Math 5621 Financial Math II
Fall 2013
Final Exam Solutions
December 6 to December 11, 2013

This is an open book take-home exam. You may consult any books, notes, websites or other printed material that you wish. Having so consulted then submit your own answers as written by you.

Do NOT under any circumstances consult with any other person. Do NOT under any circumstances cut and paste any material from another source electronically into your answer. Do NOT under any circumstances electronically copy and paste from a spreadsheet that was not created entirely by you. Failure to follow these rules will be grounds for a failing grade for the course.

Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The ten questions will be equally weighted in the grading. Please return the completed exams by 5 PM Wednesday, December 11 to my mailbox in the department office, under my office door MSB408, or by email.

1. Do both parts of problem 5.14 in the textbook. Don't just copy the solution manual blindly because it has errors in both (a) and (b). A hint for the error in (b): Whoever wrote the solution manual forgot that when you sell something short you get cash in return. You should have no problem discovering the error in part (a).

   Solution

   (a) the variance of the equity return is $\mathbf{w}^T \boldsymbol{\sigma} \mathbf{w} =$

$$\left\langle \begin{array}{ccccc} 1 & 2 & 7 & -.5 & -8.5 \end{array} \right\rangle \left\langle \begin{array}{ccccc} .02^2 & 0 & 0 & 0 & 0 \\ 0 & .04^2 & .8(.04)(.07) & 0 & .3(.04)(.03) \\ 0 & .8(.04)(.07) & .07^2 & 0 & .2(.07)(.03) \\ 0 & 0 & 0 & .02^2 & 0 \\ 0 & .3(.04)(.03) & .2(.07)(.03) & 0 & .03^2 \end{array} \right\rangle \left\langle \begin{array}{c} 1 \\ 2 \\ 7 \\ -.5 \\ -8.5 \end{array} \right\rangle$$

   $= .312525$ (the solution manual miscalculated this), and the standard deviation is $\sqrt{.312525} = .559039$.

   (b) Equation (5.33) gives $N = -3340$, a short position in futures contracts. These have value $-3340(.09) = -300.6$. The effect on equity is 0, with cash increasing by 300.6 received in the short sale and the new short position being -300.6. The correlation with government bonds is .9, from the $\frac{90{,}000}{100{,}000}$ price relationship. Now the variance is

$$\left\langle\begin{array}{cccccc} 4.006 & 2 & 7 & -.5 & -8.5 & -3.006 \end{array}\right\rangle \cdot$$

$$\left\langle\begin{array}{cccccc} .02^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & .04^2 & .8(.04)(.07) & 0 & .3(.04)(.03) & .9\,(.04)\,(.08) \\ 0 & .8(.04)(.07) & .07^2 & 0 & .2(.07)(.03) & .5\,(.07)\,(.08) \\ 0 & 0 & 0 & .02^2 & 0 & 0 \\ 0 & .3(.04)(.03) & .2(.07)(.03) & 0 & .03^2 & 3\,(.03)\,(.08) \\ 0 & .9\,(.04)\,(.08) & .5\,(.07)\,(.08) & 0 & .3\,(.03)\,(.08) & .08^2 \end{array}\right\rangle \left\langle\begin{array}{c} 4.006 \\ 2 \\ 7 \\ -.5 \\ -8.5 \\ -3.006 \end{array}\right\rangle$$

$= .260703755$ and the standard deviation is $\sqrt{.260703755} = .510592$

2. Consider a put option with an exercise price of 17, expiring three years from today, on an underlying asset which pays no dividends, has a value of 15 today, and a standard deviation of annual return equal to .50. Use a binomial model with $N = 6$ steps and probabilities $q_u = q_d = \frac{1}{2}$ at each step. (Do **NOT** use a binomial model using the formulas in the textbook.) Use a risk-free annual rate of return of 0.5% for a three-year horizon.

   (a) What would be wrong with using u and d determined by the formulas in the textbook, given the other requirements in this question?
   Solution See risk neutral pricing class notes The textbook u and d come from an assumption that $ud = 1$ and that q_u and q_d take specific complicated values. Here, I have told you to assume that $q_u = q_d = \frac{1}{2}$.

   (b) What is the value of the put option today if it is an American put option?
   Solution See spreadsheet 6.355929786

   (c) Logically, why is the value in (b) greater than 2, the amount I could realize by exercising the option immediately?
   Solution See risk neutral pricing class notes It includes the present value of the possible choice to exercise on even more favorable terms at some point in the future.

   (d) What is the first time that it might possibly be optimal to exercise this American put option, according to this binomial model?
   Solution See spreadsheet, at time 1.5 at the bottom node the current exercise value exceeds the present value of possible future choices to exercise on more favorable terms in the future.

   (e) At time $t = .5$, if you are at the up node of the tree will the value of the risk-free bonds in the replicating portfolio for a put option, after rebalancing the portfolio, be larger for an American put option or for a European put option? By how much?
   Solution See spreadsheet, it is larger by 0.046232128 for the American option.

   (f) Logically, why is the value of the risk-free bonds in the replicating portfolio in (e) larger for whichever option you chose in the answer?

Solution This is tricky: (1) The value of the option is higher for the American because there are more options (choices) in the future with the American, and options (choices) have value. (2) The short position in the underlying is larger for the American because the Δ is larger in absolute value (because the value of the future options in the American increase with lower values of the underlying, making the difference in the two possible future put values larger, increasing the Δ compared to the European). (3) The value of the risk free bonds is the option value plus the absolute value of the short position, so (1) and (2) force it to be larger for the American.

3. With the following expected returns and covariance matrix what are the weights w_1,w_2, and w_3 of each of the three assets in the optimal portfolio assuming the risk free rate is .02? You don't have to prove your answer but you do have to show how you calculated it.

	j =	**1**	**2**	**3**
	$\mathbf{r}_j$=	.0435	.124	.200
	$\boldsymbol{\sigma}_{i,j}$=			
i =	**1**	.01	.009	0
	2	.009	.03	.02
	3	0	.02	.06

Solution see class notes on CAPM

The weight vector will be $\mathbf{w} = \frac{\boldsymbol{\sigma}^{-1}(\mathbf{r}-r_f\mathbf{1})}{\mathbf{1}^T\boldsymbol{\sigma}^{-1}(\mathbf{r}-r_f\mathbf{1})}$

$$\boldsymbol{\sigma}^{-1} = \left\langle \begin{array}{ccc} .01 & .009 & 0 \\ .009 & .03 & .02 \\ 0 & .02 & .06 \end{array} \right\rangle^{-1} = \left\langle \begin{array}{ccc} 153.173 & \text{-59.0810} & 19.6937 \\ \text{-59.0810} & 65.6455 & -21.8818 \\ 19.6937 & -21.8818 & 23.9606 \end{array} \right\rangle$$

$$\boldsymbol{\sigma}^{-1}(\mathbf{r} - r_f\mathbf{1}) = \left\langle \begin{array}{ccc} 153.173 & \text{-59.0810} & 19.6937 \\ \text{-59.0810} & 65.6455 & -21.8818 \\ 19.6937 & -21.8818 & 23.9606 \end{array} \right\rangle \left\langle \begin{array}{c} .0235 \\ .104 \\ .18 \end{array} \right\rangle = \left\langle \begin{array}{c} 1.00001 \\ 1.50000 \\ 2.50000 \end{array} \right\rangle$$

So the weights are $\left\langle \begin{array}{c} 1.00001 \\ 1.50000 \\ 2.50000 \end{array} \right\rangle \div (1.00001 + 1.50000 + 2.50000) = \left\langle \begin{array}{c} .2 \\ .3 \\ .5 \end{array} \right\rangle$

4. Given the following covariance matrix

	j =	**1**	**2**
i =	**1**	.04	$-.09$
	2	$-.09$	.06

calculate the covariance between a portfolio that has 25% in asset 1 and 75% in asset 2 and another portfolio that has 95% in asset 1 and 5% in asset 2.

Solution see 6.14

$$cov(X,Y) = \mathbf{w}_X^T \boldsymbol{\sigma} \mathbf{w}_Y = \left\langle \begin{array}{cc} .25 & .75 \end{array} \right\rangle \left\langle \begin{array}{cc} .04 & -.09 \\ -.09 & .06 \end{array} \right\rangle \left\langle \begin{array}{c} .95 \\ ..05 \end{array} \right\rangle$$

$$= \left\langle \begin{array}{cc} -.0575 & .0225 \end{array} \right\rangle \left\langle \begin{array}{c} .95 \\ .05 \end{array} \right\rangle = -.0535$$

5. Alpha Gaming has a current price of \$4 per share. You believe that the appropriate market capitalization rate for Alpha is 10%. Its annual sales are \$2, 400, 000, 000. Total annual expenses including depreciation, amortization, interest, and taxes are \$2, 100, 000, 000. On a book value basis debt is \$720, 000, 000. The payout ratio is 75%. The price/book ratio is 200%. There are 400 million shares outstanding. (a) What present value of growth opportunities (PVGO) is implied by Alpha's market valuation? (b) Grow or Die: what is the maximum possible growth rate Alpha Gaming can attain without raising any new equity capital?

   Solution

   (a) From class notes

$$\begin{aligned} SharePrice &= \frac{EPS}{r} + \frac{PVGO}{shares} \text{ so} \\ PVGO &= shares\left(SharePrice - \frac{EPS}{r}\right) \\ &= 400\left(4 - \frac{\frac{2400-2100}{400}}{.10}\right) \\ &= -1,400 \text{ million or} \\ &= -3.50 \text{ per share} \end{aligned}$$

   This means that the market is expecting either the company or its current profit margins, or both, to shrink in the future. "Grow or Die" is a pressing reality for this company if the market is correct.

   (b) "Without raising any new equity capital" means that we are asking what growth rate can be attained without any new stock issuance but with debt increasing at the assumed growth rate: in the class notes this was the "sustainable growth rate."

$$\begin{aligned} g &= \frac{NetIncome \cdot PlowbackRatio}{Equity_{beginning}} \text{ with } Equity \text{ on a book value basis.} \\ &= \frac{NetIncome \cdot PlowbackRatio}{(Equity_{ending} - NetIncome \cdot PlowbackRatio)} \\ &= \frac{(2400-2100)\cdot(1-.75)}{\left(\frac{4\cdot 400}{2} - (2400-2100)\cdot(1-.75)\right)} \\ &= .1034 \end{aligned}$$

   Of course, the company can grow at more than 10.34% but it will need to raise more equity or grow debt faster than the assumed growth rate, or both, to do so. A common but less accurate calculation would be

$$
\begin{aligned}
g &= \frac{NetIncome \cdot PlowbackRatio}{Equity} \\
&= \frac{(2400 - 2100) \cdot (1 - .75)}{\frac{4 \cdot 400}{2}} \\
&= .09375
\end{aligned}
$$

6. A company has net assets with a market value of $\$7,500,000$ and a financial structure involving 40% debt. The company believes that its current optimal financial structure should inolve 45% debt. The company is considering a new project that requires an investment of $\$2,375,000$. The company believes that afer taking on the project it will have an optimal capital structure requiring 50% debt. If the company's after tax $WACC$ is 15%, its marginal cost of new debt is 6% before tax, and its marginal tax rate is 40%, then what after tax rate of return does the project need to earn in order to to be acceptable, assuming that it will be financed optimally?
Solution

   Use subscripts b for the company before the project, p for the project itself, and a for the company after the project.

   To be acceptable the project needs to earn $WACC_p = \rho_p \left(1 - .40 \frac{\Delta B}{\Delta(S+B)}\right)$ by equation (15.12) where ρ_p is the cost of capital for the project if it used no debt and $\frac{\Delta B}{\Delta(S+B)}$ is the proportion of debt in the optimal project financing. Since the optimal financing before the project was $45\%(7,500) = 3,375$ debt and after the project $50\%(7,500 + 2,375) = 4,937.5$ debt, then the optimal debt for the project must be $4,937.5 - 3,375 = 1,562.5$. Then $\frac{\Delta B}{\Delta(S+B)}$ for the project is $\frac{1,562.5}{2,375} = .6579$ so $WACC_p = \rho_p (1 - .40 \cdot .6579) = .7368\rho_p$. We still need to figure out ρ_p.

   To do that we use the relationship $\rho_a = w_b\rho_b + w_p\rho_p$ where $w_b = \frac{7500}{7500+2375} = .7595$ and $w_p = \frac{2375}{7500+2375} = .2405$, and $\rho_b = \frac{WAAC_b}{(1 - .40 \cdot .40)} = \frac{.15}{.84} = .1786$ by (15.12) and the comments following it. The relationship comes from

CAPM:

$$\begin{aligned}
\beta_a &= w_b\beta_b + w_p\beta_p \text{ by linearity of covariances so} \\
\rho_a &= r_f + \beta_a (r_M - r_f) \\
&= (w_b + w_p) r_f + \left(w_b\beta_b + w_p\beta_p\right)(r_M - r_f) \\
&= w_b (r_f + \beta_b (r_M - r_f)) + w_p \left(r_f + \beta_p (r_M - r_f)\right) \\
&= w_b\rho_b + w_p\rho_p \text{ so we now know that} \\
\rho_p &= \frac{\rho_a - w_b\rho_b}{w_p} \\
&= \frac{\rho_a - .7595 \cdot .1786}{.2405} \\
&= \frac{\rho_a}{.2405} - .5640 \text{ so} \\
WACC_p &= .7368\rho_p \\
&= 3.0636\rho_a - .4156
\end{aligned}$$

Now we only need to figure out what ρ_a is. For that use

$$\begin{aligned}
\rho_a (1 - .40 \cdot .50) &= WACC_a \text{ at the optimal financial position} \\
&\quad \text{of 50\% debt} \\
&= .50k_a + .50 \cdot .60 \cdot .06 \\
&= .50 (w_b k_b + w_p k_p) + .50 \cdot .60 \cdot .06 \text{ where } k_b \\
&\quad \text{and } k_p \text{ are at their optimal financial positions} \\
&= .50 \left(.7595 \frac{WACC_b - .45 \cdot .60 \cdot .06}{.55} + \right. \\
&\quad \left. .2405 \frac{WAAC_p - .6579 \cdot .60 \cdot .06}{.3421} \right) + .50 \cdot .60 \cdot .06 \\
&\quad \text{where } WACC_b \text{ and } WAAC_p \text{ are at optimal positions} \\
&= .50 \left(.7595 \frac{.1786 (1 - .45 \cdot .40) - .45 \cdot .60 \cdot .06}{.45} + \right. \\
&\quad \left. .2405 \frac{3.0636\rho_a - .4156 - .6579 \cdot .60 \cdot .06}{.3421} \right) + .50 \cdot .60 \cdot .06 \\
.8\rho_a &= 1.0769\rho_a - .02649 \\
\rho_a &= .0957 \\
WACC_p &= 3.0636\rho_a - .4156 \\
&= .1224 \text{ which is the answer to the question}
\end{aligned}$$

7. With a WAAC or Opportunity Cost of Capital of 17.5% (a) is a project with the following cash flows financially acceptable? (b) Is it acceptable to your boss who (irrationally) won't accept "any project with payouts that have less than a 20% return"? In justifying your answer, be sure to

calculate (c) the Net Present Value (d) the IRR and (e) one other measure of the rate of return that helps you to answer (a) and (b). Finally, (f) be sure to explain to your boss why your answer to (b) fits his rule about 20%

t	$\mathbf{CF}_t$
0	-2
1	0
2	1
3	-10
4	3
5	3
6	5
7	5
8	5
9	5
10	-6

Solution

(a) The project is financially acceptable because (c) it has a positive $NPV = 0.34148125$ (see solution speadsheet).

(b) The project is probably not acceptable to my irrational boss because (d) the $IRR = 0.190001617 < .20$ (use EXCEL Solver, an internet root-finder, or any other method you choose to see that this is the only positive solution to the IRR polynomial, but an accumulation of the cash flows at the IRR shows that the project mixes investment and financing so the IRR is an invalid tool in this case.)

But (e) even the $ModifiedIRR = 0.189387845 < .20$ (see solution spreadsheet ... modify the cash flows yourself, as shown in class, then the EXCEL IRR function succeeds, and an accumulation verifies that at the $ModifiedIRR$ the modified cash flows represent a pure investment project.)

Beware: the EXCEL MIRR function does not give the correct $ModifiedIRR$ as defined in class! It gives a lazy analyst's answer that fails to look at the net project position (investment versus financing) at each point in time. This is a good lesson. As an analyst, *never* rely on a label in a piece of software! You are responsible to verify for yourself what the tools you choose to use are doing. In this case, an EXCEL Help query on its MIRR function gives you the information that it is using a lazy shortcut.

(f) The spreadsheet shows an alternative method (but a mistaken one!) to get to a $ModifiedIRR = 0.246068 > .20$ if I want to help my boss get to the correct decision despite his irrational fixation on .20. However, this is not correct because it discounts a net investment postion at .175 for some points in time. We can justify a modification as not violatiing the .20 only if it discounts only net financing positions at .175 instead of the $ModifiedIRR$.

We might try one other way to convince our boss, in an honest way, to make the right decision. The spreadsheet shows a third modification of cash flows using .175 when it's a financing position, switching to the boss's .20 when it is an investment position and then calculating a residual $ResidualIRR = 0.14556367$ for the investment positions prior to the big -10 investment. We say to the boss: Look this project is an experiment for the first three years. If we like the result, we'll invest the -10 at time 3 (along with the accumulated 1.861119669 from the experimental period) and get a 20% return until we've repaid the entire investment, at which point the remaining cash flows will provide .175 financing to the rest of the company until they need to repay us so we can meet the -6 at the end. (If we don't like the result of the first three years we will just forego the -10 investment and walk away.) Unfortunately, to reap this harvest, you'll need to settle for just a $ResidualIRR = 0.14556367$ for the fairly small investment involved in the three year experiment. What do you say?

8. Assume your company has three classes of securities in its financing structure: \$500 million (market value) of senior perpetual debt with a market yield of 10%; \$5 billion (market value) of junior high yield (junk) perpetual debt with a market yield of 22%; and \$250 million (market value) of common equity with a market capitalization rate of 30%. Assume a corporate tax rate of 40% and that, because of the high proportion of junk financing, the tax authorities grant tax deductibility to only 35% of the interest on the high yield financing.

   (a) What is the firm's weighted average cost of capital (WACC)? **Solution** *Since the given facts included a market capitalization rate we can compute the* $WACC$ *directly from (15.19) in the text (generalized to include the junk debt) as* $WACC = \frac{250}{5750}.30 + \frac{500}{5750}(1-.40).10 + \frac{5000}{5750}(1-.35(.40)).22 = .1827826$ *or about 18.3%*

   (b) What can you conclude (if anything) about the cost of capital for an all-equity firm with the same operating risks? If you answer "nothing" give reasons.
   Solution *This is like exercise 15.1 but with junk debt instead of the preferred stock in the exercise, i.e. it is like equations (15.1) through (15.11) in the text, adjusted for the presence of the junk bonds. Let* J *stand for the market value of the junk bonds and* X the portion of its interest that is deductible *and then, following the solution manual*

for 15.1, the value of the levered firm is

$$\begin{aligned}
V_L &= V_U + \tau_c B + X\tau_c J \\
\textit{where } V_U &= \textit{value of the unlevered (all equity) firm,} \\
\textit{so } V_U &= V_L - \tau_c B - X\tau_c J \\
\textit{But } V_U &= \frac{\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c)}{\rho} \\
\textit{where } EBIT(1-T) &= \textit{cash flow from operations (perpetual)} \\
\textit{and } \rho &= \textit{the cost of capital for the unlevered firm.} \\
\textit{Thus, } \rho &= \frac{\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c)}{V_L - \tau_c B - X\tau_c J} \\
\textit{But } V_L &= \frac{\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c)}{WAAC} \textit{ so} \\
\mathbb{E}\left[EBIT(1-T)\right](1-\tau_c) &= WAAC \cdot V_L \textit{ and so} \\
\rho &= \frac{WAAC \cdot V_L}{V_L - \tau_c B - X\tau_c J} \\
&= \frac{WACC}{1 - \tau_c \frac{B}{V_L} - X\tau_c \frac{J}{V_L}} \\
&= \frac{.1827826}{1 - .40\frac{500}{5750} - .35(.40)\frac{5000}{5750}} \\
&= .2167 \textit{ or about } 21.7\%
\end{aligned}$$

This entire analysis uses Modigliani-Miller style assumptions except for the taxes. Thus if considerations involving (1) expected value of future loss of deductions on debt (beyond what's already assumed) (2) expected value of future defaults or (3) expected value of financial flexibility are important (as they are in fact likely to be for such a highly levered firm), then we have overstated the cost of capital for an all equity firm. Nothing in the given facts allows us to estimate the amount of this overstatement.

9. Your nuclear research department just discovered a way to turn lead into gold. With the price of gold at \$1300 per ounce this week you are quite excited and are making plans. You've already learned, for example, that you'll need to plan on annual spending of 1% of the value of any gold you produce just to store it safely and insure it. It's going to take you 12 years and a lot of money to implement the nuclear technology before you get your first output of gold, however, so you need to make an assumption about the price of gold 12 years from now in order to evaluate whether to go ahead with the investment today. The best experts you can find tell you that in their opinion the price of gold has a beta of 0, will be flat for the next two years while the market digests the Fed's tapering plans, but then it will advance 10% a year for 3 years reflecting the inflation of the dollar that must come sooner or later, followed by a steady 5%

annual increase thereafter. The annual risk free rate for a 12 year horizon is 1.75%. What is the present value today of an ounce of gold produced 12 years from now?
SOLUTION

Always trust the market price more than any expert's opinion, unless you are in the business of speculating (outguessing the market). Here your business is gold production, not speculation, so trust the market price of gold. With storage and insurance costs of 1% of the value of the gold per year the market is telling you that one ounce of gold twelve years from now can be produced without fail by putting $\$(.99)^{-12}\, Price - per - ounce - today$ worth of gold into insured storage today. It is a replicating portfolio guaranteed to pay off for one ounce of gold in twelve years. So the present value today of an ounce of gold produced twelve years from now is $\$(.99)^{-12}\, Price - per - ounce - today = \$1.1281781 \times 1300 \ = \$1466.63$.

10. The Black-Scholes formula for the price of a call option is

$$c = S\Phi(d_1) - e^{-rT} K\Phi(d_2)$$

where d_1 and d_2 are expressions that you can evaluate. Once you know d_1 the value of $\Phi(d_1)$ can be obtained from a spreadsheet function of normal probabiity values (or a published table of them.) Presumably, then, $\Phi(d_1)$ must be the probability of some event. Explain what that event is and why $\Phi(d_1)$ is its probability.
SOLUTION

$\Phi(d_1)$ is not the probability of any event. It is the conditional expected value $\mathbb{E}\left[e^{-rT}\frac{S_T}{S}|S_T > K\right]$. It is just an accident of the mathematical form of the lognormal density function that this complicated expected value can be found in a table of probability values. (note: using concepts not covered in class, it is possible to identify $\Phi(d_1)$ as the probability that $S_T > K$ under an alternative make-believe risk-neutral probability measure that corresponds to using S_t rather than a risk-free investment account to discount future cash flows, also known as "using S_t as the numeraire" .)