**Math 5621 Financial Math II**
**Fall 2015**
**Final Exam Solutions**
December 11 to December 16, 2015

This is an open book take-home exam. You may consult any books, notes, websites or other printed material that you wish. Having so consulted then submit your own answers as written by you.

Do NOT under any circumstances consult with any other person. Do NOT under any circumstances cut and paste any material from another source electronically into your answer. Do NOT under any circumstances electronically copy and paste from a spreadsheet that was not created entirely by you. Failure to follow these rules will be grounds for a failing grade for the course.

Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The four questions will be equally weighted in the grading. Please return the completed exams by 5 PM Wednesday, December 16 to my mailbox in the department office, under my office door MSB408, or by email.

1. A commodities trading firm has the following market value balance sheet (in millions of $):

| | ASSETS | | LIABILITIES |
|---|---|---|---|
| short-term | 150 | short term | 150 |
| treasury bonds | 200 | short commodity positions | 650 |
| long commodity positions | 650 | equity | 200 |
| | 1,000 | | 1,000 |

The standard deviations and correlations between returns on the asset and liability holdings are:

$\sigma(sta) = .02$ $\rho(sta, tb) = 0$ $\rho(sta, lcp) = 0$ $\rho(sta, stl) = 0$ $\rho(sta, scp) = 0$

$\sigma(tb) = .02$ $\rho(tb, lcp) = .8$ $\rho(tb, stl) = 0$ $\rho(tb, scp) = .8$

$\sigma(lcp) = .25$ $\rho(lcp, stl) = 0$ $\rho(lcp, scp) = -.7$

$\sigma(stl) = .02$ $\rho(stl, scp) = 0$

$\sigma(scp) = .35$

(a) What is the standard deviation of returns on equity

Solution

By equation (5.32), dividing everything in the balance sheet by the equity 200 so that weights add up to 1, the variance of the equity return is

$$\mathbf{w}^T\boldsymbol{\sigma}\mathbf{w} = \left\langle\ .75 \quad 1.0 \quad 3.25 \quad -.75 \quad -3.25\ \right\rangle$$

$$\text{(a)} \left\langle \begin{array}{ccccc} .02^2 & 0 & 0 & 0 & 0 \\ 0 & .02^2 & .8(.02)(.25) & 0 & .8(.02)(.35) \\ 0 & .8(.02)(.25) & .25^2 & 0 & -.7(.25)(.35) \\ 0 & 0 & 0 & .02^2 & 0 \\ 0 & .8(.02)(.35) & -.7(.25)(.35) & 0 & .35^2 \end{array} \right\rangle \left\langle \begin{array}{c} .75 \\ 1.0 \\ 3.25 \\ -.75 \\ -3.25 \end{array} \right\rangle$$

$= 3.23842$ and the standard deviation is $\sqrt{3.23842} = 1.7996$

(b) Suppose the firm wants to hedge by taking a position in treasury futures. If the price for a futures contract is $V_{tf} = \$90,000$ for each $\$100,000$ treasury future contract and

$\sigma(tf) = .35$ $\rho(tf, sta) = 0$ $\rho(tf, tb) = .9$ $\rho(tf, lcp) = .5$ $\rho(tf, stl) = 0$ $\rho(tf, scp) = -.3$

then should the treasury futures position be long or short? How many contracts should they buy or sell? How much is the standard deviation of equity reduced?

Solution

Equation (5.33) gives $N = -\frac{1}{.09(.35)}\left(200\,(.9)\,(.02) + 650\,(.5)\,(.25) - 650\,(-.3)\,(.35)\right) = -4,860$, a short position in futures contracts. These have value $-4,860(.09) = -437.4$. The effect on equity is 0, with cash increasing by 437.4 received in the short sale and the new short position being $-437.4$. Now the variance is (remembering to divide by 200 equity)

$$\left\langle\ 2.937 \quad 1.0 \quad 3.25 \quad -.75 \quad -3.25 \quad -2.187\ \right\rangle$$

$$\left\langle \begin{array}{cccccc} .02^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & .02^2 & .8(.02)(.25) & 0 & .8(.02)(.35) & .9\,(.02)\,(.35) \\ 0 & .8(.02)(.25) & .25^2 & 0 & -.7(.25)(.35) & .5\,(.25)\,(.35) \\ 0 & 0 & 0 & .02^2 & 0 & 0 \\ 0 & .8(.02)(.35) & -.7(.25)(.35) & 0 & .35^2 & -.3\,(.35)\,(.35) \\ 0 & .9\,(.02)\,(.35) & .5\,(.25)\,(.35) & 0 & -.3\,(.35)\,(.35) & .35^2 \end{array} \right\rangle \left\langle \begin{array}{c} 2.937 \\ 1.0 \\ 3.25 \\ -.75 \\ -3.25 \\ -2.187 \end{array} \right\rangle$$

$= 2.65565$ and the standard deviation is $\sqrt{2.65565} = 1.6296$

2. For simplicity's sake, your investment firm wants to consider the entire world market for investments to consist of (a) the entire American market (North and South) as a single asset (b) the entire European market as a single asset (c) the entire Asian market as a single asset. (They might be making a big mistake to ignore the African market, but that's the decision they made.) The risk-free rate is .0025 and the expected rates of return for each of the three assets and the covariance matrix among them are

$$\begin{array}{ccccc} & \mathbf{i} = & (\mathbf{a}) & (\mathbf{b}) & (\mathbf{c}) \\ & \mathbf{r}_i = & .0440 & .0122 & .1052 \\ & \boldsymbol{\sigma}_{i,j} = & & & \\ \mathbf{j} = & (\mathbf{a}) & .04 & -.003 & .048 \\ & (\mathbf{b}) & -.003 & .0225 & .012 \\ & (\mathbf{c}) & .048 & .012 & .16 \end{array}$$

What is the optimal portion of each world asset in your firm's investment portfolio?

Solution

see class notes on CAPM

The weight vector will be $\mathbf{w} = \frac{\boldsymbol{\sigma}^{-1}(\mathbf{r}-r_f\mathbf{1})}{\mathbf{1}^T\boldsymbol{\sigma}^{-1}(\mathbf{r}-r_f\mathbf{1})}$

$$\boldsymbol{\sigma}^{-1} = \left\langle \begin{array}{ccc} .04 & -.003 & .048 \\ -.003 & .0225 & .012 \\ .048 & .012 & .16 \end{array} \right\rangle^{-1} = \left\langle \begin{array}{ccc} 42.4028 & 12.9564 & -13.6926 \\ 12.9564 & 50.2552 & -7.6561 \\ -13.6926 & -7.6561 & 10.932 \end{array} \right\rangle$$

$$\boldsymbol{\sigma}^{-1}(\mathbf{r} - r_f\mathbf{1}) = \left\langle \begin{array}{ccc} 42.4028 & 12.9564 & -13.6926 \\ 12.9564 & 50.2552 & -7.6561 \\ -13.6926 & -7.6561 & 10.932 \end{array} \right\rangle \left\langle \begin{array}{c} .0415 \\ .0097 \\ .1027 \end{array} \right\rangle = \left\langle \begin{array}{c} .479163 \\ .238885 \\ .480209 \end{array} \right\rangle$$

So the weights are $\left\langle \begin{array}{c} .479163 \\ .238885 \\ .480209 \end{array} \right\rangle \div (.479163 + .238885 + .480209) = \left\langle \begin{array}{c} .4 \\ .2 \\ .4 \end{array} \right\rangle$,

(a) 40% (b) 20% (c) 40%

3. This problem involves a European put option expiring in $T$ years with strike price $K$ on an asset whose value today is $S_0$. The risk free rate (continuously compounded) is $r$.

   (a) Write down the Black-Scholes formula for $V_0$ the value today of the European put option

Solution

By put-call parity

$$\begin{aligned} V_0 + S_0 &= C_0 + e^{-rT}K \\ \text{where } S_0 &= \text{value of underlying and} \\ C_0 &= S_0\Phi\left(\frac{\ln\frac{S_0}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) - e^{-rT}K\Phi\left(\frac{\ln\frac{S_0}{K} + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right) \\ & \quad \text{is the value of the corresponding call option} \\ \text{so } V_0 &= e^{-rT}K\left[1 - \Phi\left(\frac{\ln\frac{S_0}{K} + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right)\right] \\ & \quad -S_0\left[1 - \Phi\left(\frac{\ln\frac{S_0}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right)\right] \end{aligned}$$

(b) Write down the $N$-stage binomial tree formula for $V_0$ the value today of the European put option.

Solution

By the definition of a european put option, the value is the sum of the present values of each possible $(K - S_T) > 0$ multiplied by the probability of that value occurring at time $T$ in the tree:

$$\begin{aligned}
V_0 &= \sum_{j=0}^{j_K-1} e^{-rT}\left(K - u^j d^{N-j} S_0\right)\left(\frac{1}{2}\right)^N \frac{N!}{j!\,(N-j)!} \\
\text{where } j_K &> K \text{ is the first such value with } u^{j_K} d^{N-j_K} S_0 > K \\
u &= e^{\frac{T}{N}\left(r-\frac{1}{2}\sigma^2\right)+\sigma\sqrt{\frac{T}{N}}};\ d = e^{\frac{T}{N}\left(r-\frac{1}{2}\sigma^2\right)-\sigma\sqrt{\frac{T}{N}}} \\
\text{and } \left(\frac{1}{2}\right)^N \frac{N!}{j!\,(N-j)!} &= \text{binomial probability of } j \text{ up } N-j \text{ down}
\end{aligned}$$

so $V_0 = e^{-rT} K \sum_{j=0}^{j_K-1}\left(\frac{1}{2}\right)^N \frac{N!}{j!\,(N-j)!} - S_0 \sum_{j=0}^{j_K-1} e^{-rT} u^j d^{N-j}\left(\frac{1}{2}\right)^N \frac{N!}{j!\,(N-j)!}$

(c) Explain why the term involving $S_0$ in the Black-Scholes formula for the value today of the European put option is NOT multiplied by $e^{-rT}$.

Solution

Approximately

$$\left[1 - \Phi\left(\frac{\ln\frac{S_0}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right)\right] \approx \sum_{j=0}^{j_K-1} e^{-rT} u^j d^{N-j}\left(\frac{1}{2}\right)^N \frac{N!}{j!\,(N-j)!}$$

or exactly, using the lognormal and after integrating and simplifying as in the class notes,

$$\begin{aligned}
\left[1 - \Phi\left(\frac{\ln\frac{S_0}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right)\right] &= \int_{-\infty}^{\frac{K}{S_0}} e^{-rT} e^x \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - T\left(r-\frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}}\right)^2} dx \\
&= \mathbb{E}\left[e^{-rT}\frac{S_T}{S_0} | S_T < K\right] \mathbb{P}\left[S_T < K\right]
\end{aligned}$$

So the term involving $S_0$ actually does have a factor $e^{-rT}$ contained within it. The factor $e^{-rT}$ is disguised within the integration that gives rise to the coefficient $\left[1 - \Phi\left(\frac{\ln\frac{S_0}{K}+rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right)\right]$.

4. Look at formula (14.18) on page 504 of the textbook for the value of an all-equity firm experiencing finite supernormal growth

   (a) Is the term

   $$\frac{EBIT_1\left(1-\tau_c\right)}{k_u}\left\{\frac{Kr-k_uK}{k_u-Kr}\left[1-\left(\frac{1+Kr}{1+k_u}\right)^N\right]\right\}$$

   equal to the present value today of $V_N$, the present value at the horizon time $N$ of all the free cash flow after the horizon time $N$?

   Solution

   No

   (b) Given a reason for or analysis of why or why not in part a.

   Solution

   The textbook is assuming that maintenance investment exactly equals depreciation plus amortization, so

   $$\begin{aligned}
EBIT_1\left(1-\tau_c\right) &= \text{current level of after-tax free cash-flow}\\
K &= \text{ratio of new investment to free cash-flow}\\
r-k_u &= \text{rate of cash return minus cost of capital}\\
EBIT_1\left(1-\tau_c\right)\left(Kr-k_uK\right) &= \text{amount of cash return minus cost of}\\
&\quad\ \text{capital on new investment at current leve}\\
\frac{EBIT_1\left(1-\tau_c\right)}{k_u}\left(Kr-k_uK\right) &= \text{perpetuity for cash return minus cost of}\\
&\quad\ \text{capital on new investment at current leve}\\
\frac{1}{k_u-Kr}\left[1-\left(\frac{1+Kr}{1+k_u}\right)^N\right] &= N\text{ year annuity, growing at rate } Kr\\
\frac{EBIT_1\left(1-\tau_c\right)}{k_u}\left\{\frac{Kr-k_uK}{k_u-Kr}\left[1-\left(\frac{1+Kr}{1+k_u}\right)^N\right]\right\} &= \text{PV of cash returns minus cost of capital}\\
&\quad\ \text{in perpetuity on new investments startin}\\
&\quad\ \text{at current level and growing at rate } Kr \text{ fo}\\
&\quad\ N \text{ years.}
\end{aligned}$$

   This includes, of course, the present value today of the present value at the horizon time $N$ of all the free cash flow after the horizon time $N$ on new investments made from time 1 to time $N$. But it also includes the present value today of free cash flow from time 1 to time $N$ on new investments made from time 1 to time $N$. And it excludes the present value today of the present value at the horizon time $N$

of the free cash flow after the horizon time $N$ at the current level of after-tax free cash flow. Another way to see this is to just look at the term $\frac{EBIT_1(1-\tau_c)}{k_u}$. It is the perpetuity for the current level of free cash flow for all times before and after the horizon time $N$. If I subtract that away from the current value in formula (14.18) I must still have remaining the present value of free cash flows prior to the horizon time $N$ from new investments made prior to that time, and I will be subtracting away the present value of some free cash flows after the horizon time $N$.