Yield Curve Models Seminar

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Spring 2008

$B(t)$ is the value of a risk-free bank account or money market account at time t where $B(0) = 1$
$dB(t) = r_t B(t) dt, \quad B(0) = 1$

$$B(t) = e^{\int_0^t r_s ds}$$

$B(t)$ is the first example of a *numeraire* (to be defined later).
r_t is called the instantaneous spot rate, or the short rate, at t. "Spot" means "goes into effect at the time in question." "Short" means "instantaneous, or for the shortest actual period available."

The value of a payment of 1 at time T as viewed from time $t < T$ is $\frac{B(t)}{B(T)}$

The discount factor between times t and T is

$$D(t, T) = \frac{B(t)}{B(T)} = e^{-\int_t^T r_s ds}$$

$D(t, T)$ can be stochastic, ie. random, if r_s is a random process for $s > t$, ie a collection of random variables, one for each s.

$P(t, T)$ is the value at t of a risk free zero-coupon bond paying-off 1 at T

If r_t is deterministic (no randomness, known for certain) then $P(t, T) = D(t, T)$

With stochastic r_t we will arrive later at the conclusion that $P(t, T) = \mathbb{E}_{rn}\left[D(t, T)\right]$, where $\mathbb{E}_{rn}$ indicates that it is not really the expected value $\mathbb{E}\left[D(t, T)\right]$, but instead the expected value under some new and different probability measure, called risk-neutral, perhaps related to the original (real world) probability measure but not equal to it.

$T - t$ is the time to maturity, the amount of time measured in years between t and T.

To get a yield curve of interest rates from zero coupon bond values requires a day-count convention and a compounding type

A day-count convention is a function $\tau(T - t)$ that defines the amount of time (in years) between t and T when t and T are other than exactly a whole of number of years apart

Examples are Actual Time (less than a day), (Actual Days)/365, (Actual Days)/360, or (30 Days per Month)/360.

Continuously compounded spot rate is $R(t, T) = -\frac{\ln P(t,T)}{\tau(t,T)}$, so $P(t, T) = e^{-R(t,T)\tau(t,T)}$

Simply compounded spot rate is $L(t, T) = \frac{1-P(t,T)}{\tau(t,T)P(t,T)}$, so $P(t, T) = [1 + L(t, T)\tau(t, T)]^{-1}$. (We use the letter L because simple compounding is the convention in the LIBOR swap markets.)

Annually compounded spot rate is $Y(t, T) = \left[\frac{1}{P(t,T)}\right]^{1/\tau(t,T)} - 1$, so $P(t, T) = [1 + Y(t, T)]^{-\tau(t,T)}$

k-thly compounded spot rate is $Y^k(t,T) = k\left\{\left[\frac{1}{P(t,T)}\right]^{1/[k\tau(t,T)]} - 1\right\}$,
so $P(t,T) = \left[1 + \frac{Y^k(t,T)}{k}\right]^{-k\tau(t,T)}$

$\lim_{k\to\infty} Y^k(t,T) = R(t,T)$ and

$$
\begin{aligned}
r_t &= \lim_{T\to t^+} R(t,T) \\
&= \lim_{T\to t^+} L(t,T) \\
&= \lim_{T\to t^+} Y(t,T) \\
&= \lim_{T\to t^+} Y^k(t,T)
\end{aligned}
$$

so a market that always includes zero coupon bonds that are near maturity can support a money-market account $B(t)$ as originally defined.

The zero-coupon curve at time t is the function

$$\begin{aligned} T &\longrightarrow L(t,T) \text{ for } t < T \leq t+1 \\ &\longrightarrow Y(t,T) \text{ for } T > t+1 \end{aligned}$$

Again, this is a convenient choice because of the trading conventions in the LIBOR swap markets. A purely theoretical approach would probably choose to model the same compounding type for all maturities, either $R(t,T)$ for theoretical ease or $Y(t,T)$ to avoid throwing the constant e in the face of practical people.

The zero-bond curve at time t is the function $T \longrightarrow P(t,T)$ for $T > t$

The value of a forward-rate agreement expiring at time T, maturing at time S, for a fixed rate K, against a floating rate $L(T,S)$ (resetting in T, maturing in S), on a notional (or nominal) amount N is given by

$$\mathbf{FRA}(S,T,S,\tau(T,S),N,K) \quad = \quad N\tau(T,S)\left[K - L(T,S)\right]$$

payment at time S

$$\mathbf{FRA}(t,T,S,\tau(T,S),N,K) \quad =$$

$$= \quad N\left[P(t,S)\tau(T,S)K - P(t,T) + P(t,S)\right]$$

value at time t **STOP! See why!**

The simply-compounded forward (as opposed to "spot") interest rate prevailing at time t for expiry $T > t$ and maturity $S > T$ is the unique K that makes this 0, $K = F(t;T,S) = \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$, so for any other K, $\mathbf{FRA}(t,T,S,\tau(T,S),N,K) = NP(t,S)\tau(T,S)\left[K - F(t;T,S)\right]$.

Compare this with the payment at time S. $F(t;T,S)$ is a sort of estimate at time t for the random variable $L(T,S)$. We'll eventually conclude that $F(t;T,S) = \mathbb{E}_{rn}\left[L(T,S)\right]$, where again $\mathbb{E}_{rn}$ indicates that it's not the true $\mathbb{E}$, but a distorted "risk-neutral" one.

The instantaneous forward (as opposed to "spot") rate for T at t is $f(t,T) = \lim_{S\to T^+} F(t;T,S) = -\frac{\partial \ln P(t,T)}{\partial T}$ **STOP! See why!**, so $P(t,T) = e^{-\int_t^T f(t,s)ds}$. **STOP!** How does $f(t,s)$ differ from r_s? How does $P(t,T)$ differ from $D(t,T)$?

The instantaneous forward rates $f(t,T)$ are a key element in any interest rate model because the expression $-\frac{\partial \ln P(t,T)}{\partial T}$ provides a link between the model (of the future) and prices actually observed in the market (today).

Forward rate agreements **FRA** are the building blocks for a set of key financial instrument that trade in huge volumes in the market and whose pricing is one reason for the need to have yield curve models: Interest Rate Swaps (IRS).

Interest Rate Swaps

Consider pretermined sets of dates $\mathcal{T} = \{T_\alpha, T_{\alpha+1}, ..., T_\beta\}$ and year-fractions $\boldsymbol{\tau} = \{\tau_{\alpha+1} = \tau(T_{\alpha+1} - T_\alpha), ..., \tau_\beta = \tau(T_\beta - T_{\beta-1})\}$, called a tenor-structure. A Receiver Interest Rate Swap (**RFS**), where **F** stands for "forward", is a contract that pays $N\tau_i\,(K - L(T_{i-1}, T_i))$ at each time $T_i > T_\alpha$ in the tenor. Each payment is the same as the payment on a certain **FRA**, so the value at time $t \leq T_\alpha$ of the swap contract is

$$\begin{aligned}
\mathbf{RFS}\,(t, \mathcal{T}, \boldsymbol{\tau}, N, K) &= \sum_{i=\alpha+1}^{\beta} \mathbf{FRA}\,(t, T_{i-1}, T_i, \tau_i, N, K) \\
&= N \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)(K - F(t; T_{i-1}, T_i)) \\
&= NP(t, T_\beta) + N \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) - NP(t, T_\alpha)
\end{aligned}$$

STOP! WHY?

A Payor Interest Rate Swap (**PFS**) reverses the payments, so it pays $N\tau_i \left(L(T_{i-1}, T_i) - K\right)$ at each time $T_i > T_\alpha$ in the tenor. The terminology can vary, but generally to "own", "buy", or "be long" an interest rate swap refers to the **RFS**, to "owe", "sell", or "be short" an interest rate swap refers to the **PFS**. The generic interest rate swap contract entitles one to "receive the fixed rate" K in exchange for paying the floating rate $L(T_{i-1}, T_i)$ for each period in the tenor starting at the expiry date T_α and running until the contract ends at the maturity date T_β. So if instead one "pays the fixed rate" K in exchange for receiving the floating rate $L(T_{i-1}, T_i)$ then one has sold the interest rate swap contract.

T_α is the "expiry date", $\left\{T_{\alpha+1}, \ldots, T_{\beta-1}\right\}$ are the "reset dates", and T_β is the "maturity date."

The series of payments $\{N\tau_i K\}$ at each time $T_i > T_\alpha$ in the tenor is called the "fixed leg" of the swap contract. The series of payments $\{N\tau_i L(T_{i-1}, T_i)\}$ is called the "floating leg." The fixed leg equals the interest payments on a fixed coupon bond; the floating leg on floating rate note. In actual markets, the fixed and floating legs often have different tenor structures, i.e. different reset dates and day-count conventions.

The value at t of a fixed coupon bond is
$\mathbf{CB}(t, \mathcal{T}, \boldsymbol{\tau}, N, K) = NP(t, T_\beta) + N\sum_{i=\alpha+1}^{\beta} \tau_i KP(t, T_i)$
$= NP(t, T_\alpha) + \mathbf{RFS}\,(t, \mathcal{T}, \boldsymbol{\tau}, N, K)$ **STOP! WHY?** The value of a floating rate note at any reset date is just N, so its value at $t < T_\alpha$ is $NP(t, T_\alpha)$ **STOP! WHY?**

At a given time t if we want the value of an interest rate swap to be 0 then K must take the value called the "forward swap rate"

$$\begin{aligned} S_{\mathcal{T},\boldsymbol{\tau}}(t) &= S_{\alpha,\beta}(t) \\ &= \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)} \\ &= \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F(t; T_{j-1}, T_j)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F(t; T_{j-1}, T_j)}} \end{aligned}$$ **STOP! WHY?**

The two main interest rate derivatives traded in the market are "caps" and "swaptions." An ability to value them is an important reason to have a good yield curve model. In fact, a modern trend is to have models called "market models" that directly model the prices of either caps or swaptions (and, if need be, back into a corresponding yield curve). Amazingly, both kinds of market models are in active use and they cannot be mutually consistent!

A cap is like a **PFS** but it pays ONLY when positive: $\mathbf{Cap}(t, \mathcal{T}, \boldsymbol{\tau}, N, K)$ pays $N\tau_i(L(T_{i-1}, T_i) - K)_+$ at each time $T_i > T_\alpha$ in the tenor structure. Why "cap"? Suppose a debtor takes on a floating rate obligation but at the same time purchases a cap with identical tenor. Then at each time in the tenor structure its net interest obligation will be $N\tau_i L(T_{i-1}, T_i) - N\tau_i(L(T_{i-1}, T_i) - K)_+ = N\tau_i \min(L(T_{i-1}, T_i), K)$. The interest obligation effectively has been capped at rate K.

There's also a contract called a "floor" that is like a **RFS** but pays ONLY when positive: $\mathbf{Flr}(t, \mathcal{T}, \boldsymbol{\tau}, N, K)$ pays $N\tau_i(K - L(T_{i-1}, T_i))_+$ at each time $T_i > T_\alpha$ in the tenor structure. Why "floor"??

Suppose a lender puts out funds at a floating rate but at the same time purchases a floor with identical tenor. Then at each time in the tenor structure its net interest receipts will be
$N\tau_i L(T_{i-1}, T_i) + N\tau_i (K - L(T_{i-1}, T_i))_+ = N\tau_i \max(L(T_{i-1}, T_i), K)$.
The interest receipts have an effective floor at the rate K.

The payments under a cap or a floor are just the total of the payments at each point in the tenor structure, so a cap can be considered to be just a combination of simpler contracts called "caplets" each with a one-period tenor structure. A floor an be considered to be just a combination of "floorlets."

It was Fischer Black's idea to express the prices of caplets and floorlets as call or put options on the forward rate, by analogy with the Black-Scholes formula for the price of a call or put option on an underlying asset, even though forward rates, per se, have no return, risk free or otherwise, and may not evolve in any geometric Brownian fashion. The market quotes prices on caps and floors using this analogy, quoting a Black-Scholes implied volatility rather than some price per notional amount.

$$\textbf{Cap}^{Black}(t, \mathcal{T}, \boldsymbol{\tau}, N, K, \sigma_{\alpha,\beta}) =$$

$$= N \sum_{i=\alpha+1}^{\beta} P(t, T_i)\tau_i Bl(K, F(t; T_{i-1}, T_i), v_i, 1)$$

$$\textbf{Flr}^{Black}(t, \mathcal{T}, \boldsymbol{\tau}, N, K, \sigma_{\alpha,\beta}) =$$

$$= N \sum_{i=\alpha+1}^{\beta} P(t, T_i)\tau_i Bl(K, F(t; T_{i-1}, T_i), v_i, -1)$$

where $Bl(K, F, v, s) = sF\Phi(sd_1(K, F, v)) - sK\Phi(sd_2(K, F, v))$,

$$d_j(K, F, v) = \frac{\ln(F/K) + (-1)^{j-1}v^2/2}{v}, \text{ and } v_i = \sigma_{\alpha,\beta}\sqrt{T_{i-1} - t}$$

are all just as in the usual Black-Scholes formulae.

NOTE THE ABSENCE OF DRIFT!

A cap or floor is said to be "At The Money" (ATM) if

$$K = K_{ATM} = S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}$$

A cap (or floor) is ATM if and only if it's price equals the price of the corresponding floor (or cap). **WHY?** Well, the difference between cap payoff and floor payoff is just the payoff on a swap. **VERIFY!**

A cap is "In The Money" (ITM) if $K < S_{\alpha,\beta}(t)$ and "Out of The Money" (OTM) if $K > S_{\alpha,\beta}(t)$. A floor is ITM or OTM in the reverse case.

A graph of $\sigma_{\alpha,\beta}$ against T_β, called the maturity, with everything else fixed (for example, a fixed T_α such as 3 months all $K = S_{\alpha,\beta}(t)$) is called a cap volatility curve and represents the state of prices in the cap market, a little bit like the yield curve for bonds.

The other main kind of traded interest rate derivative is the "Swaption", short for "Swap Option." It gives the right, but not the obligation, to enter into an interest rate swap contract at some time or times in the future. A European payor swaption $\mathbf{PS}(t, u, \mathcal{T}, \boldsymbol{\tau}, N, K)$ gives the right, but not the obligation, to enter into the payor forward swap $\mathbf{PFS}(u, \mathcal{T}, \boldsymbol{\tau}, N, K)$, with tenor $T_\beta - T_\alpha$ and tenor structure $\mathcal{T}, \boldsymbol{\tau}$, at the swaption maturity time u. Confusingly, the swaption maturity u is usually the expiry date, in other words first reset date, of the forward swap. So at the maturity of the swaption at date T_α one has the right, but not the obligation, to enter into a spot payor swap $\mathbf{PFS}(T_\alpha, \mathcal{T}, \boldsymbol{\tau}, N, K)$. This we'll denote by $\mathbf{PS}(t, \mathcal{T}, \boldsymbol{\tau}, N, K)$.
The value of this if exercised at time T_α is

$$\mathbf{PFS}(T_\alpha, \mathcal{T}, \boldsymbol{\tau}, N, K) = N \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)(F(T_\alpha; T_{i-1}, T_i) - K)$$

Of course it won't be exercised unless it is positive so the actual value at time T_α is

$$\mathbf{PS}(T_\alpha, \mathcal{T}, \boldsymbol{\tau}, N, K) = N \left\{ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)(F(T_\alpha; T_{i-1}, T_i) - K) \right\}_+$$

and at time t

$$\mathbf{PS}(t, \mathcal{T}, \boldsymbol{\tau}, N, K) =$$

$$= NP(t, T_\alpha) \left\{ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)(F(T_\alpha; T_{i-1}, T_i) - K) \right\}_+$$

and this cannot be decomposed into swaptionlets or swapletions. Notice that the Payor Swaption value must be less than the value of the corresponding cap.

For yield curve modeling this has just introduced the ugly complication of *terminal correlation* among interest rates, to go along with *instantaneous corrleation* among changes in interest rates.

The market followed Fischer Black's lead for swaptions, too, and expresses swaption prices in terms of implied volatility of a call on the swap rate

$$\mathbf{PS}^{Black}(t, \mathcal{T}, \boldsymbol{\tau}, N, K, \sigma_{\alpha,\beta}) =$$

$$= NBl(K, S_{\alpha,\beta}(t), \sigma_{\alpha,\beta}\sqrt{T_\alpha - t}, 1) \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$$

THIS IS NOT THE SAME $\sigma_{\alpha,\beta}$ AS QUOTED IN CAP PRICES!

The receiver swaption is the same idea with the right, not obligation, to enter into a receiver swap and

$$\mathbf{RS}^{Black}(t, \mathcal{T}, \boldsymbol{\tau}, N, K, \sigma_{\alpha,\beta}) =$$

$$= NBl(K, S_{\alpha,\beta}(t), \sigma_{\alpha,\beta}\sqrt{T_\alpha - t}, -1) \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$$

A swaption is said to be "At The Money" (ATM) if

$$K = K_{ATM} = S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}$$

A payor (or receiver) swaption is ATM if and only if it's price equals the price of the corresponding receiver (or payor) swaption. **WHY?** Well, the difference between payor swaption payoff and receiver swaption payoff is just the payoff on a swap. **VERIFY!**

A payor swaption is "In The Money" (ITM) if $K < S_{\alpha,\beta}(t)$ and "Out of The Money" (OTM) if $K > S_{\alpha,\beta}(t)$. A receiver swaption is ITM or OTM in the reverse case.

A 3 dimensional graph of $\sigma_{\alpha,\beta}$ against maturity and tenor (T_α and $T_\beta - T_\alpha$), with everything else fixed (for example, all $K = S_{\alpha,\beta}(t)$) is called a swaption volatility surface and represents the state of prices in the swaptions market, a little bit like the yield curve for bonds.

The risk neutral (no arbitrage) framework

A market is a K+1 dimensional adapted process with a bank account.
Probability space $\Omega, \mathcal{F}, \mathbb{Q}_0$ represents the randomness in the world.
Right-continuous filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ represents time.
$\mathbf{X}_t = \left\langle X_t^0, X_t^1, ..., X_t^K \right\rangle$ are the market prices at time t of $K+1$ non-dividend-paying assets. In what follows, think of it as a column vector.
$X_t^0 = B_t$ is the value of the bank account, so $dX_t^0 = r_t X_t^0 dt$ with $X_0^0 = 1$ and $\frac{1}{X_t^0} = D(0, t)$
One generally thinks of $\mathbf{X}_t$ as a submartingale: $\mathbb{E}\left[X_t^k | \mathcal{F}_u\right] \geq X_u^k$ for $u < t$ and all k, but I'm not sure this assumption is essential.

The risk neutral (no arbitrage) framework

A trading strategy (or portfolio) is a K+1 dimensional locally bounded and predictable process
$\boldsymbol{\phi}_t = \left\langle \phi_t^0, \phi_t^1, ..., \phi_t^K \right\rangle$ where ϕ_t^k represents how many units of X^k to own at time t.

"Predictable" means that I don't need to know anything about the future at t and beyond in order to know $\boldsymbol{\phi}_t$.

Technically, it means that $\boldsymbol{\phi}_t$ is $\mathcal{F}_u$-measurable for all $u < t$, which means that all the probabilities associated with $\boldsymbol{\phi}_t$ are encompassed by the $\mathcal{F}_u$ with $u < t$.

The value and gain processes associated with $\boldsymbol{\phi}_t$ are

$$
\begin{aligned}
V_t(\boldsymbol{\phi}) &= \boldsymbol{\phi}_t \mathbf{X}_t = \sum_{k=0}^{K} \phi_t^k X_t^k \\
G_t(\boldsymbol{\phi}) &= \int_0^t \boldsymbol{\phi}_u d\mathbf{X}_u = \sum_{k=0}^{K} \int_0^t \phi_u^k dX_u^k
\end{aligned}
$$

The risk neutral (no arbitrage) framework

The trading strategy $\boldsymbol{\phi}_t$ is said to be self-financing if $V_t(\boldsymbol{\phi}) = V_0(\boldsymbol{\phi}) + G_t(\boldsymbol{\phi})$, or equivalently

$$dV_t(\boldsymbol{\phi}) = d(\boldsymbol{\phi}_t \mathbf{X}_t) = \boldsymbol{\phi}_t d\mathbf{X}_t = \sum_{k=0}^{K} \phi_t^k dX_t^k,$$

so $\boldsymbol{\phi}_t$ is self-financing provided that, even though $\boldsymbol{\phi}_t$ varies with t, $d(\boldsymbol{\phi}_t \mathbf{X}_t) = \boldsymbol{\phi}_t d\mathbf{X}_t$

It is a theorem that $\boldsymbol{\phi}_t$ is self-financing if and only if

$$\begin{aligned} D(0,t)V_t(\boldsymbol{\phi}) &= V_0(\boldsymbol{\phi}) + \int_0^t \boldsymbol{\phi}_u d(D(0,u)\mathbf{X}_u), \text{ or equivalently} \\ d(D(0,t)V_t(\boldsymbol{\phi})) &= \boldsymbol{\phi}_t d(D(0,t)\mathbf{X}_t) \end{aligned}$$

The risk neutral (no arbitrage) framework

Proof: by Itô's product rule:

$$d\left(D(0,t)V_t\left(\boldsymbol{\phi}\right)\right) =$$

$$\begin{aligned}
&= D(0,t)dV_t\left(\boldsymbol{\phi}\right) + dD(0,t)V_t\left(\boldsymbol{\phi}\right) + dD(0,t)dV_t\left(\boldsymbol{\phi}\right) \\
&= D(0,t)\boldsymbol{\phi}_t d\mathbf{X}_t + dD(0,t)\boldsymbol{\phi}_t\mathbf{X}_t + dD(0,t)\boldsymbol{\phi}_t d\mathbf{X}_t \\
&\quad \text{if and only if } \boldsymbol{\phi}_t \text{ is self-financing} \\
&= \boldsymbol{\phi}_t\left\{D(0,t)d\mathbf{X}_t + dD(0,t)\mathbf{X}_t + dD(0,t)d\mathbf{X}_t\right\} \\
&= \boldsymbol{\phi}_t d\left(D(0,t)\mathbf{X}_t\right), \text{ by Itô's product rule again}
\end{aligned}$$

The risk neutral (no arbitrage) framework

An equivalent martingale measure is a probability measure $\mathbb{Q}$ on $\Omega, \mathcal{F}$ that satisfies

- $D(0,u)\mathbf{X}_u$ is a martingale on $\Omega, \mathcal{F}, \mathbb{Q}$, meaning $\mathbb{E}^{\mathbb{Q}}\left[D(0,u)\mathbf{X}_u | \mathcal{F}_t\right] = D(0,t)\mathbf{X}_t$ for all $t < u$.
- A technical condition: $\mathbb{Q}$ is equivalent to $\mathbb{Q}_0$, meaning that for any $A \in \mathcal{F}$, $\mathbb{Q}(A) = 0$ if and only if $\mathbb{Q}_0(A) = 0$.
- A technical condition: $\frac{d\mathbb{Q}}{d\mathbb{Q}_0}$ is square integrable, meaning that there is a function $f(\omega)$ from Ω to $\mathbb{R}$ that gives

$$\mathbb{Q}(A) = \int_A f(\omega) d\mathbb{Q}_0(\omega) \text{ for any } A \in \mathcal{F} \text{and } \int_\Omega (f(\omega))^2 \, d\mathbb{Q}_0(\omega) < \infty$$

A market allows arbitrage if there is a self-financing strategy $\boldsymbol{\phi}_t$ that has $V_0(\boldsymbol{\phi}) = 0$ and $\mathbb{Q}_0\left[V_t(\boldsymbol{\phi}) < 0\right] = 0$ for some $t > 0$, but $\mathbb{E}^{\mathbb{Q}_0}\left[V_t(\boldsymbol{\phi})\right] > 0$.

It is a theorem that a market is free of arbitrage if and only if it has an equivalent martingale measure. This is also called the risk-neutral measure.

The risk neutral (no arbitrage) framework

The connection between the two concepts, risk neutral and no arbitrage, is the martingale representation theorem. It says that martingale measures create self-financing strategies and vice-versa.

A contingent claim H is a positive random variable. A contingent claim is attainable if there is a self-financing strategy $\boldsymbol{\phi}$ and a time u such that $H = V_u(\boldsymbol{\phi})$. For any time $t < u$ the price π_t of H at $t < u$ is $\pi_t = V_t(\boldsymbol{\phi})$. So far, the price depends upon $\boldsymbol{\phi}$. More than one self-financing strategy could mean more than one price.

Theorem: If there is an equivalent martingale measure $\mathbb{Q}$, π_t for an attainable claim H is unique and is given by $\pi_t = \mathbb{E}^{\mathbb{Q}}[D(t,u)H|\mathcal{F}_t]$. If there are more than one equivalent martingale measures, they all give the same price for an attainable claim.

A market is called complete if all contingent claims are attainable. In that case, the equivalent martingale measure is unique.

EMM exists$\Longleftrightarrow$No arbitrage $\Longrightarrow$ Attainable prices unique

Complete $\Longleftrightarrow$ EMM unique

A risk-neutral measure $\mathbb{Q}$ might not be easy to work with (or even to characterize).

Change of numeraire idea

A numeraire Z_t is any non-dividend paying asset, not necessarily the bank account. We usually think of a numeraire as the value of some self-financing strategy $Z_t = V_t(\boldsymbol{\psi})$ that we can work with. Now let's look at the values X_t^k in terms of Z_t: the relative values ("present values") are $\frac{X_t^k}{Z_t}$. For any other trading strategy $\boldsymbol{\phi}_t$ it works out for $\frac{1}{Z_t}$ just as it did for $\frac{1}{B_t} = D(0,t)$ that by Itô's product rule

$$d\left(\frac{V_t(\boldsymbol{\phi})}{Z_t}\right) = \frac{1}{Z_t} dV_t(\boldsymbol{\phi}) + d\left(\frac{1}{Z_t}\right) V_t(\boldsymbol{\phi}) + d\left(\frac{1}{Z_t}\right) dV_t(\boldsymbol{\phi})$$

$$= \frac{1}{Z_t} \boldsymbol{\phi}_t d\mathbf{X}_t + d\left(\frac{1}{Z_t}\right) \boldsymbol{\phi}_t \mathbf{X}_t + d\left(\frac{1}{Z_t}\right) \boldsymbol{\phi}_t d\mathbf{X}_t, \text{ iff. } \boldsymbol{\phi}_t \text{ self-financing}$$

$$= \boldsymbol{\phi}_t \left\{ \frac{1}{Z_t} d\mathbf{X}_t + d\left(\frac{1}{Z_t}\right) \mathbf{X}_t + d\left(\frac{1}{Z_t}\right) d\mathbf{X}_t \right\} = \boldsymbol{\phi}_t d\left(\frac{\mathbf{X}_t}{Z_t}\right), \text{ by Itô's}$$

product rule. "Present values" relative to any numeraire work just the same as present values relative to the bank account for doing risk-neutral finance. Arbitrage/no arbitrage is independent of numeraire; yes/no under one numeraire means yes/no under all numeraires.

Change of numeraire idea

Assume that a market has at least one numeraire N_t and probability measure $\mathbb{Q}_N$ equivalent to $\mathbb{Q}_0$ such that for any asset S_t the price relative to the numaire N_t is a $\mathbb{Q}_N$-martingale $\frac{S_t}{N_t} = \mathbb{E}^{\mathbb{Q}_N}\left[\frac{S_u}{N_u}|\mathcal{F}_t\right]$ for all $0 \leq t \leq u$. Now let U_t be any other arbitrary numeraire. Then there exists a probability measure $\mathbb{Q}_U$ equivalent to $\mathbb{Q}_0$ such that the price of any attainable claim H, payable at u, relative to the numeraire U_t is a $\mathbb{Q}_U$-martingale $\frac{\pi_t^H}{U_t} = \mathbb{E}^{\mathbb{Q}_U}\left[\frac{H}{U_u}|\mathcal{F}_t\right]$ for all $0 \leq t \leq u$. Furthermore, $\mathbb{Q}_U$ is given by $\frac{d\mathbb{Q}_U}{d\mathbb{Q}_N}(u) = \frac{U_u N_0}{U_0 N_u}$.

In other words, if there is an equivalent martingale measure for one numeraire N_t (for example, the bank account) then we can use any other numeraire U_t that we care to in order to perform risk-neutral pricing of attainable claims. Furthermore, the two measures are related simply by the ratio of the numeraires.

Change of numeraire idea

Proof: For any asset S_t we know that $\mathbb{E}^{\mathbb{Q}_N}\left[\frac{S_u}{N_u}\right] = \mathbb{E}^{\mathbb{Q}_N}\left[\frac{S_u}{N_u}|\mathcal{F}_0\right] = \frac{S_0}{N_0}$, by assumption. Since the numeraire U_t works just like the bank account for risk-neutral purposes, we know that a risk-neutral measure $\mathbb{Q}_U$ exists (EMM$\Longleftrightarrow$No arbitrage, and arbitrage is independent of numeraire). So $\mathbb{E}^{\mathbb{Q}_U}\left[\frac{U_0}{N_0}\frac{S_u}{U_u}\right] = \mathbb{E}^{\mathbb{Q}_U}\left[\frac{U_0}{N_0}\frac{S_u}{U_u}|\mathcal{F}_0\right] = \frac{U_0}{N_0}\frac{S_0}{U_0} = \frac{S_0}{N_0}$ and $\mathbb{E}^{\mathbb{Q}_N}\left[\frac{S_u}{N_u}\right] = \mathbb{E}^{\mathbb{Q}_U}\left[\frac{U_0}{N_0}\frac{S_u}{U_u}\right]$. But the definition of $\frac{d\mathbb{Q}_U}{d\mathbb{Q}_N}$ implies that $\mathbb{E}^{\mathbb{Q}_N}\left[\frac{S_u}{N_u}\right] = \mathbb{E}^{\mathbb{Q}_U}\left[\frac{S_u}{N_u}\frac{d\mathbb{Q}_N}{d\mathbb{Q}_U}\right]$. Since S_t could be any asset, the fact that the right hand sides are equal implies that $\frac{U_0}{N_0}\frac{1}{U_u} = \frac{1}{N_u}\frac{d\mathbb{Q}_N}{d\mathbb{Q}_U}$ and $\frac{d\mathbb{Q}_U}{d\mathbb{Q}_N} = \frac{U_u N_0}{U_0 N_u}$.

In addition, $\pi_t^H = \mathbb{E}^{\mathbb{Q}_U}\left[\frac{U_t}{U_u}H|\mathcal{F}_t\right]$ (risk neutral pricing in U_t)
$= U_t\mathbb{E}^{\mathbb{Q}_U}\left[\frac{H}{U_u}|\mathcal{F}_t\right]$ by the usual conditional expectation property.

Change of numeraire idea

We now have proved two key facts about numeraires:

FACT ONE: The price of an asset divided by a reference asset (numeraire) is a martingale (no drift) in the measure associated with that numeraire.

Example: $\frac{S_t}{B_t} = e^{-\int_0^t r_s ds} S_t$ has to be a martingale in the measure (risk-neutral) associated with the bank account. Let $b(t, S_t)$ stand for the drift (the dt term) in the stochastic process for S_t in the risk-neutral measure.

$$
\begin{aligned}
d\left(e^{-\int_0^t r_s ds} S_t\right) &= (-r_t)\, e^{-\int_0^t r_s ds} S_t dt + e^{-\int_0^t r_s ds} dS_t \\
&= e^{-\int_0^t r_s ds} \left\{\left(-r_t S_t + b(t, S_t)\right) dt + (\ldots) dW_t\right\} \\
\text{so } b(t, S_t) &= r_t S_t,
\end{aligned}
$$

risk-neutral drift is the risk-free rate.

Change of numeraire idea

Example: The forward LIBOR rate $F_2(t) = \frac{P(t,T_1) - P(t,T_2)}{P(t,T_2)(T_2 - T_1)}$. Let the zero coupon bond $P(t, T_2)$ be the numeraire. Then the forward LIBOR rate $F_2(t)$ must be a martingale in the measure associated with that zero coupon bond as numeraire. If we can write down a differential expression for it, the drift term (the coefficient of dt) will have to be 0.

Example: The forward swap rate $S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{\alpha+1}^{\beta}(T_i - T_{i-1})P(t,T_i)}$. Let the portfolio of zero coupon bonds appearing in the denominator be the numeraire. Then the forward swap rate $S_{\alpha,\beta}(t)$ must be a martingale in the measure with that portfolio as numeraire. If we can write down its differential, the drift term will have to be 0.

FACT TWO: $price_t = \mathbb{E}^{\mathbb{Q}_B}\left[B_t \frac{Payoff_u}{B_u} | \mathcal{F}_t\right]$ is the same no matter what numeraire B_t you choose. I.e. $price_t = \mathbb{E}^{\mathbb{Q}_S}\left[S_t \frac{Payoff_u}{S_u} | \mathcal{F}_t\right]$ for any other numeraire S_t, you just have to be careful to replace *all three* occurences of B with the new numeraire S.

Change of numeraire idea

We will prove a third fact and fourth fact that are a little more technical. Stated roughly they say:

FACT THREE: The drifts of an asset under the measures associated with two different numeraires are related as

$$\begin{aligned} Drift^{Num2}_{Asset} &= Drift^{Num1}_{Asset} \\ &+ Vol_{Asset} \times Corr_{Market} \times \left(\frac{Vol_{Num2}}{Num2} - \frac{Vol_{Num1}}{Num1} \right) \end{aligned}$$

FACT FOUR: The brownian shocks to an asset under the measures associated with two different numeraires are related as

$$Vol_{Asset} \times BrShock^{Num2}_{Corr} = Vol_{Asset} \times BrShock^{Num1}_{Corr}$$

$$- Vol_{Asset} \times Corr_{Market} \times \left(\frac{Vol_{Num2}}{Num2} - \frac{Vol_{Num1}}{Num1} \right) dt$$

Vol_{Asset}, $BrShock_{Corr}$, and Vol_{Num} are vectors and $Corr_{Market}$ is a matrix.

Change of numeraire idea

To get the idea here, first remember the idea of the ordinary (non-vector) Girsanov Theorem:

$$\begin{aligned} \text{If } dX_t &= f(X_t)dt + \sigma(X_t)dW_t \text{ for } dW_t \text{ Brownian under } \mathbb{P} \\ \text{and } dX_t &= f^*(X_t)dt + \sigma(X_t)dW_t^* \text{ for } dW_t^* \text{ Brownian under } \mathbb{P}^* \\ \text{then } dW_t^* &= -\frac{f^*(X_t) - f(X_t)}{\sigma(X_t)}dt + dW_t \text{ must hold.} \end{aligned}$$

Notice that $\sigma(X_t)$ is the same no matter which measure is used. Girsanov says that this dW_t^* will in fact be Brownian for $\mathbb{P}^*$ defined by

$$\begin{aligned} \frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_t} &= \exp\left\{ -\frac{1}{2}\int_0^t \left(\frac{f^*(X_s) - f(X_s)}{\sigma(X_s)} \right)^2 ds \right. \\ &\qquad \left. + \int_0^t \left(\frac{f^*(X_s) - f(X_s)}{\sigma(X_s)} \right) dW_s \right\} \end{aligned}$$

Change of numeraire idea

This means

$$\mathbb{P}^*[A] = \int_A \frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} d\mathbb{P} \text{ for any } A \in \mathcal{F}_t \text{ and}$$

$$\mathbb{E}^*[X|\mathcal{F}_t] = \mathbb{E}\left[X \frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t}\right] \text{ for any RV } X$$

Itô's lemma (see next slide) gives

$$d\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} = \left(\frac{f^*(X_t) - f(X_t)}{\sigma(X_t)}\right)\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} dW_t$$

so $\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t}$ is a martingale under $\mathbb{P}$.

Change of numeraire idea

To see how Itô's lemma applied on the previous slide, define $Z_t = \int_0^t \left(\frac{f^*(X_s)-f(X_s)}{\sigma(X_s)}\right) dW_s$ so $\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t}$ can be considered to be a funtion of t and Z_t, and we apply Itô's lemma to that function

$$\begin{aligned} \frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} &= \exp\left\{-\frac{1}{2}\int_0^t \left(\frac{f^*(X_s)-f(X_s)}{\sigma(X_s)}\right)^2 ds + Z_t\right\} \\ d\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} &= \frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t}\left\{-\frac{1}{2}\left(\frac{f^*(X_t)-f(X_t)}{\sigma(X_t)}\right)^2 dt \right. \\ & \left. \quad +dZ_t + \frac{1}{2}dZ_t dZ_t\right\} \end{aligned}$$

But $dZ_t = \left(\frac{f^*(X_t)-f(X_t)}{\sigma(X_t)}\right) dW_t$, and
$dZ_t dZ_t = \left(\frac{f^*(X_t)-f(X_t)}{\sigma(X_t)}\right)^2 dW_t dW_t = \left(\frac{f^*(X_t)-f(X_t)}{\sigma(X_t)}\right)^2 dt$ which validates the result on the previous slide.

Change of numeraire idea

Let's set this up for our case where the market is a $(K+1)$-vector of assets $\mathbf{X}_t$, usually correlated, depending on a $(K+1)$-vector $\langle 0, W_1, ..., W_K \rangle$ of K independent Brownians, and $\mathbb{P}$ and $\mathbb{P}^*$are determined by two numeraires S and U:

$$\begin{aligned} \text{Let } d\mathbf{X}_t &= \boldsymbol{\mu}_t^S(\mathbf{X}_t)dt + \boldsymbol{\sigma}_t(\mathbf{X}_t)\mathbf{C}d\mathbf{W}_t^S \text{ for } d\mathbf{W}_t^S \text{ Brownian under } \mathbb{Q}^S \\ \text{and } d\mathbf{X}_t &= \boldsymbol{\mu}_t^U(\mathbf{X}_t)dt + \boldsymbol{\sigma}_t(\mathbf{X}_t)\mathbf{C}d\mathbf{W}_t^U \text{ for } d\mathbf{W}_t^U \text{ Brownian under } \mathbb{Q}^U \end{aligned}$$

where S_t and U_t are two numeraires, $d\mathbf{X}_t$, $\boldsymbol{\mu}_t^S(\mathbf{X}_t)$, $\boldsymbol{\mu}_t^U(\mathbf{X}_t)$, $d\mathbf{W}_t^S$ and $d\mathbf{W}_t^U$ are $(K+1)$-vectors (column), the non-zero components of $d\mathbf{W}_t^S$ and $d\mathbf{W}_t^U$ are sets of independent Brownians, $\mathbf{C}$ is a matrix that models correlations among the random shocks $d\mathbf{X}_t$ to the assets $\mathbf{X}_t$ that determine the market, and $\boldsymbol{\sigma}_t(\mathbf{X}_t)$ is a diagonal matrix that models the volatility of the random shocks $d\mathbf{X}_t$ relative to the vector $\mathbf{C}d\mathbf{W}_t^S$ or $\mathbf{C}d\mathbf{W}_t^U$ of correlated Brownian motions. $\boldsymbol{\sigma}_t(\mathbf{X}_t)$ and $\mathbf{C}$ are the same no matter which measure is used, just as in the one-dimensional case, the 0-th column of $\mathbf{C}$ is zeroes except for a leading 1,and the first entry in $\boldsymbol{\sigma}_t(\mathbf{X}_t)$ is 1. What is the Radon-Nikodym derivative process $\frac{d\mathbb{Q}^U}{d\mathbb{Q}^S}|_{\mathcal{F}_t}$ that makes $d\mathbf{W}_t^U$ Brownian under $\mathbb{Q}^U$?

Change of numeraire idea

To digest the notation, note that for any random column vector process $\mathbf{V}_t$, the exterior product $d\mathbf{V}_t\,(d\mathbf{V}_t)'$ is a matrix recording the instantaneous covariance among the random shocks $d\mathbf{V}_t$ to that vector. $()'$ just indicates the transpose. So, for example, $d\mathbf{W}_t\,(d\mathbf{W}_t)' = \hat{\mathbf{I}}dt$ where $\hat{\mathbf{I}}$ is the identity matrix except for a 0 first entry. Now let $\hat{\boldsymbol{\rho}}$ be the matrix of instantaneous correlations among the random shocks $\mathbf{C}d\mathbf{W}_t^S$ that drive the market, and $\boldsymbol{\rho} = \hat{\boldsymbol{\rho}}$ but with a top entry of 1. Then

$$
\begin{aligned}
\hat{\boldsymbol{\rho}}dt &= \left(\mathbf{C}d\mathbf{W}_t^S\right)\left(\mathbf{C}d\mathbf{W}_t^S\right)' = \mathbf{C}d\mathbf{W}_t^S\left(d\mathbf{W}_t^S\right)'\mathbf{C}' \\
&= \mathbf{C}\left(\hat{\mathbf{I}}dt\right)\mathbf{C}' = \mathbf{C}\hat{\mathbf{I}}\mathbf{C}'dt, \text{ so } \boldsymbol{\rho} = \mathbf{C}\mathbf{C}'
\end{aligned}
$$

The entries in $\hat{\boldsymbol{\rho}}$ are the derivatives with respect to t of the quadratic covariances among the components of $\mathbf{C}d\mathbf{W}_t^S$. The components c_{ij} of $\mathbf{C}$ satisfy $\sum_{k=0}^{K} c_{ik}c_{ki} = 1$ for all i. In fact, $d\mathbf{W}_t^S$ can be constructed such that $\mathbf{C}$ is lower triangular with 1's along the diagonal. Since $\mathbf{C}$ is independent of which numeraire is chosen $d\mathbf{W}_t^U$ will work the same.

Change of numeraire idea

For the value A_t of any particular asset defined (in terms of the column vector of assets $\mathbf{X}_t$ that define the market) by a row vector $\mathbf{A}$ of components, so $A_t = \mathbf{A}\mathbf{X}_t$, we have
$dA_t = \mu_t^S(A_t)dt + \sigma_t^A \mathbf{C} d\mathbf{W}_t^S = \mu_t^U(A_t)dt + \sigma_t^A \mathbf{C} d\mathbf{W}_t^U$ where
$dA_t = \mathbf{A}d\mathbf{X}_t$, $\mu_t^S(A_t) = \mathbf{A}\boldsymbol{\mu}_t^S(\mathbf{X}_t)$ and $\mu_t^U(A_t) = \mathbf{A}\boldsymbol{\mu}_t^U(\mathbf{X}_t)$ are scalars and $\sigma_t^A = \mathbf{A}\boldsymbol{\sigma}_t(\mathbf{X}_t)$ is a row vector.

So we can find dA_t for any asset A_t in terms of any numeraire S_t or U_t by studying $d\mathbf{X}_t$ representing the whole market. Then just vector product by $\mathbf{A}$.

Also if $A_t = \mathbf{V}_t(\boldsymbol{\phi})$ for a self-financing strategy $\boldsymbol{\phi}_t$ then we can write
$dA_t = \mu_t^S(\boldsymbol{\phi}_t)dt + \sigma_t^{\boldsymbol{\phi}} \mathbf{C} d\mathbf{W}_t^S = \mu_t^U(\boldsymbol{\phi}_t)dt + \sigma_t^{\boldsymbol{\phi}} \mathbf{C} d\mathbf{W}_t^U$ where
$dA_t = \boldsymbol{\phi}_t d\mathbf{X}_t$, $\mu_t^S(\boldsymbol{\phi}_t) = \boldsymbol{\phi}_t\boldsymbol{\mu}_t^S(\mathbf{X}_t)$, $\mu_t^U(\boldsymbol{\phi}_t) = \boldsymbol{\phi}_t\boldsymbol{\mu}_t^U(\mathbf{X}_t)$ and $\sigma_t^{\boldsymbol{\phi}} = \boldsymbol{\phi}_t\boldsymbol{\sigma}_t(\mathbf{X}_t)$ because $dA_t = \boldsymbol{\phi}_t d\mathbf{X}_t$ defines the self-financing concept.

Change of numeraire idea

In this matrix-vector setup the correct generalization of Girsanov is a function of the entire market

$$
\begin{aligned}
\frac{d\mathbb{Q}^U}{d\mathbb{Q}^S}|_{\mathcal{F}_t} &= \exp\left\{ -\frac{1}{2}\int_0^t \left| (\sigma_s(\mathbf{X}_s)\mathbf{C})^{-1} \left[\mu_s^U(\mathbf{X}_s) - \mu_s^S(\mathbf{X}_s) \right] \right|^2 ds \right. \\
&\quad \left. + \int_0^t \left((\sigma_s(\mathbf{X}_s)\mathbf{C})^{-1} \left[\mu_s^U(\mathbf{X}_s) - \mu_s^S(\mathbf{X}_s) \right] \right)' d\mathbf{W}_s^U \right\} \text{ so} \\
d\frac{d\mathbb{Q}^U}{d\mathbb{Q}^S}|_{\mathcal{F}_t} &= \left[\mu_t^U(\mathbf{X}_t) - \mu_t^S(\mathbf{X}_t) \right]' \left[(\sigma_t(\mathbf{X}_t)\mathbf{C})^{-1} \right]' \frac{d\mathbb{Q}^U}{d\mathbb{Q}^S}|_{\mathcal{F}_t} d\mathbf{W}_t^S
\end{aligned}
$$

by Itô's lemma. But we know (from the proof of FACT ONE) that

$$
\begin{aligned}
\frac{d\mathbb{Q}^U}{d\mathbb{Q}^S}|_{\mathcal{F}_t} &= \frac{S_0 U_t}{S_t U_0} \text{ so} \\
d\frac{d\mathbb{Q}^U}{d\mathbb{Q}^S}|_{\mathcal{F}_t} &= \frac{S_0}{U_0}\sigma_t^{S/U}\mathbf{C}d\mathbf{W}_t^S \text{ where } \sigma_t^{S/U} = \text{ some row vector} \\
\text{because } \frac{U_t}{S_t} &= \text{ a } \mathbb{Q}^S \text{ martingale by FACT ONE}
\end{aligned}
$$

Change of numeraire idea

Equate the coefficients of $d\mathbf{W}_t^S$ in the two different expressions for $d\frac{d\mathbb{Q}^U}{d\mathbb{Q}^S}|_{\mathcal{F}_t}$

$$\left[\mu_t^U(\mathbf{X}_t) - \mu_t^S(\mathbf{X}_t)\right]' \left[(\sigma_t(\mathbf{X}_t)\mathbf{C})^{-1}\right]' \frac{d\mathbb{Q}^U}{d\mathbb{Q}^S}|_{\mathcal{F}_t} = \frac{S_0}{U_0}\sigma_t^{S/U}\mathbf{C}$$

and substitute $\frac{d\mathbb{Q}^U}{d\mathbb{Q}^S}|_{\mathcal{F}_t} = \frac{S_0 U_t}{S_t U_0}$ so

$$\begin{aligned}\left[\mu_t^U(\mathbf{X}_t) - \mu_t^S(\mathbf{X}_t)\right]' \left[(\sigma_t(\mathbf{X}_t)\mathbf{C})^{-1}\right]' \frac{U_t}{S_t} &= \sigma_t^{S/U}\mathbf{C} \\ \left[(\sigma_t(\mathbf{X}_t)\mathbf{C})^{-1}\right]\left[\mu_t^U(\mathbf{X}_t) - \mu_t^S(\mathbf{X}_t)\right] &= \frac{S_t}{U_t}\mathbf{C}'\left(\sigma_t^{S/U}\right)'\end{aligned}$$

$$\mu_t^U(\mathbf{X}_t) = \mu_t^S(\mathbf{X}_t) + \frac{S_t}{U_t}\sigma_t(\mathbf{X}_t)\rho\left(\sigma_t^{S/U}\right)' \text{ (remember } \rho = \mathbf{C}\mathbf{C}'\text{)}$$

gives the change in the drifts for the whole market when the numeraire changes. We can also determine $\sigma_t^{S/U}$.

Change of numeraire idea

Write both numeraires as stochastic processes involving one measure

$$\begin{aligned} dS_t &= \mu_t^S(S_t)dt + \boldsymbol{\sigma}_t^S \mathbf{C} d\mathbf{W}_t^S \\ dU_t &= \mu_t^S(U_t)dt + \boldsymbol{\sigma}_t^U \mathbf{C} d\mathbf{W}_t^S \end{aligned}$$

where $\boldsymbol{\sigma}_t^U$ and $\boldsymbol{\sigma}_t^S$ are vectors. Now find $\boldsymbol{\sigma}_t^{S/U}$ by calculating

$$\begin{aligned} d\frac{U_t}{S_t} &= \frac{1}{S_t}dU_t + U_t d\frac{1}{S_t} + dU_t d\frac{1}{S_t} \text{ and} \\ d\frac{1}{S_t} &= -\frac{1}{S_t^2}dS_t + \frac{1}{S_t^3}dS_t dS_t \text{ (Itô's lemma)} \end{aligned}$$

and substitute the expressions for dS_t and dU_t. Since $\frac{U_t}{S_t}$ is a $\mathbb{Q}^S$ martingale (FACT ONE) we only need to worry about terms involving $\mathbf{C}d\mathbf{W}_t^S$

$$\begin{aligned} d\frac{U_t}{S_t} &= \left(\frac{\boldsymbol{\sigma}_t^U}{S_t} - \frac{U_t}{S_t^2}\boldsymbol{\sigma}_t^S\right)\mathbf{C}d\mathbf{W}_t^S \text{ from which} \\ \boldsymbol{\sigma}_t^{S/U} &= \frac{U_0}{S_0}\left(\frac{\boldsymbol{\sigma}_t^U}{S_t} - \frac{U_t}{S_t}\frac{\boldsymbol{\sigma}_t^S}{S_t}\right) \text{ (go back to def. of } \boldsymbol{\sigma}_t^{S/U}) \end{aligned}$$

Change of numeraire idea

Put this value for $\sigma_t^{S/U}$ into the formula for the change of market drifts when the numeraire changes

$$\mu_t^U(\mathbf{X}_t) = \mu_t^S(\mathbf{X}_t) + \frac{U_t}{S_t}\sigma_t(\mathbf{X}_t)\rho\frac{U_0}{S_0}\left(\frac{\sigma_t^S}{U_t} - \frac{S_t}{U_t}\frac{\sigma_t^U}{U_t}\right)'$$

$$\mu_t^U(\mathbf{X}_t) = \mu_t^S(\mathbf{X}_t) + \sigma_t(\mathbf{X}_t)\rho\frac{U_0}{S_0}\left(\frac{\sigma_t^U}{U_t} - \frac{\sigma_t^S}{S_t}\right)'$$

To simplify things assume $U_0 = S_0$

$$\mu_t^U(\mathbf{X}_t) = \mu_t^S(\mathbf{X}_t) + \sigma_t(\mathbf{X}_t)\rho\left(\frac{\sigma_t^U}{U_t} - \frac{\sigma_t^S}{S_t}\right)' \text{ for the total market.}$$

For any asset A_t just multiply by its component vector $\mathbf{A}$

$$\mu_t^U(A_t) = \mu_t^S(A_t) + \sigma_t^A\rho\left(\frac{\sigma_t^U}{U_t} - \frac{\sigma_t^S}{S_t}\right)'$$

which, finally, is the proof for FACT THREE.

Change of numeraire idea

For FACT FOUR go back to

$$\mu_t^U(\mathbf{X}_t)dt + \sigma_t(\mathbf{X}_t)\mathbf{C}d\mathbf{W}_t^U = \mu_t^S(\mathbf{X}_t)dt + \sigma_t(\mathbf{X}_t)\mathbf{C}d\mathbf{W}_t^S$$

because they both $= d\mathbf{X}_t$. Now rearrange

$$\sigma_t(\mathbf{X}_t)\mathbf{C}d\mathbf{W}_t^U = \sigma_t(\mathbf{X}_t)\mathbf{C}d\mathbf{W}_t^S - \left(\mu_t^U(\mathbf{X}_t) - \mu_t^S(\mathbf{X}_t)\right)dt$$

and substitute from the total market form of FACT THREE to get a total market expression for FACT FOUR

$$\sigma_t(\mathbf{X}_t)\mathbf{C}d\mathbf{W}_t^U = \sigma_t(\mathbf{X}_t)\mathbf{C}d\mathbf{W}_t^S - \ \boldsymbol{\sigma}_\mathbf{t}(\mathbf{X}_\mathbf{t})\boldsymbol{\rho}\left(\frac{\sigma_t^U}{U_t} - \frac{\sigma_t^S}{S_t}\right)' dt$$

For any asset A_t hit both sides with $\mathbf{A}$ to get FACT FOUR

$$\sigma_t^A\mathbf{C}d\mathbf{W}_t^U = \sigma_t^A\mathbf{C}d\mathbf{W}_t^S - \ \sigma_t^A\boldsymbol{\rho}\left(\frac{\sigma_t^U}{U_t} - \frac{\sigma_t^S}{S_t}\right)' dt$$

Change of numeraire idea

In fact, we can hit both sides of the total market form of FACT THREE with $\sigma_{\mathbf{t}}(\mathbf{X_t})^{-1}$. It's diagonal and unless the market is degenerate (has extra non-random investments, but we can fix that at the beginning) will have all diagonal entries non-zero, so it must be invertible. This gives

$$\mathbf{C}d\mathbf{W}_t^U = \mathbf{C}d\mathbf{W}_t^S - \boldsymbol{\rho}\left(\frac{\boldsymbol{\sigma}_t^U}{U_t} - \frac{\boldsymbol{\sigma}_t^S}{S_t}\right)' dt$$

the correlated Brownian underpinning for FACT FOUR.

We'll use this later in the form

$$\mathbf{C}d\mathbf{W}_t^U - \mathbf{C}d\mathbf{W}_t^S = -\boldsymbol{\rho}\left(\frac{\boldsymbol{\sigma}_t^U}{U_t} - \frac{\boldsymbol{\sigma}_t^S}{S_t}\right)' dt$$

Change of numeraire idea

For the exercise, to find dU_t in the measure corresponding to U_t, use FACT THREE with the bank account B_t as the first numeraire

$$\mu_t^U(U_t) = \mu_t^B(U_t) + \boldsymbol{\sigma}_t^U \boldsymbol{\rho} \left(\frac{\boldsymbol{\sigma}_t^U}{U_t} - D(0,t)\boldsymbol{\sigma}_t^B \right)'$$

But the drift of any asset, including B_t, under the bank account (risk-neutral) measure $\mathbb{Q}^B = \mathbb{Q}$ is r_t times the value of the asset, so

$$\begin{aligned} \mu_t^U(U_t) &= r_t U_t + \boldsymbol{\sigma}_t^U \boldsymbol{\rho} \left(\frac{\boldsymbol{\sigma}_t^U}{U_t} - D(0,t)\boldsymbol{\sigma}_t^B \right)' \text{ and} \\ dU_t &= \left(r_t U_t + \boldsymbol{\sigma}_t^U \boldsymbol{\rho} \left(\frac{\boldsymbol{\sigma}_t^U}{U_t} - D(0,t)\boldsymbol{\sigma}_t^B \right)' \right) dt + \boldsymbol{\sigma}_t^U \mathbf{C} d\mathbf{W}_t^U \end{aligned}$$

But $\boldsymbol{\rho} \left(\boldsymbol{\sigma}_t^B\right)'$ is the $\mathbf{0}$-vector because $r_t B_t dt = dB_t = \mu_t^B(B_t)dt + \boldsymbol{\sigma}_t^B \mathbf{C} d\mathbf{W}_t^B$ making $\boldsymbol{\sigma}_t^B \mathbf{C} = \mathbf{0}$, so

$$dU_t = \left(r_t U_t + \frac{\boldsymbol{\sigma}_t^U \boldsymbol{\rho} \left(\boldsymbol{\sigma}_t^U\right)'}{U_t} \right) dt + \boldsymbol{\sigma}_t^U \mathbf{C} d\mathbf{W}_t^U$$

Change of numeraire idea

This can be rewritten as

$$\begin{aligned} dU_t &= \left(r_t + \frac{\sigma_t^U \boldsymbol{\rho} \left(\sigma_t^U\right)'}{U_t^2} \right) U_t dt + \sigma_t^U \mathbf{C} d\mathbf{W}_t^U \text{ or} \\ \mu_t^U(U_t) &= \left(r_t + \frac{\sigma_t^U \boldsymbol{\rho} \left(\sigma_t^U\right)'}{U_t^2} \right) U_t \end{aligned}$$

The answer to the exercise is that for any numeraire its drift in the measure corresponding to that numeraire is the numeraire itself, times the risk free rate plus the "correlated square" volatility per unit of numeraire.

Change of numeraire idea

An expression such as $\frac{\sigma_t^U}{U_t}$ is reminiscent of the derivative of a logarithm, $\frac{\sigma_t^U}{U_t} - \frac{\sigma_t^S}{S_t}$ of the derivative of the difference of two logarithms, i.e. the derivative of the logarithm of a ratio. To make this precise, for any asset A_t (including U_t or S_t) use Itô's lemma to calculate

$$\begin{aligned}
d\ln(A_t) &= \frac{dA_t}{A_t} - \frac{1}{2}\frac{dA_t dA_t}{A_t^2} \\
&= \frac{\mu_t^U(A_t)}{A_t}dt + \frac{\sigma_t^A}{A_t}\mathbf{C}d\mathbf{W}_t^U - \frac{1}{2}\frac{\left(\sigma_t^A\mathbf{C}d\mathbf{W}_t^U\right)\left(\sigma_t^A\mathbf{C}d\mathbf{W}_t^U\right)'}{A_t^2} \\
&= \frac{\mu_t^U(A_t)}{A_t}dt + \frac{\sigma_t^A}{A_t}\mathbf{C}d\mathbf{W}_t^U - \frac{1}{2}\frac{\sigma_t^A\mathbf{C}\mathbf{I}\mathbf{C}'\left(\sigma_t^A\right)'}{A_t^2}dt \\
&= \left(\frac{\mu_t^U(A_t)}{A_t} - \frac{1}{2}\frac{\sigma_t^A\boldsymbol{\rho}\left(\sigma_t^A\right)'}{A_t^2}\right)dt + \frac{\sigma_t^A}{A_t}\mathbf{C}d\mathbf{W}_t^U
\end{aligned}$$

Change of numeraire idea

But the exercise told us that for a numeraire $\mu_t^U(U_t) = \left(r_t + \frac{\sigma_t^U \rho \left(\sigma_t^U\right)'}{U_t^2}\right) U_t$. Plug that into the expression for $d\ln(U_t)$ and see that in the measure corresponding to any numeraire the drift of the logarithm of that numeraire is the risk free rate plus half a "correlated square" volatility per unit of numeaire:

$$
\begin{aligned}
d\ln(U_t) &= \left(r_t + \frac{1}{2}\frac{\sigma_t^U \rho \left(\sigma_t^U\right)'}{U_t^2}\right) dt + \frac{\sigma_t^U}{U_t}\mathbf{C}d\mathbf{W}_t^U \\
\text{Similarly, } d\ln(S_t) &= \left(r_t + \frac{1}{2}\frac{\sigma_t^S \rho \left(\sigma_t^S\right)'}{S_t^2}\right) dt + \frac{\sigma_t^S}{S_t}\mathbf{C}d\mathbf{W}_t^S. \\
\text{So } d\ln\left(\frac{U_t}{S_t}\right) &= D_t dt + \frac{\sigma_t^U}{U_t}\mathbf{C}d\mathbf{W}_t^U - \frac{\sigma_t^S}{S_t}\mathbf{C}d\mathbf{W}_t^S \text{ where} \\
D_t &= \frac{1}{2}\left(\frac{\sigma_t^U \rho \left(\sigma_t^U\right)'}{U_t^2} - \frac{\sigma_t^S \rho \left(\sigma_t^S\right)'}{S_t^2}\right)
\end{aligned}
$$

Change of numeraire idea

So

$$\begin{aligned} d\ln\left(\frac{U_t}{S_t}\right) &= D_t dt + \left(\frac{\sigma_t^U}{U_t} - \frac{\sigma_t^S}{S_t}\right)\mathbf{C}d\mathbf{W}_t^U + \frac{\sigma_t^S}{S_t}\left(\mathbf{C}d\mathbf{W}_t^U - \mathbf{C}d\mathbf{W}_t^S\right) \\ &= \left(\frac{\sigma_t^U}{U_t} - \frac{\sigma_t^S}{S_t}\right)\mathbf{C}d\mathbf{W}_t^U + \left[D_t - \frac{\sigma_t^S}{S_t}\boldsymbol{\rho}\left(\frac{\sigma_t^U}{U_t} - \frac{\sigma_t^S}{S_t}\right)'\right]dt \end{aligned}$$

If now for any asset A_t we multiply $d\ln(A_t)d\ln(\frac{U_t}{S_t})$ and substitute the expressions just developed for $d\ln(A_t)$ and $d\ln(\frac{U_t}{S_t})$, we can ignore the drifts (the coefficients of dt), because $dtdt = 0$ and $d\mathbf{W}_t^U dt = 0$.

Change of numeraire idea

$$
\begin{aligned}
d\ln(A_t)d\ln(\frac{U_t}{S_t}) &= \frac{\sigma_t^A}{A_t}\mathbf{C}d\mathbf{W}_t^U\left(\left(\frac{\sigma_t^U}{U_t}-\frac{\sigma_t^S}{S_t}\right)\mathbf{C}d\mathbf{W}_t^U\right)' \\
&= \frac{\sigma_t^A}{A_t}\mathbf{C}d\mathbf{W}_t^U\left(d\mathbf{W}_t^U\right)'\mathbf{C}'\left(\frac{\sigma_t^U}{U_t}-\frac{\sigma_t^S}{S_t}\right)' \\
&= \frac{\sigma_t^A}{A_t}\boldsymbol{\rho}\left(\frac{\sigma_t^U}{U_t}-\frac{\sigma_t^S}{S_t}\right)' dt \text{ so} \\
A_t d\ln(A_t)\left(d\ln(\frac{U_t}{S_t})\right) &= \sigma_t^A\boldsymbol{\rho}\left(\frac{\sigma_t^U}{U_t}-\frac{\sigma_t^S}{S_t}\right)' dt \text{ or also} \\
A_t\frac{d}{dt}\left[\ln(A_t),\ln(\frac{U_t}{S_t}\right]_t &= \sigma_t^A\boldsymbol{\rho}\left(\frac{\sigma_t^U}{U_t}-\frac{\sigma_t^S}{S_t}\right)'
\end{aligned}
$$

These expression for the ubiquitous $\sigma_t^A\boldsymbol{\rho}\left(\frac{\sigma_t^U}{U_t}-\frac{\sigma_t^S}{S_t}\right)'$ factors can be put back into the expressions for FACT THREE and FACT FOUR

Change of numeraire idea
FACT THREE

$$\begin{aligned}\mu_t^U(A_t)dt &= \mu_t^S(A_t)dt + A_t d\ln(A_t) d\ln(\frac{U_t}{S_t}) \\ \mu_t^U(A_t) &= \mu_t^S(A_t) + A_t \frac{d}{dt}\left[\ln(A_t), \ln(\frac{U_t}{S_t})\right]_t\end{aligned}$$

FACT FOUR

$$\begin{aligned}\sigma_t^A \mathbf{C} d\mathbf{W}_t^U &= \sigma_t^A \mathbf{C} d\mathbf{W}_t^S - A_t d\ln(A_t) d\ln(\frac{U_t}{S_t}) \\ &= \sigma_t^A \mathbf{C} d\mathbf{W}_t^S - A_t \frac{d}{dt}\left[\ln(A_t), \ln(\frac{U_t}{S_t})\right]_t dt\end{aligned}$$

Change of numeraire idea

Another version of FACT THREE: if $\mu_t^S(A_t) = A_t m_t^{AS}$ for m_t^{AS} some proportional drift (as in a lognormal)

$$\begin{aligned} \text{then } \mu_t^U(A_t)dt &= A_t m_t^{AS} dt + A_t d\ln(A_t) d\ln(\frac{U_t}{S_t}) \\ &= A_t \left(m_t^{AS} dt + d\ln(A_t) d\ln(\frac{U_t}{S_t}) \right) \end{aligned}$$

so $\mu_t^U(A_t)$ also has a proportional drift $\mu_t^U(A_t) = A_t m_t^{AU}$ where

$$\begin{aligned} m_t^{AU} dt &= m_t^{AS} dt + d\ln(A_t) d\ln(\frac{U_t}{S_t}) \text{ or} \\ m_t^{AU} &= m_t^{AS} + \frac{d}{dt}\left[\ln(A_t), \ln(\frac{U_t}{S_t})\right]_t \end{aligned}$$

Change of numeraire idea

Another version of FACT FOUR: if $\sigma_t^A \mathbf{C} d\mathbf{W}_t^S = A_t \mathbf{v}_t^{AS} \mathbf{C} d\mathbf{W}_t^S$ for $\mathbf{v}_t^{AS}$ some vector of proportional volatilities (as in a multivariate lognormal)

$$\begin{aligned} \text{then } \sigma_t^A \mathbf{C} d\mathbf{W}_t^U &= \sigma_t^A \mathbf{C} d\mathbf{W}_t^S - A_t d\ln(A_t) d\ln(\frac{U_t}{S_t}) \\ &= A_t \left(\mathbf{v}_t^{AS} \mathbf{C} d\mathbf{W}_t^S - d\ln(A_t) d\ln(\frac{U_t}{S_t}) \right) \end{aligned}$$

so $\sigma_t^A \mathbf{C} d\mathbf{W}_t^U$ also has proportional volatilities $\sigma_t^A \mathbf{C} d\mathbf{W}_t^U = A_t \mathbf{v}_t^{AU} \mathbf{C} d\mathbf{W}_t^U$ where

$$\begin{aligned} \mathbf{v}_t^{AU} \mathbf{C} d\mathbf{W}_t^U &= \mathbf{v}_t^{AS} \mathbf{C} d\mathbf{W}_t^S - d\ln(A_t) d\ln(\frac{U_t}{S_t}) \\ &= \mathbf{v}_t^{AS} \mathbf{C} d\mathbf{W}_t^S - \frac{d}{dt} \left[\ln(A_t), \ln(\frac{U_t}{S_t}) \right]_t dt \end{aligned}$$

Change of numeraire idea

To summarize

FACT ONE: In the measure associated with a numeraire any asset divided by that numeraire is a martingale.

FACT TWO: In the measure associated with a numeraire the price of any attainable claim (payoff) is the expected value of the "present value" of the claim, "discounted" using that numeraire.

FACT THREE: When changing numeraires change the drift of any asset by *adding* the volatilities of the asset (relative to the market) multiplied (with correlation) by the differences of the pro-rata volatilities of the numeraires (relative to the market).

FACT FOUR: When changing numeraires change the random shocks to any asset by *subtracting* the volatilities of the asset (relative to the market) multiplied (with correlation) by the differences of the pro-rata volatilities of the numeraires (relative to the market).