Math 5621
Financial Mathematics II
Mathematics of Corporate Finance
Fall 2010
Final Examination Solutions
December 10 - 15, 2010

This is a take-home examination due back to me by 5 PM Wednesday, December 15, in my department mail box, under my office door, or by email. You may consult any written source, including textbooks, notes, solution manuals, websites, or anything else written. You may NOT consult with any other person, which would result in your failing the course. Be sure to put your name on all papers submitted. Please show all of your work and give all reasoning and calculations associated with your answers; give me a chance to give partial credit on an incorrect answser. The seven questions will be equally weighted in the grading.

1. Consider a put option with an exercise price of 50, expiring four years from today, on an underlying asset whch pays no dividends, has a market value of 40 today, and a standard deviation of return equal to 0.40. Use a binomial model with $N = 8$ steps and probabilities $q_u = q_d = \frac{1}{2}$ at each step. (BE CAREFUL HERE: do not just copy formulas from the textbook, there will be no partial credit for use of formulas inappropriate to these instructions.) Use a risk-free annual rate of return of 2%. (BE CAREFUL HERE: an annual rate of return is not the same thing as a continuously compounded annual rate of return, a force of interest.)

   (a) What would be wrong with using the formulas for u and d in the textbook?

   The textbook gives formulas for u and d that correspond to values of q_u and q_d that are not equal to $\frac{1}{2}$.

   (b) What is the value of the put option today if it is an American option?

   17.40 (see spreadsheet)

   (c) Why is this value greater than 10, which I could get for exercising right away?

   The expected value of the present value of exercising the option in the future is greater than 10, using the risk-free rate and make-believe risk-neutral probabilitities. But these give proper values today since a replicating portfolio can be constructed to assure future payoffs equal to the ones generated using risk-neutral probabilities.

   (d) What is the earliest time it might possibly be best to exercise this American put option?

   $t = 1$ (see spreadsheet) is the first time that a node of the tree (the bottom one) has the current exercise value in excess of the expected value of the present value of exercising in the future.

(e) At time $t = 1$, if you are at the up-down node of the tree, will the value of the risk-free bonds in the replicating portfolio for a put option, after rebalancing the portfolio, be larger for the American put option or for the corresponding European put option? By how much?

Yes, by 1.97 (see spreadsheet)

(f) Why are the two values in (**e.**) different?

(1) The value of the option is higher for the American because there are more options (choices) in the future with the American, and options (choices) have value. (2) The short position in the underlying is larger for the American because the Δ is larger in absolute value (because the value of the future options in the American increase with lower values of the underlying, making the difference in the two possible future put values larger, increasing the American compared to the European). (3) The value of the risk free bonds is the option value plus the absolute value of the short position, so (1) and (2) force it to be larger for the American.

2. Suppose that the current price in the market for blank silicon wafers used as raw material for chip manufacturing is \$0.50 per wafer. Your engineering staff tell you that their best and most reliable consultants forecast that the price of blank silicon wafers will rise an average rate of 2% per year for the next 3 years, 6% per year for the following 2 years, and reach long run equilibrium at 3% per year thereafter forever. You think that the forecast makes a lot of sense. You expect to be using $400,000$ blank silicon wafer per year in your manufacturing operation for each of the next 20 years. Assume that blank silicon wafers have a $\beta = 0$, that the risk free rate is 1% for the next three years and 3% thereafter forever, and that any excess stock of silicon wafers from year to year can be stored for a negligible cost. For each of the the next 20 years you have purchased a European call option expiring at the end of that year on $400,000$ blank silicon wafers with a strike price of \$0.75 per wafer to hedge your exposure to a rise in price. For each of the next 20 years you have sold short a European put option expiring at the end of that year on $400,000$ blank silicon wafers with a strike price of \$0.75 per wafer in order to help finance the call option purchase. What is the value *today* of your net position in all of these options?

   Since the cost of storing silicon wafers is assumed to be negligible, a position (long or short) in silicon wafers can be treated like a position in any other underlying asset. Current market values are superior to any expert forecast as a measure of the present value today of any future position in a tradable underlying asset. That means we can use option pricing theory for this problem, with the current market value of silicon wafers as the present value today of any future position in silicon wafers.

Put-call parity gives directly the value of the options positions described:

$$\begin{aligned} & \text{(value of call today)} - \text{(value of put today)} \\ = \; & \text{(value of underlying today)} - \text{(present value at risk-free rate of future strike price)} \end{aligned}$$

We can add this up over each of next 20 year's options:

$$\begin{aligned} & \text{Position Value} \\ = \; & (20)(.50)(400,000) - \left(\frac{1}{1.01} + \frac{1}{1.01^2} + \frac{1}{1.01^3}(1 + \frac{1}{1.03} + \frac{1}{1.03^2} + ... + \frac{1}{1.03^{17}}) \right) (.75)(400,000) \\ = \; & 4,000,000 - \left(\frac{1 - \frac{1}{1.01^3}}{.01} + \frac{1}{1.01^3} \frac{1 - \frac{1}{1.03^{17}}}{.03} \right) (.75)(400,000) \\ = \; & -\$715,967 \end{aligned}$$

3. The risk free rate is 2%. Portfolio A has an expected return of 11% and a standard deviation of returns of 64%. Portfolio B has an expected return of 8% and a standard deviation of returns of 25%.

   (a) From a risk/reward perspective which of the two portfolios is superior? Why?
   Sharpe Ratio of A $= \frac{.11-.02}{.64} = .141$, Sharpe Ratio of B $= \frac{.08-.02}{.25} = .240$, which is higher, so Portfolio B is superior from a risk-reward perspective.

   (b) Suppose the returns from the two portfolios have a correlation coefficient of 0.4. What is the optimal allocation ratio to each of A and B in a new portfolio to be constructed as a combination of a portion of A and a portion of B?
   Let Portfolio P be α of Portfolio A and $1 - \alpha$ of Portfolio B.

$$\begin{aligned} r_P &= .11\alpha + .08(1-\alpha) \\ &= .03\alpha + .08 \\ r_P - .02 &= .03\alpha + .06 \\ \sigma_P &= \left((.64)^2 \alpha^2 + (.25)^2 (1-\alpha)^2 + 2\,(.4)\,(.64)\,(.25)\,\alpha\,(1-\alpha) \right)^{.5} \\ &= \left(.4096\alpha^2 + .0625\,(1-\alpha)^2 + .128\left(\alpha - \alpha^2\right) \right)^{.5} \\ &= \left(.3441\alpha^2 + .003\alpha + .0625 \right)^{.5} \end{aligned}$$

   The Sharpe Ratio of P is maximum (so P is optimal) at the value of α where

$$0 = \frac{d \frac{r_P - .02}{\sigma_P}}{d\alpha}.$$

Taking the derivative

$$
\begin{aligned}
0 &= \frac{\frac{d(r_P-.02)}{d\alpha}\sigma_P - (r_P - .02)\frac{d\sigma_P}{d\alpha}}{\sigma_P^2} \\
0 &= \frac{(.03)\,\sigma_P - (.03\alpha + .06)\,(.5)\,\sigma_P^{-1}\,(.6882\alpha + .003)}{\sigma_P^2} \\
0 &= \frac{(.03)\,\sigma_P^2 - (.03\alpha + .06)\,(.3441\alpha + .0015)}{\sigma_P^3} \\
0 &= (.03)\,\sigma_P^2 - (.03\alpha + .06)\,(.3441\alpha + .0015) \\
0 &= (.03)\,\left(.3441\alpha^2 + .003\alpha + .0625\right) \\
&\quad - (.03\alpha + .06)\,(.3441\alpha + .0015) \\
0 &= -(.020601)\,\alpha + .001785 \\
\alpha &= .08665 \\
1-\alpha &= .91335
\end{aligned}
$$

This is the same result as using the formula derived in class

$$
\begin{aligned}
\alpha &= \frac{(r_A - r_f)\,\sigma_B^2 - (r_B - r_f)\,\rho\sigma_A\sigma_B}{(r_A - r_f)\,\sigma_B^2 + (r_B - r_f)\,\sigma_A^2 - (r_A + r_B - 2r_f)\,\rho\sigma_A\sigma_B} \\
&= \frac{(.11 - .02)\,(.25)^2 - (.08 - .02)\,(.4)\,(.64)\,(.25)}{(.11 - .02)\,(.25)^2 + (.08 - .02)\,(.64)^2 - (.11 + .08 - 2\,(.02))\,(.4)\,(.64)\,(.25)}
\end{aligned}
$$

The Sharpe Ratio of P is

$$
\frac{(.08665)\,(.11) + (.91335)\,(.08) - .02}{\left((.08665)^2\,(.64)^2 + (.91335)^2\,(.25)^2 + 2\,(.4)\,(.08665)\,(.91335)\,(.64)\,(.25)\right)^{.5}} = .245
$$

which is superior to either Portfolio A or Portfolio B alone.

4. Gimmel Inc. has a beta of 0.5 on its equity, 40% debt in its capital structure, with the debt being valued by the market as essentially risk-free at a 5% pre-tax annual yield. The expected return on the entire market is 13%. Gimmel is considering a project called Gamma to develop a chain of high-end urban retail outlets for its products that it expects will produce a cash yield of 25% annually on an after tax basis. The main competitor will be Himmel Inc., which is thought to have similar risk characteristics as project Gamma. Himmel's equity beta is 2.2 and it has 25% debt in its capital structure. Assume that the marginal tax rate for both companies is 45% and that Gamma will be funded with 40% debt and 60% from retained earnings. From a purely financial perspective should Gimmel proceed with the Gamma project? Give a specific financial analysis and reason.

   For Himmel, $r_S = .05 + 2.2(.13 - .05) = .226$, $WAAC = .25(1 - .45)(.05) + .75(.226) = .1764$, and $.1764 = \rho(1 - .45(.25))$ so $\rho = .1988$ using (15.12).

For Gamma at 40% debt, required $WACC = .1988(1 - .45(.40)) = .1630$ so with projected after-tax cash yield of 25%, Gamma easily exceeds the required $WACC$ and from a financial perspective can proceed.

5. The Black-Scholes formula for the price of a European call option is

$$c = S\Phi(d_1) - e^{-rT} K\Phi(d_2)$$

where d_1 and d_2 are certain expressions that you can evaluate. Once you know d_1, the value $\Phi(d_1)$ can be obtained from a table of normal probability distribution values (or a computer algorithm to obtain those values). Therefore, $\Phi(d_1)$ must be the probability of some event occuring or some state of affairs being realized. Correct? Explain exactly what that event or state of affairs is.

$\Phi(d_1)$ is not the probability of any event, at least not under real-world probabilities and not under our make-believe risk-neutral probabilities. It is the expected value $\mathbb{E}_q\left[e^{-rT}\frac{1}{S(T)}Max\left[(S(T)-K)_+\right]\right]$ under our make-believe risk-neutral probabilities, where r is the instantaneous risk-free rate. It is just an accident of the mathematical form of the lognormal density function that after completing a square in the integral for this complicated expected value the answer can be found in a table of probability values. (note: using concepts not covered in class, it is possible to identify $\Phi(d_1)$ as the probability $\mathbb{P}_S\left[S(T) > K\right]$ under the make-believe probability measure $\mathbb{P}_S$ that corresponds to using $S(t)$ rather than the risk-free bank account as numeraire.)

6. A stock has a dividend yield of 2% and the company pays 7.5% interest on its long term debt. The ROE based on beginning of year equity is 16%. The are 10 million shares outstanding. The market to book ratio is 1.25 and the share price is \$40. The interest payments on the long term debt amount to \$2.50 per share. What is the maximum possible growth rate the company can finance without using any new external sources of financing of any kind?

We are looking for the internal growth rate:

$$\begin{aligned}
g &= \frac{PB \cdot NI}{NA} \text{ using beginning of year } NA = BV + D \\
&= PB \cdot ROE \cdot \frac{BV}{BV + D} \text{ using } ROE \text{ on begining of year } BV \\
&= \frac{NI - DIV}{NI} \cdot ROE \cdot \frac{1}{1 + \frac{D}{BV}} \\
&= \left(ROE - \frac{DIV}{BV} \right) \cdot \frac{1}{1 + \frac{\frac{Int}{BV}}{\frac{Int}{D}}} \\
&= \left(ROE - \frac{d \cdot MV}{BV} \right) \cdot \frac{1}{1 + \frac{\frac{Int}{MV}\frac{MV}{BV}}{\frac{Int}{D}}} \\
&= (.16 - (.02) \cdot (1.25)) \cdot \frac{1}{1 + \frac{\frac{2.50}{40}(1.25)}{.075}} \\
&= .0661
\end{aligned}$$

7. For years, a company has plowed back 60% of earnings while making a 20% return on equity and maintaining a 3% dividend yield. The debt ratio has remained constant. The market has priced the shares as if the growth rate corresponding to this financial performance could continue forever. By what % and in what direction will the share price change if the company suddenly announces, in a complete surprise to the market, that is has no further opportunities for profitable growth beyond its current scale of operations, it now plans no further growth at all, and will begin to pay out all of its earnings as dividends every year?

   Under the scenario described, all of the current $PVGO$, present value of growth opportunities per-share, will disappear from the stock price at the time of the surprise announcement. So we get a decline in price:

$$
\begin{aligned}
-\frac{PVGO}{P} &= -\frac{1}{P}\left(P-\frac{eps}{k_S}\right) \\
&= -\frac{1}{P}\left(P-\frac{\frac{eps(1-PB)}{1-PB}}{d+g}\right) \\
&= -\frac{1}{P}\left(P-\frac{\frac{\text{div}}{1-PB}}{d+g}\right) \\
&= -\frac{1}{P}\left(P-\frac{\text{div}}{(1-PB)\,(d+PB\cdot ROE)}\right) \\
&= -\left(1-\frac{d}{(1-PB)\,(d+PB\cdot ROE)}\right) \\
&= -\left(1-\frac{.03}{(1-.60)\,(.03+.60\,(.20))}\right) \\
&= -.50 \\
&= 50\%\text{ price decline}
\end{aligned}
$$