put your name on all pages submitted and please show all of your work so that I can understand your reasoning. The four questions will be equally weighted in the grading. The exam is due back to me by 5 PM on Wednesday May 5, in my mailbox at the department offices, under my office door MSB 408, or by email.

1. Consider a put option with strike price of 22 and a 5 year expiry on a stock whose current market price is 20 and that has a standard deviation of annual return equal to .50. Build a binomial model for the value of the put using two steps per year and with probability of stock price increase or decrease both equal to 1/2. Assume that the risk free rate (annual, convertible continuously) is .02.

   Solution 1 *See solution worksheet for the binomial model.*

   (a) If it is an American put, would you exercise immediately and pocket the 2 that is available upon exercise or would you hold on to the put? Why, exactly? **Solution** *Hold on. The binomial model demonstrates a value of* 8.70 *for holding onto the put, using the risk-free rate and make-believe risk-neutral probabilitities. The binomial model gives proper values today since a replicating portfolio can be constructed to assure future payoffs equal to the ones on the binomial tree. They reflect the possibility of opportunities in the future for much larger exercise gains, offset to some extent by the possibility of lower or no gains at all.*

   (b) How much more is the American put worth today than the European put? Why should it be worth more? **Solution** .38 *is demonstrated in the binomial model. This reflects the value of the choice (option) to exercise the American option at times not available for exercise of the European option. Such choices (options) have value, captured in the binomial model by the points (shaded) at which early exercise has more value than holding onto the option.*

   (c) After one down step in the model, what is the difference (if any) between the amount of risk free investment in the replicating portfolio for the American put versus the amount of risk free investment in the replicating portfolio for the European put? Why should there be a difference? **Solution** 1.26 *is demonstrated in the binomial model. The difference stems from (1) The value of the option is higher for* *the American because there are more options (choices) in the future with the American, and options (choices) have value. (2) The short position in the underlying is larger for the American because the* Δ *is larger in absolute value (because the value of the future options in the American increase with lower values of the underlying, making the difference in the two possible future put values larger, increasing the* Δ *compared to the European). (3) The value of the risk free bonds is the option value plus the absolute value of the short position, so (1) and (2) force it to be larger for the American.*

(d) After one down step in the model, what is the difference (if any) between the amount of risk free investment in the replicatiing portfolio for the European put in the binomial model versus the amount of risk free investment in the replicating portfolio for the European put according to the Black-Scholes formula? Why should there be a difference? **Solution** 2.68 *is demonstrated in the solution worksheet. The difference stems from (1) continuous modeling for the Black-Scholes value versus discrete modeling for the binomial value. (2) the universally adopted convention of using* $\mu = r_f - \frac{1}{2}\sigma^2 = -.105$ *for the binomial modeling, same as for the Black-Scholes, whereas the technically correct value in the binomial model, consistent with Black-Scholes in the limit, would be* $\mu = r_f - \frac{\ln\left\{\frac{1}{2}\left(e^{\sqrt{\frac{T}{N}}\sigma} + e^{-\sqrt{\frac{T}{N}}\sigma}\right)\right\}}{\frac{T}{N}} = -.1025$, *as seen in the class notes.*

2. Assume your company has three classes of securities in its financing structure: \$500 million (market value) of senior perpetual debt with a market yield of 10%; \$5 billion (market value) of junior high yield (junk) perpetual debt with a market yield of 22%; and \$250 million (market value) of common equity with a market capitalization rate of 30%. Assume a corporate tax rate of 40% and that, because of the high proportion of junk financing, the tax authorities grant tax deductibility to only 35% of the interest on the high yield financing.

(a) What is the firm's weighted average cost of capital (WACC)? **Solution** *Since the given facts included a market capitalization rate we can compute the* $WACC$ *directly from (15.19) in the text (generalized to include the junk debt) as* $WACC = \frac{250}{5750}.30 + \frac{500}{5750}(1 - .40).10 + \frac{5000}{5750}(1 - .35(.40)).22 = .1827826$ *or about 18.3%*

(b) What can you conclude (if anything) about the cost of capital for an all-equity firm with the same operating risks? If you answer "nothing" give reasons. **Solution** *This is like exercise 15.1 but with junk debt instead of the preferred stock in the exercise, i.e. it is like equations (15.1) through (15.11) in the text, adjusted for the presence of the junk bonds. Let* J *stand for the market value of the junk bonds and* X the portion of its interest that is deductible *and* *then, following the solution manual for 15.1, the value of the levered firm is*

$$\begin{aligned}
V_L &= V_U + \tau_c B + X\tau_c J \\
\textit{where } V_U &= \textit{value of the unlevered (all equity) firm,} \\
\textit{so } V_U &= V_L - \tau_c B - X\tau_c J \\
\textit{But } V_U &= \frac{\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c)}{\rho} \\
\textit{where } EBIT(1-T) &= \textit{cash flow from operations (perpetual)} \\
\textit{and } \rho &= \textit{the cost of capital for the unlevered firm.} \\
\textit{Thus, } \rho &= \frac{\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c)}{V_L - \tau_c B - X\tau_c J} \\
\textit{But } V_L &= \frac{\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c)}{WAAC} \textit{ so} \\
\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c) &= WAAC \cdot V_L \textit{ and so} \\
\rho &= \frac{WAAC \cdot V_L}{V_L - \tau_c B - X\tau_c J} \\
&= \frac{WACC}{1 - \tau_c \frac{B}{V_L} - X\tau_c \frac{J}{V_L}} \\
&= \frac{.1827826}{1 - .40\frac{500}{5750} - .35(.40)\frac{5000}{5750}} \\
&= .2167 \textit{ or about } 21.7\%
\end{aligned}$$

This entire analysis uses Modigliani-Miller style assumptions except for the taxes. Thus if considerations involving (1) expected value of future loss of deductions on debt (beyond what's already assumed) (2) expected value of future defaults or (3) expected value of financial flexibility are important (as they are in fact likely to be for such a highly levered firm), then we have overstated the cost of capital for an all equity firm. Nothing in the given facts allows us to estimate the amount of this overstatement.

3. If a portfolio is to be constructed out of only two stocks, A and B, with $\sigma_A = .15$, $\sigma_B = .4$, and $\rho_{AB} = .6$, and if the risk free rate $r_f = .02$ and the expected returns on A and B are $r_A = .06$ and $r_B = .15$:

   (a) What is the proportion of A and B in the optimal portfolio that can be constructed from the two? **Solution** *By the formula demonstrated in class for the solution to the problem of creating the highest Sharpe ratio from a portfolio of two assests (or by setting $w_B = 1 - w_A$ and setting the derivative of the Sharpe ratio with respect to w_A equal to* *zero to find the maximum)*

$$\begin{aligned} w_A &= \frac{(r_A - r_f)\,\sigma_B^2 - (r_B - r_f)\,\rho_{AB}\sigma_A\sigma_B}{(r_A - r_f)\,\sigma_B^2 + (r_B - r_f)\,\sigma_A^2 - (r_A + r_B - 2r_f)\,\rho_{AB}\sigma_A\sigma_B} \\ &= .53666 \\ w_B &= \frac{(r_B - r_f)\,\sigma_A^2 - (r_A - r_f)\,\rho_{AB}\sigma_A\sigma_B}{(r_A - r_f)\,\sigma_B^2 + (r_B - r_f)\,\sigma_A^2 - (r_A + r_B - 2r_f)\,\rho_{AB}\sigma_A\sigma_B} \\ &= .46334 \end{aligned}$$

Those of you who arrived at $w_A = .7025$ and $w_B = .2975$ by all using the same ridiculous method starting with Sharpe ratio $= .36$, you got no credit at all because a simple check shows that

$$\begin{aligned} \frac{.7025(.06) + .2975(.15) - .02}{\left((.7025)^2\,(.15)^2 + (.2975)^2\,(.4)^2 + 2\,(.7025)\,(.2975)\,(.6)\,(.15)\,(.4)\right)^{\frac{1}{2}}} &= .3326, \\ \text{not} &= .36 \end{aligned}$$

If I could prove that you all colluded on the same obviously incorrect answer you could be subject to failing the course and perhaps other disciplinary action.

(b) If the expected return on the market and its standard deviation are $r_M = .095$ and $\sigma_M = .20$, would you prefer to hold just the market portfolio or to hold the portfolio that you constucted in part a.? Explain why. **Solution** *The Sharpe Ratio for the market is $\frac{.095-.02}{.2} = .375$. The Sharpe Ratio for the constructed portfolio is $\frac{.53666(.06)+.46334(.15)-.02}{\sqrt{(.53666)^2(.15)^2+(.46334)^2(.4)^2+2(.53666)(.46334)(.6)(.15)(.4)}} = .337$. The market portfolio has the higher Sharpe Ratio and is preferable.*

4. Capital Asset Pricing Model (CAPM)

(a) What correlations ρ_{AM} and ρ_{BM} between the returns on stocks A and B and the returns on the market would make the facts given in question 3. consistent with CAPM? **Solution** *According to CAPM, $r_A - r_f = \rho_{AM}\frac{\sigma_A}{\sigma_M}(r_M - r_f)$ so $\rho_{AM} = \frac{\sigma_M(r_A-r_f)}{\sigma_A(r_M-r_f)} = \frac{.20(.06-.02)}{.15(.095-.02)} = .7111$; similarly $\rho_{BM} = \frac{.20(.15-.02)}{.40(.095-.02)} = .8667$.*

(b) If both of those correlations instead were $\rho_{AM} = \rho_{BM} = 0.75$, what would CAPM predict the expected return to be on the portfolio you constructed in 3.a.? **Solution** *According to CAPM the portfolio re-*

turn would be

$$\begin{aligned} w_A r_A + w_B r_B &= w_A \left(r_f + \rho_{AM} \frac{\sigma_A}{\sigma_M} (r_M - r_f) \right) + w_B \left(r_f + \rho_{BM} \frac{\sigma_B}{\sigma_M} (r_M - r_f) \right) \\ &= (.53666) \left(.02 + .75 \frac{.15}{.20} (.095 - .02) \right) \\ &\quad + (.46334) \left(.02 + .75 \frac{.40}{.20} (.095 - .02) \right) \\ &= .094766 \end{aligned}$$

(c) In that case (4.b. above) is the portfolio you constucted in 3.a. still the optimal one that can be built from these two assets? Why or why not? (You can do an easy check on this, without going all through the optimization calculation again.) **Solution** *If it is optimal, it should have the highest Sharpe Ratio. The Sharpe Ratio is* $\frac{.094766 - .02}{\sqrt{(.53666)^2 (.15)^2 + (.46334)^2 (.4)^2 + 2 (.53666)(.46334)(.6)(.15)(.4)}} = .30851$.Make a slight change in the weights to .54 *and* .46 *giving a Sharpe ratio of*

$$\begin{aligned} &\frac{.54 \left(r_f + \rho_{AM} \frac{\sigma_A}{\sigma_M} (r_M - r_f) \right) + .46 \left(r_f + \rho_{BM} \frac{\sigma_B}{\sigma_M} (r_M - r_f) \right) - .02}{\sqrt{(.54)^2 (.15)^2 + (.46)^2 (.4)^2 + 2 (.54) (.46) (.6) (.15) (.4)}} \\ = &\frac{.54 \left(.02 + .75 \frac{.15}{.20} (.095 - .02) \right)}{\sqrt{(.54)^2 (.15)^2 + (.46)^2 (.4)^2 + 2 (.54) (.46) (.6) (.15) (.4)}} \\ &+ \frac{+.46 \left(.02 + .75 \frac{.40}{.20} (.095 - .02) \right) - .02}{\sqrt{(.54)^2 (.15)^2 + (.46)^2 (.4)^2 + 2 (.54) (.46) (.6) (.15) (.4)}} \\ = &.30867 \end{aligned}$$

which is higher, meaning that the original one could not have been optimal (note the importance of keeping a fair amount of accuracy if you want to reason this way).