

**Math 324**  
 Advanced Financial Mathematics  
 Spring 2008  
 Final Exam Solutions  
 May 2, 2008

This is an open book take-home exam. You may work with textbooks and notes but do not consult any other person. Show all of your work and put your name on all papers. The exam is due back by 5 PM on Thursday May 8. You may place it in my box in the faculty mailroom or under my office door.

1. Let  $W(t)$  be a Brownian motion and  $\mathcal{F}(t)$  the associated filtration. Show in two different ways that

$$e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}$$

is an  $\mathcal{F}(t)$  martingale where  $\sigma$  is a non-negative constant.

1. (a) First, show it using Itô's lemma (Itô-Doebelin formula)
- (b) Second, show it without using Itô's lemma in any way, using only the definitions of  $W(t)$  and of a martingale

Solution:

$$\begin{aligned} de^{\sigma W(t) - \frac{1}{2}\sigma^2 t} &= e^{\sigma W(t) - \frac{1}{2}\sigma^2 t} \left( -\frac{1}{2}\sigma^2 dt + \sigma dW(t) + \frac{1}{2}\sigma^2 dt \right) \\ &= \sigma e^{\sigma W(t) - \frac{1}{2}\sigma^2 t} dW(t) \end{aligned}$$

which is a martingale because coefficient of  $dt$  is zero. Secondly,

$$\begin{aligned} \mathbb{E} \left[ e^{\sigma W(t) - \frac{1}{2}\sigma^2 t} | \mathcal{F}(s) \right] &= \mathbb{E} \left[ e^{\sigma W(s) - \frac{1}{2}\sigma^2 s} e^{\sigma(W(t) - W(s)) - \frac{1}{2}\sigma^2(t-s)} | \mathcal{F}(s) \right] \\ &= e^{\sigma W(s) - \frac{1}{2}\sigma^2 s} e^{-\frac{1}{2}\sigma^2(t-s)} \mathbb{E} \left[ e^{\sigma(W(t) - W(s))} | \mathcal{F}(s) \right] \\ &= e^{\sigma W(s) - \frac{1}{2}\sigma^2 s} \end{aligned}$$

because  $W(t) - W(s)$  is normal with variance  $t - s$  and mean of the lognormal is  $\exp(\frac{1}{2}\text{variance})$ . But this proves the martingal property.

2. Let  $W(t)$  be a Brownian motion and  $\mathcal{F}(t)$  the associated filtration. Without assuming any knowledge of the moments of the standard normal distribution, use Itô's lemma and your knowledge of stochastic calculus to show that  $\mathbb{E} [W^6(t)] = 15t^3$ . (Hint: first use stochastic calculus to figure out what  $\mathbb{E} [W^2(t)]$  and  $\mathbb{E} [W^4(t)]$  are equal to.)

Solution:

$$\begin{aligned}
 dW^2(t) &= 2W(t)dW(t) + dt \\
 W^2(t) &= 2 \int_0^t W(s)dW(s) + \int_0^t ds \\
 \mathbb{E}[W^2(t)] &= \int_0^t ds = t \\
 dW^4(t) &= 4W^3(t)dW(t) + 6W^2(t)dt \\
 W^4(t) &= 4 \int_0^t W^3(s)dW(s) + 6 \int_0^t W^2(s)ds \\
 \mathbb{E}[W^4(t)] &= 6 \int_0^t s ds = 3t^2 \\
 dW^6(t) &= 6W^5(t)dW(t) + 15W^4(t)dt \\
 W^6(t) &= 6 \int_0^t W^5(s)dW(s) + 15 \int_0^t W^4(s)ds \\
 \mathbb{E}[W^6(t)] &= 15 \int_0^t 3s^2 ds = 15t^3
 \end{aligned}$$

3. Derive the general form for all solutions  $S(t)$  to the equation

$$dS(t) = \sigma(t)S(t)dW(t) + \alpha(t)S(t)dt$$

Solution:

$$\begin{aligned}
 d \ln S(t) &= \frac{1}{S(t)}dS(t) - \frac{1}{2} \frac{1}{S^2(t)}dS(t)dS(t) \\
 &= \sigma(t)dW(t) + \alpha(t)dt - \frac{1}{2}\sigma^2(t)dt \\
 \ln S(t) - \ln S(0) &= \int_0^t \sigma(s)dW(s) + \int_0^t \left( \alpha(s) - \frac{1}{2}\sigma^2(s) \right) ds \\
 S(t) &= S(0)e^{\int_0^t \sigma(s)dW(s) + \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s)) ds}
 \end{aligned}$$

4. You know that the price  $V(t)$  at time  $t$  for a derivative security that has payoff  $V(T)$  at time  $T > t$  is given by

$$V(t) = \tilde{\mathbb{E}} \left[ e^{-\int_t^T R(u)du} V(T) \mid \mathcal{F}(t) \right]$$

where  $R(t)$  is the risk-free short-term interest rate. Using formulas where necessary, explain the difference between  $\mathbb{E}$  and  $\tilde{\mathbb{E}}$  and what the connection is between  $\tilde{\mathbb{E}}$  and the process  $S(t)$  for the underlying stock price.

Solution:  $\tilde{\mathbb{E}}$  is defined under a measure  $\tilde{\mathbb{P}}$  equivalent to the original  $\mathbb{P}$  but for which  $e^{-\int_0^t R(u)du}V(t)$  is a martingale. The connection with  $S(t)$  is that where

$dS(t) = \sigma(t)S(t)dW(t) + \alpha(t)S(t)dt$ , i.e.  $S(t) = S(0)e^{\int_0^t \sigma(s)dW(s) + \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s))ds}$ ,  
we define for any measurable set  $A$ ,  $\tilde{\mathbb{P}}(A) = \int_{\Omega} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega)$  where

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(\omega) = e^{-\int_0^T \frac{\alpha(s) - R(s)}{\sigma(s)} dw(s) - \frac{1}{2} \int_0^T \left(\frac{\alpha(s) - R(s)}{\sigma(s)}\right)^2 ds}$$

and the relation between Brownian motions is

$$d\tilde{W}(t) = dW(t) + \frac{\alpha(t) - R(t)}{\sigma(t)} dt$$

5. In the same situation as question 4 above, explain why the existence of a portfolio process  $X(t) = V(t)$  with

$$dX(t) = \Delta(t)dS(t) + R(t)[X(t) - \Delta(t)S(t)] dt$$

depends upon the assumption that  $\sigma(t) > 0$  almost surely, where  $\sigma(t)$  is the volatility process in the geometric Brownian motion process  $S(t)$  for the underlying stock price. Be sure to explain what any new symbols that you introduce are, where they come from, and what justifies your using them.

Solution: The martingale representation theorem gives us a process  $\tilde{\Gamma}(t)$  with

$$d\left(e^{-\int_0^t R(s)ds} V(t)\right) = \tilde{\Gamma}(t)d\tilde{W}(t)$$

But if  $X(t)$  exists as given then

$$\begin{aligned} d\left(e^{-\int_0^t R(s)ds} X(t)\right) &= -R(t)e^{-\int_0^t R(s)ds} dt X(t) + e^{-\int_0^t R(s)ds} dX(t) + 0 \\ &= -R(t)e^{-\int_0^t R(s)ds} X(t) dt \\ &\quad + e^{-\int_0^t R(s)ds} \{\Delta(t)[\sigma(t)S(t)dW(t) + \alpha(t)S(t)dt] \\ &\quad + R(t)[X(t) - \Delta(t)S(t)] dt\} \\ &= e^{-\int_0^t R(s)ds} \Delta(t)S(t) \{\sigma(t)dW(t) + (\alpha(t) - R(t)) dt\} \\ &= e^{-\int_0^t R(s)ds} \Delta(t)S(t)\sigma(t)d\tilde{W}(t) \end{aligned}$$

So if  $X(t)$  exists as given then

$$\Delta(t) = \frac{\tilde{\Gamma}(t)}{e^{-\int_0^t R(s)ds} S(t)\sigma(t)} \text{ must hold and for that}$$

$$\sigma(t) \neq 0 \text{ must hold almost surely.}$$

**6.** Let the random variable  $A$  be the value at time  $T$  of an asset and assume that  $A$  is almost-surely positive, where  $A$  is  $\mathcal{F}(T)$  measurable in the filtration determined by a Brownian motion  $W(t)$ . Assume that there is a risk free rate process  $R(t)$  and a unique risk-neutral measure. Show that there exist random processes  $V(t)$ ,  $\alpha(t)$  and  $\sigma(t)$  so that  $dV(t) = \alpha(t)V(t)dt + \sigma(t)V(t)dW(t)$  (i.e.  $V(t)$  is a generalized geometric Brownian motion) and  $V(T) = A$ . This means that all positive assets measurable in the filtration generated by a Brownian motion can be represented by a generalized geometric Brownian motion based on the original Brownian motion.

Solution: Define  $V(t) = e^{-\int_t^T R(s)ds} A$  so  $V(T) = A$  and  $V(t) \neq 0$  almost surely. Since a risk-neutral  $\tilde{\mathbb{P}}$  exists,  $e^{-\int_0^t R(s)ds} V(t)$  is  $\tilde{\mathbb{P}}$ -martingale and the representation theorem gives a process  $\tilde{\Gamma}(t)$  with

$$d\left(e^{-\int_0^t R(s)ds} V(t)\right) = \tilde{\Gamma}(t)d\tilde{W}(t) \text{ so}$$

$$-R(t)e^{-\int_0^t R(s)ds} dtV(t) + e^{-\int_0^t R(s)ds} dV(t) = \tilde{\Gamma}(t)d\tilde{W}(t)$$

$$dV(t) = R(t)V(t)dt + e^{\int_0^t R(s)ds} \tilde{\Gamma}(t)d\tilde{W}(t)$$

Define  $\sigma(t) = e^{\int_0^t R(s)ds} \frac{\tilde{\Gamma}(t)}{V(t)}$  and  $\alpha(t) = R(t) + \Theta(t)\sigma(t)$  where  $d\tilde{W}(t) = \Theta(t)dt + dW(t)$  and then  $dV(t) = \alpha(t)V(t)dt + \sigma(t)V(t)dW(t)$

**7.** Do exercise 6.1 from the textbook.

Solution:

$$\begin{aligned}Z(t) &= e^0 = 1 \\dZ(u) &= Z(u) \left\{ \sigma(u)dW(u) + \left(b(u) - \frac{1}{2}\sigma^2(u)\right)du + \frac{1}{2}\sigma^2(u)du \right\} \\&= \sigma(u)Z(u)dW(u) + b(u)Z(u)du \\Y(t) &= x \\dY(u) &= \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)}du + \frac{\gamma(u)}{Z(u)}dW(u) \\d(Y(u)Z(u)) &= dY(u)Z(u) + Y(u)dZ(u) + dY(u)dZ(u) \\&= (a(u) - \sigma(u)\gamma(u))du + \gamma(u)dW(u) + \sigma(u)(Y(u)Z(u))dW(u) \\&\quad + b(u)(Y(u)Z(u))du + \sigma(u)\gamma(u)du \\&= (a(u) + b(u)(Y(u)Z(u)))du + (\gamma(u) + \sigma(u)(Y(u)Z(u)))dW(u)\end{aligned}$$