Math 5621 Financial Math II
Fall 2011
Final Exam Solutions - With corrections Dec. 19, 2011
December 9 to December 14, 2011

This is an open book take-home exam. You may consult any books, notes, websites or other printed material that you wish. Having so consulted then submit your own answers as written by you.

Do NOT under any circumstances consult with any other person. Do NOT under any circumstances cut and paste any material from another source electronically into your answer. Do NOT under any circumstances electronically copy a spreadsheet that was not created by you. Failure to follow these rules will be grounds for a failing grade for the course.

Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The eight questions will be equally weighted in the grading. Please return the completed exams by 5:30 PM Wednesday, December 14 to my mailbox in the department office, under my office door MSB408, or by email.

1. Build a binomial pricing model using the following assumptions:$r_f = .02$, $\sigma = .22$, $T = 2$, $N = 4$, $S_0 = 50$, and $q_u = 1/2$. (Do NOT use any other choice for q_u). Use the model to price an American Put option on S with strike price 65 expiring at $T = 2$. What is the value of the put? What is the value of the position held in S_0 at time 0 in the replicating portfolio?

   Solution

   See spreadsheet on course website: The value of the put is 15.69 which is derived from the tree. There are several instances of early exercise of the American put noted on the tree.

   The value of the position in S_0 at time 0 is a short position -41.36 which is equal to the value of S_0 multiplied by the delta in the tree at time 0, namely the difference in the V values one step ahead divided by the difference in the S values one step ahead.

$$\begin{aligned} \text{delta} &= \frac{V_u - V_d}{S_u - S_d} \\ \text{position} &= \frac{V_u - V_d}{S_u - S_d} S_0 \end{aligned}$$

   Since the V values rise as S declines, the delta is negative, indicating a short position.

2. Using the same assumptions as question #1, according to the Black-Scholes formula what is the value of the position held in S_0 at time 0 in the hedging portfolio for a European Put expiring at $T = 2$?

   Solution

See spreadsheet on the course website: The value of the position in S_0 at time 0 is a short position -35.60. It is equal to the value of S_0 multiplied by the delta in the Black-Scholes formula. The delta is the part of the Black-Scholes formula multiplied by S_0. For a European put option use put-call parity to see that the delta is the delta of the European call option minus 1. The delta of the European call option is the usual

$$\begin{aligned}
\Phi\left(d_1\right) &= \Phi\left(\frac{\ln\frac{S_0}{K}+(r_f+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\\
\text{delta} &= 1-\Phi\left(d_1\right)\\
\text{position} &= \left[1-\Phi\left(d_1\right)\right]S_0
\end{aligned}$$

The value of the position is not as short as in question #1 because in question #1 the replicating portfolio needs to reproduce some American early exercise values that produce greater differences than would a tree of values for a European put option. In addition, there is always some approximation error between a tree with a small number of steps N and a Black-Scholes value equivalent to $N=\infty$.

3. An investor's entire portfolio consists of 25% in the risk free investment, with a return of .03, and the balance invested in two stocks S_1 and S_2 where $r_1=.08$, $r_2=.12$, $\sigma_1^2=.04$, $\sigma_2^2=.09$, and $\rho_{12}=.6$. What is the expected return r_P and the variance of return σ_P^2 on the investor's entire portfolio? Don't just fill values into a formula for how much of each of the two stocks is in the portfolio unless you also prove that formula.

   Since it is the investor's entire portfolio, it must have a maximum Sharpe ratio. So we need to find the proportions w and $.75-w$ of S_1 and S_2 that maximize the Sharpe ratio of the portfolio. To do so, maximize the Sharpe ratio by setting its derivative with respect to w equal to 0.

$$\begin{aligned}
0 &= \frac{d}{dw}\frac{r_P-.03}{\sigma_P}\\
0 &= \frac{d}{dw}\frac{.25\left(.03\right)+w\left(.08\right)+\left(.75-w\right)\left(.12\right)-.03}{\left[w^2\left(.04\right)+\left(.75-w\right)^2\left(.09\right)+2w\left(.75-w\right)\left(.6\right)\left(.04\right)^{.5}\left(.09\right)^{.5}\right]^{.5}}\\
0 &= \frac{\left\{\begin{array}{c} -.04\left[w^2\left(.04\right)+\left(.75-w\right)^2\left(.09\right)+2w\left(.75-w\right)\left(.6\right)\left(.04\right)^{.5}\left(.09\right)^{.5}\right]^{.5}\\ -\left[\left(.0675\right)-\left(.04\right)w\right]\left(.5\right)\left[w^2\left(.04\right)+\left(.75-w\right)^2\left(.09\right)+2w\left(.75-w\right)\left(.6\right)\left(.04\right)^{.5}\left(.09\right)^{.5}\right]^{-.5}\cdot\\ \left[2w\left(.04\right)-2\left(.75-w\right)\left(.09\right)+2\left(.75-w\right)\left(.6\right)\left(.04\right)^{.5}\left(.09\right)^{.5}-2w\left(.6\right)\left(.04\right)^{.5}\left(.09\right)^{.5}\right]\end{array}\right\}}{\left[w^2\left(.04\right)+\left(.75-w\right)^2\left(.09\right)+2w\left(.75-w\right)\left(.6\right)\left(.04\right)^{.5}\left(.09\right)^{.5}\right]}\\
0 &= -.04\left[w^2\left(.04\right)+\left(.75-w\right)^2\left(.09\right)+2w\left(.75-w\right)\left(.6\right)\left(.04\right)^{.5}\left(.09\right)^{.5}\right]\\
&\quad -\left[\left(.0675\right)-\left(.04\right)w\right]\left(.5\right)\left[\begin{array}{c}2w\left(.04\right)-2\left(.75-w\right)\left(.09\right)\\ +2\left(.75-w\right)\left(.6\right)\left(.04\right)^{.5}\left(.09\right)^{.5}-2w\left(.6\right)\left(.04\right)^{.5}\left(.09\right)^{.5}\end{array}\right]
\end{aligned}$$

$$\begin{aligned}
0 &= -.02295w + .00070875 \\
w &= \frac{.00070875}{.002295} \\
w &= .3088 \\
.75 - w &= .4412 \\
r_P &= .25\,(.03) + w\,(.08) + (.75 - w)\,(.12) \\
r_P &= .0851 \\
\sigma_P^2 &= w^2\,(.04) + (.75 - w)^2\,(.09) + 2w\,(.75 - w)\,(.6)\,(.04)^{.5}\,(.09)^{.5} \\
\sigma_P^2 &= .0311
\end{aligned}$$

4. The cash flows from three projects A, B, and C are shown in the table. A and B are mutually exclusive. You can do neither or you can do one, you can't do both. C is independent of A and B. The discount rate is 10% for all three projects. Which project or combination of projects is preferable according to (a) the net present value method (b) the internal rate of return method (c) the modified internal rate of return method? Show the calculations for each method. .

Year	A	B	C
0	-1	-1	-1
1	0	1	0
2	2	0	0
3	-1	1	3

Solution

(a) $NPV_A = \frac{-1}{1.1^3} + \frac{2}{1.1^2} - 1 = -.0984$, $NPV_B = \frac{1}{1.1^3} + \frac{1}{1.1} - 1 = .6604$, $NPV_C = \frac{3}{1.1^3} - 1 = 1.254$. B is preferable to A and has positive NPV; so does C; so choose B and C together.

(b) IRR_A comes from solutions of $-\left(\frac{1}{1+r}\right)^3 + 2\left(\frac{1}{1+r}\right)^2 - 1 = 0$. There are three solutions: $r = -261.80\%$, -38.20%, and 0% (use EXCEL or Google a cubic solver). They are all less than 10%. Starting at -1, A begins as an investment project and, finishing at -1, A ends as a financing project. So A cannot be decided on pure IRR grounds; the IRR is acceptable for a financing project but not for an investment project and the project itself is mixed.

$IRR_B = 46.56\%$ and being greater than 10% is acceptable.

$IRR_C = 44.22\%$ and being greater than 10% is acceptable.

$IRR_{A\&C} = 32.47\%$ and being greater than 10% is acceptable.

$IRR_{B\&C} = 45.05\%$ and being greater than 10% is acceptable.

Tentatively choose B since it has highest IRR. But now look at changing your mind to $B\&C$. Being greater than 10% it is acceptable. Similarly, $A\&C$ is acceptable. $B\&C$ and $A\&C$ are both acceptable but they are mutually exclusive so choose the one of them with the higher IRR, namely choose $B\&C$.

(c) Cash flow for modified IRR for A is -1 0 1.090909 (from $2 + \frac{-1}{1.1}$). $ModIRR_A = 4.45\%$. This is less than 10% and is unacceptable (compare with IRR discussion above; the investment aspect of the project seems to prevail.) For all the other possibilities, modified IRR is the same as regular IRR above. The analysis comes out the same: choose $B\&C$.

An alternative way to look at A is the set up on page 33 of the text or problem 2.7: $\left[2 - (1 + ModIRR_A)^2\right](1.10) - 1 = 0$ which gives the same answer.

5. A company has net assets with a market value of $5,000,000 and a financial structure involving 50% debt. The company believes that its financial structure is optimal. The company is considering a new project that requires an investment of $1,250,000. Taking on the project will leave the company's overall relative operating risk exactly where it is before taking on the project. If the company's after tax $WACC$ is 15%, its marginal cost of new debt is 6% before tax, and its marginal tax rate is 40%, then what after tax rate of return does the project need to earn in order to to be acceptable, assuming that it will be financed optimally?

Solution

Compute an all-equity cost of capital for the company from

$$\begin{aligned} WAAC &= \rho\left(1 - \tau_c \frac{B}{B+S}\right) \\ .15 &= \rho\left(1 - .4\frac{.5\,(5,000,000)}{5,000,000}\right) \\ \rho &= .1875 \end{aligned}$$

Since the project will not change the operating risk this is also an appropriate all-equity cost of capital for the project. Since the company believes that it is optimally financed at 50% debt and the project has the same operating risk as the company then the optimal financing for the project is also 50% debt. Thus, the after tax rate of return the project needs to earn must be at least the $WAAC$ corresponding to 50% debt financing

$$\begin{aligned} WAAC &= \rho\left(1 - \tau_c \frac{B}{B+S}\right) \\ &= .1875\,(1 - .4(.5)) \\ &= .15 \end{aligned}$$

6. How would each of the following actions affect a firm's quick ratio?

Solution

Quick ratio is: (cash + s.t. securities + receivables)/current liabilities

(a) Sell inventory for cash: cash up, inventory doesn't matter, INCREASE Quick ratio

(b) Borrow short term from a bank to pay a supplier: s.t. debt up so current liabilities up, payables down so current liabilities down, NO EFFECT on Quick ratio

(c) collect an overdue bill from a customer: cash up, no effect on receivables (since it was overdue it was written off receiveables), INCREASE Quick ratio

(d) Buy more inventory for cash: cash down, inventory doesn't matter, DECREASE Quick ratio

7. Your firm's planning model has four scenarios for the future, A, B, C, and D. In scenario A, which your model assigns probability .4, the market returns .15 and your firm returns .20. In scenario B, which your model assigns probability .35, the market returns .05 and your firm returns zero. In scenario C, which your model assigns probability .2, the market return is .20 while your firm returns .50. In scenario D, which your model assigns the remaining probability, the market returns $-.15$ while the return on your frm is $-.30$. The risk free rate is 2%. Calculate the expected return and variance of return for your firm and for the market. Calculate the covariance of the return for your firm with the market return. What expected return will the market demand for the firm? If you believe the model is true in all respects, what prediction would you make for the price of the firm's shares relative to the market?

Solution

p	F	$\mathbb{E}[F]$	$F-.165$	$\mathbb{V}[F]$	M	$\mathbb{E}[M]$	$M-.11$	$\mathbb{V}[M]$	Cov_{FM}
.4	.2	.08	.035	.00049	.15	.06	.04	.00064	.00056
.35	0	0	(.165)	.00953	.05	.0175	(.06)	.00126	.00346
.2	.50	.10	.335	.02245	.20	.04	.09	,00162	.00603
.05	(.3)	(.015)	(.465)	.01081	(.15)	(.0075)	(.26)	.00338	.00605
Σ		.165		.04328		.11		.00690	.01610

$r_F = .165$, $\sigma_F^2 = .04328$, $r_M = .11$, $\sigma_M^2 = .0069$, $Cov_{FM} = .01610$

CAPM sugggests that the appropriate value for the firm's return is

$$\begin{aligned} r_F &= r_f + \beta_F (r_M - r_f) \\ &= .02 + \frac{.01610}{.0069} (.11 - .02) \\ &= .23 \end{aligned}$$

So your model is expecting a lower return for the firm (.165) than the market will expect (.23). To achieve its expected return, the market will have to bid your share price to a lower level (lower price increases the expected return.)

8. Alpha Gaming has a current price of \$20 per share. Its annual sales are \$12,000,000,000. Total annual expenses including depreciation, amortization, interest, and taxes are \$10,500,000,000. On a book value basis debt is \$3,600,000,000. The payout ratio is 75%. The price/book ratio is 300%. There are 400 million shares outstanding. What is the maximum possible growth rate Alpha Gaming can sustain without increasing its debt ratio or issuing new equity capital?

Solution

The key words are "without issuing new equity" (so it must be either the sustainable growth rate or the internal growth rate that we want) and "without increasing its debt *ratio*" (so it must be the sustainable growth rate that we want.) The usual textbook formulation of sustainable growth rate is

$$
\begin{aligned}
g &= PB(ROE) \\
&= (1-.75)\,\frac{NetIncome}{BookEquity} \\
&= .25\frac{12-10.5}{\left(\frac{20(.4)}{3}\right)} \\
&= .1406
\end{aligned}
$$

Slightly more accurate would be to use beginning of year $BookEquity$ in the ROE calculation

$$
\begin{aligned}
g &= PB(ROE_{boy}) \\
&= (1-.75)\,\frac{NetIncome}{BookEquity_{boy}} \\
&= .25\frac{12-10.5}{BookEquity - PB\,(NetIncome)} \\
&= .25\frac{1.5}{\left(\frac{20(.4)}{3}\right) - .25\,(1.5)} \\
&= .1636
\end{aligned}
$$