Structure of the CAPM Covariance Matrix

James Bridgeman
University of Connecticut

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INTENTIONS

- SEE HOW CAPM COVARIANCE MATRIX:
  - ACTUALLY HOLDS THE EXPECTED RETURNS
  - ACTIVELY REFLECTS THE MARKET WEIGHTS
- MOSTLY PEDAGOGY
- A TINY BIT OF LIGHT ON LITERATURE

CAPM SET-UP

- MARKET COVARIANCE MATRIX - $\sigma_{ij} = \text{covariance}[\bar{r}_i, \bar{r}_j]$

$$\boldsymbol{\Sigma} = \begin{Bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{Bmatrix} \text{ invertible, } \sigma_{ij} = \sigma_{ji},\ \sigma_{ii} > 0, \text{ and } \sigma_{ii}\sigma_{jj} > \sigma_{ij}^2$$

- MARKET WEIGHTS & EXPECTED RETURNS

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix} \text{ market weights, } \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix},\ \mathbf{e}^T \cdot \mathbf{w} = 1,$$

$w_i > 0$ (you can't short what no one owns)

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ \dots \\ r_n \end{pmatrix} \text{ expected returns, } r_f \text{ risk-free rate, } (\mathbf{r} - r_f\mathbf{e}) \text{ market risk premia}$$

ASSUMING MARKET EFFICIENT (ON FRONTIER)

(FOR EXAMPLE, ASSUME THAT $\exists$ RATIONAL REP AGENT or THAT MARKET IS BIVAR NORMAL or ALL HAVE QUADR UTIL)

- COVARIANCE-RETURNS-WEIGHTS RELATIONSHIP

$$(\mathbf{r} - r_f \mathbf{e}) = \boldsymbol{\Sigma} \cdot \mathbf{w}$$

$$\mathbf{w} = \frac{\boldsymbol{\Sigma}^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \boldsymbol{\Sigma}^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} \text{ (denominator ensures } \mathbf{e}^T \cdot \mathbf{w} = 1\text{)}$$

- RETURNS-BETAS RELATIONSHIP

$$(\mathbf{r} - r_f \mathbf{e}) = \boldsymbol{\beta} \left(\mathbf{w}^T \cdot \mathbf{r} - r_f \right) = \boldsymbol{\beta} \left(r_M - r_f \right)$$

$$\begin{aligned} \text{where } \boldsymbol{\beta} &= \frac{\boldsymbol{\Sigma} \cdot \mathbf{w}}{\mathbf{w}^T \cdot \boldsymbol{\Sigma} \cdot \mathbf{w}} = \mathbf{Cov}\left[\bar{\mathbf{r}}, \bar{r}_M\right] = \mathbf{Cov}\left[\bar{\mathbf{r}}, \mathbf{w}^T \cdot \bar{\mathbf{r}}\right] \\ \text{and } r_M &= \mathbf{w}^T \cdot \mathbf{r} \text{ is the expected market return} \end{aligned}$$

QUICK PROOF

FOR COVARIANCE-RETURNS-WEIGHTS RELATIONSHIP - use Lagrange Multipliers

FOR BETA

$$
\begin{aligned}
(\mathbf{r} - r_f \mathbf{e}) &= \boldsymbol{\Sigma} \cdot \mathbf{w} \\
&= \frac{\boldsymbol{\Sigma} \cdot \mathbf{w}}{\mathbf{w}^T \cdot \boldsymbol{\Sigma} \cdot \mathbf{w}} \left(\mathbf{w}^T \cdot \boldsymbol{\Sigma} \cdot \mathbf{w} \right) \\
&= \beta \left(\mathbf{w}^T \cdot (\mathbf{r} - r_f \mathbf{e}) \right) \\
&= \beta \left(\mathbf{w}^T \cdot \mathbf{r} - r_f \mathbf{w}^T \cdot \mathbf{e} \right) \\
&= \beta \left(\mathbf{w}^T \cdot \mathbf{r} - r_f \right) \\
&= \beta \left(r_M - r_f \right)
\end{aligned}
$$

- EXAM QUESTION
  - Be Sure To Emphasize Effect Of Negative Covariance

$$\mathbf{\Sigma} = \begin{Bmatrix} .01 & .10 & -.20 \\ .10 & .04 & .25 \\ -.20 & .25 & .09 \end{Bmatrix}, \mathbf{r} = \begin{Bmatrix} .02 \\ .10 \\ .20 \end{Bmatrix}, r_f = .03, \text{ what is } \mathbf{w}?$$

- WAIT UNTIL GRADING TO WORK OUT THE ANSWER
- ANSWER

$$\mathbf{\Sigma}^{-1} = \begin{Bmatrix} 4.5 & 4.5 & -2.5 \\ 4.5 & 3.0 & 1.7 \\ -2.5 & 1.7 & .73 \end{Bmatrix}, \mathbf{w} = \frac{\mathbf{\Sigma}^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} = \begin{Bmatrix} -.27 \\ .80 \\ .47 \end{Bmatrix}, \text{ Rats!}$$

- WHAT WENT WRONG?
  - AHA! Forgot Requirement $\sigma_{ii}\sigma_{jj} > \sigma_{ij}^2$. Easy To Fix Next Semester.

ENLIGHTENED PERHAPS, BUT STILL LAZY

- EXAM TIME THE FOLLOWING SEMESTER
  - Still Be Sure To Emphasize Effect Of Negative Covariance

$$\mathbf{\Sigma} = \begin{Bmatrix} .010 & .015 & -.028 \\ .015 & .040 & .05 \\ -.028 & .05 & .090 \end{Bmatrix}, \mathbf{r} = \begin{Bmatrix} .02 \\ .10 \\ .20 \end{Bmatrix}, r_f = .03, \text{ what is } \mathbf{w}?$$

- STILL WAIT UNTIL GRADING TO WORK OUT THE ANSWER
- ANSWER

$$\mathbf{\Sigma}^{-1} = \begin{Bmatrix} -13 & 33 & -23 \\ 33 & -1.4 & 11 \\ -23 & 11 & -2.1 \end{Bmatrix}, \mathbf{w} = \frac{\mathbf{\Sigma}^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} = \begin{Bmatrix} -1.92 \\ 2.03 \\ .89 \end{Bmatrix}, \text{ Worse!}$$

- WHAT WENT WRONG THIS TIME?
  - God Knows, Just Avoid The Negative Covariances Next Semester

JUST PLAIN STUBBORNLY LAZY

- THIRD SEMESTER'S A CHARM?
  - Let Everything Be Positive, Including All Of The Risk Premia

$$\mathbf{\Sigma} = \begin{Bmatrix} .010 & .015 & .028 \\ .015 & .040 & .05 \\ .028 & .05 & .090 \end{Bmatrix}, \mathbf{r} = \begin{Bmatrix} .04 \\ .10 \\ .20 \end{Bmatrix}, r_f = .03, \text{ what is } \mathbf{w}?$$

- CONFIDENTLY WAIT TO WORK OUT THE ANSWER
- ANSWER

$$\mathbf{\Sigma}^{-1} = \begin{Bmatrix} 791 & 36 & -266 \\ 36 & 83.5 & -57.6 \\ -266 & -57.6 & 126 \end{Bmatrix}, \mathbf{w} = \frac{\mathbf{\Sigma}^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} = \begin{Bmatrix} 1.47 \\ .15 \\ -.62 \end{Bmatrix}, \text{ No!}$$

- THERE'S MORE TO THIS THAN MEETS THE EYE!
- WHAT DOES Σ HAVE TO LOOK LIKE TO BE A MARKET?

A FAMILIAR PHENOMENON IN THE LITERATURE

- BEST & GRAUER (1985, 1992)
  - Efficient Markets with all $w_i > 0$ form a segment of the Frontier with length $\longrightarrow 0$ as number of assets $n \longrightarrow \infty$
- Brennan & Lo (2010)
  - For any given $(\mathbf{r} - r_f \mathbf{e})$, *Impossible* $\boldsymbol{\Sigma}$ are those with some $w_i < 0$. Then $\mathbb{P}\left[\boldsymbol{\Sigma}\ \textit{Impossible}\right] \nearrow$ geometrically with n for reasonable distribution assumptions on $\boldsymbol{\Sigma}$.
- LEDOIT & WOLF (2004, 2013, 2014)
  - Introduce statistical Shrinkage techniques to transform Empirical $\boldsymbol{\Sigma}$ into a Frontier Portfolio
- LEVY & ROLL (2010)
  - Empirical $\boldsymbol{\Sigma}$ has "high" $\mathbb{P}$ of being "close" to a Frontier Portfolio, with "close" attained by $\pm\epsilon$ on σ_{ii} and r_i. Note ρ_{ij} can remain fixed.
- BOYLE (2012, 2014)
  - On large class of $\boldsymbol{\Sigma}$, Frontier Portfolio equivalent to $\boldsymbol{\Sigma}$ Almost Positive

IS THERE A SIMPLE-MINDED WAY TO SEE ALL THIS?

- GIVEN $\mathbf{w}$ and $\mathbf{r} - r_f\mathbf{e}$ WITH ALL $w_i > 0$
- WHAT DO Σ THAT SIT ON FRONTIER LOOK LIKE?

$$\begin{aligned}
\mathbf{w} &= \frac{\mathbf{\Sigma}^{-1} \cdot (\mathbf{r} - r_f\mathbf{e})}{\mathbf{e}^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mathbf{r} - r_f\mathbf{e})} \text{ so} \\
\mathbf{\Sigma} \cdot \mathbf{w} &= \frac{(\mathbf{r} - r_f\mathbf{e})}{\mathbf{e}^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mathbf{r} - r_f\mathbf{e})} \text{ and for any } k \\
(k\mathbf{\Sigma}) \cdot \mathbf{w} &= \frac{k\,(\mathbf{r} - r_f\mathbf{e})}{\mathbf{e}^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mathbf{r} - r_f\mathbf{e})} \\
&= \frac{(\mathbf{r} - r_f\mathbf{e})}{\mathbf{e}^T \cdot (k\mathbf{\Sigma})^{-1} \cdot (\mathbf{r} - r_f\mathbf{e})}
\end{aligned}$$

So it is enough to find all $\mathbf{\Sigma}$ such that

$$\mathbf{\Sigma} \cdot \mathbf{w} = (\mathbf{r} - r_f\mathbf{e})$$

and then multiply by any constant.

IS THERE A SIMPLE-MINDED WAY TO SEE ALL THIS?

- EASY TO FIND A MATRIX $\mathbf{R}$ WITH $\mathbf{R}\cdot\mathbf{w}=\mathbf{r}-r_f\mathbf{e}$

$$\text{Just let } \mathbf{R} = \left\{ \left(\mathbf{r} - r_f\mathbf{e}\right)\mathbf{e}^T \right\}$$

with n identical column vectors each equal to $(\mathbf{r} - r_f\mathbf{e})$. We can also express $\mathbf{R}$ as n row vectors

$$\mathbf{R} = \left\{ \begin{matrix} \left(r_1 - r_f\right)\mathbf{e}^T \\ \left(r_2 - r_f\right)\mathbf{e}^T \\ \cdots \\ \left(r_n - r_f\right)\mathbf{e}^T \end{matrix} \right\}.$$

- UNFORTUNATELY THE CHOICE $\mathbf{\Sigma}=\mathbf{R}$ IS NOT INVERTIBLE
  - Rows and columns clearly fail to be independent
- COULD EASILY FAIL TO HAVE $\sigma_{ii} > 0$
- IS UNLIKELY TO HAVE $\sigma_{ij} = \sigma_{ji}$ OR $\sigma_{ii}\sigma_{jj} > \sigma_{ij}^2$ FOR ALL i, j

IS THERE A SIMPLE-MINDED WAY TO SEE ALL THIS?

- BUT MAYBE THERE IS A MATRIX **A** WITH $\mathbf{A}\cdot\mathbf{w}=\mathbf{0}$

$$\text{So } (\mathbf{R}+\mathbf{A})\cdot\mathbf{w} = \mathbf{R}\cdot\,\mathbf{w} = (\mathbf{r} - r_f\mathbf{e})$$

- AND WITH $\mathbf{\Sigma} = (\mathbf{R}+\mathbf{A})$ INVERTIBLE AND SATISFYING

$$\sigma_{ij}=\sigma_{ji},\ \sigma_{ii}>0, \text{ and } \sigma_{ii}\sigma_{jj}>\sigma_{ij}^2$$

- VIEW **A** AS ROW VECTORS

$$\mathbf{A} = \begin{Bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \dots \\ \mathbf{a}_n^T \end{Bmatrix}, \text{ with each } \mathbf{a}_i^T = \begin{Bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{Bmatrix}$$

- THE CONDITION FOR $\mathbf{A}\cdot\mathbf{w}=\mathbf{0}$ IS

$$\mathbf{a}_i^T \cdot \mathbf{w} = 0 \text{ for all } i$$

in other words, all n of the row vectors $\mathbf{a}_i^T$ must be in the $(n-1)$-dimensional hyperplane through the origin perpendicular to the market weight vector **w**.

THERE IS A SIMPLE-MINDED WAY TO UNDERSTAND!

- IF ALL $\boldsymbol{\Sigma}$ ARISE THIS WAY
- AND IF ALL OF THE OTHER CONDITIONS CAN BE MET
  - (PERHAPS TWO BIG IFs)
- THEN
  - The odds of an $(n-1)$-dimensional hyperplane through the origin being perpendicular to a vector $\mathbf{w}$ with all $w_i > 0$ is $\left(\frac{1}{2}\right)^{n-1}$
    - Half the lines through the origin in 2-space are perpendicular to something in first quadrant.
    - A quarter of the planes through the origin in 3-space are perpendicular to someting in the first octant.
    - And so on ...
  - The lazy pedagogue had at best 1 chance in 4 to write down a valid $\boldsymbol{\Sigma}$
    - even after he respected all of the σ constraints
  - It's immediately clear that the odds to randomly encounter a valid $\boldsymbol{\Sigma}$ disappear at least exponentially in n, independent of (reasonable) distribution assumptions

- $\sigma_{ij} = \sigma_{ji}$ means that

$$\begin{aligned} (r_i - r_f) + a_{ij} &= (r_j - r_f) + a_{ji} \\ \text{so } a_{ij} &= a_{ji} - (r_i - r_j) \end{aligned}$$

and $\mathbf{A}$ can be expressed also as consisting of n column vectors

$$\mathbf{A} = \begin{Bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \ldots & \mathbf{c}_n \end{Bmatrix} \text{ where } \mathbf{c}_j = \mathbf{a}_j - (\mathbf{r} - r_j \mathbf{e}) .$$

- $\sigma_{ii} > 0$ means that

$$\begin{aligned} (r_i - r_f) + a_{ii} &> 0 \\ \text{so } a_{ii} &> -(r_i - r_f) . \end{aligned}$$

- $\sigma_{ii}\sigma_{jj} > \sigma_{ij}^2$ means that

$$(a_{ii} + (r_i - r_f))(a_{jj} + (r_j - r_f)) > (a_{ij} + (r_i - r_f))^2$$

$$\text{so } a_{jj} > -(r_j - r_f) + \frac{(a_{ij} + (r_i - r_f))^2}{a_{ii} + (r_i - r_f)} .$$

- Looks like an induction might be possible.

INVERTIBILITY CONDITIONS

$$\Sigma = \mathbf{R}+\mathbf{A} = \begin{Bmatrix} (r_1 - r_f)\,\mathbf{e}^T + \mathbf{a}_1^T \\ (r_2 - r_f)\,\mathbf{e}^T + \mathbf{a}_2^T \\ \cdots \\ (r_n - r_f)\,\mathbf{e}^T + \mathbf{a}_n^T \end{Bmatrix}$$

IS INVERTIBLE IFF ITS n ROW VECTORS ARE INDEPENDENT

- INDEPENDENCE HOLDS IF AND ONLY IF
  1. The n row vectors $\mathbf{a}_1^T$, $\mathbf{a}_2^T$, ... ,$\mathbf{a}_n^T$ span the $(n-1)$-dimensional hyperplane perpendicular to the market weight vector $\mathbf{w}$, and
  2. $\mathbf{w}^T \cdot (\mathbf{r} - r_f\mathbf{e}) \neq 0$, i.e. the market risk premium vector $(\mathbf{r} - r_f\mathbf{e})$ is not perpendicular to the market weight vector $\mathbf{w}$.
- PROOF
  1. $(r_1 - r_f)\,\mathbf{e}^T$, ... ,$(r_n - r_f)\,\mathbf{e}^T$ are collinear so the row vectors of $\mathbf{R}+\mathbf{A}$ cannot span an n-space unless $\mathbf{a}_1^T$, ... ,$\mathbf{a}_n^T$ span an $(n-1)$-space, and the $(n-1)$-dimensional hyperplane perpendicular to the market weight vector $\mathbf{w}$ is the $(n-1)$-space they are in.

INVERTIBILITY CONDITIONS

- PROOF (continued)

  1. (prior slide)
  2. Requires a slightly fussy proof that essentially follows from the fact that $\mathbf{e}^T$ and $\mathbf{w}$ both have all components positive and $\mathbf{a}_1^T$, ... ,$\mathbf{a}_n^T$ are all perpendicular to $\mathbf{w}$.

- CONDITION 2 HOLDS IN ANY REASONABLE MODEL

$$\mathbf{w}^T \cdot (\mathbf{r} - r_f \mathbf{e}) = \left(\mathbf{w}^T \cdot \mathbf{r} - r_f \right) = (r_M - r_f) > 0$$

  where $r_M = \mathbf{w}^T \cdot \mathbf{r}$ is the expected return on the risky market.

- SO $\mathbf{\Sigma} = \mathbf{R}+\mathbf{A}$ IS INVERTIBLE IN A REASONABLE MODEL IF AND ONLY IF

  1. The n row vectors $\mathbf{a}_1^T$, $\mathbf{a}_2^T$, ... ,$\mathbf{a}_n^T$ span the $(n-1)$-dimensional hyperplane perpendicular to the market weight vector $\mathbf{w}$

HOW HARD IS IT TO CHOOSE A?

- WORK BACKWARDS TO CHOOSE $\mathbf{a}_1^T$, $\mathbf{a}_2^T$, ... ,$\mathbf{a}_n^T$
  - Suppose you already have chosen $\mathbf{a}_1^T$, $\mathbf{a}_2^T$, ... ,$\mathbf{a}_{n-1}^T$.
  - Then there is no choice about what $\mathbf{a}_n^T$ must be. By symmetry:

$$\begin{aligned} a_{n\ 1} &= a_{1\ n} - (r_n - r_1) \\ a_{n\ 2} &= a_{2\ n} - (r_n - r_2) \\ &\ldots \\ a_{n\ n-1} &= a_{n-1\ n} - (r_n - r_{n-1}) . \end{aligned}$$

Since $\mathbf{a}_n^T \cdot \mathbf{w} = 0$ the choice for $a_{n\ n}$ also is fixed. The requirement is

$$w_1 a_{n\ 1} + \ \ldots\ + w_{n-1} a_{n\ n-1} + w_n a_{n\ n} = 0$$

$$\text{so } a_{n\ n} = -\frac{1}{w_n} \left(w_1 a_{n\ 1} + \ \ldots\ + w_{n-1} a_{n\ n-1} \right),$$

which finishes the complete determination of $\mathbf{a}_n^T$.

HOW HARD IS IT TO CHOOSE A?

- WORK BACKWARDS TO CHOOSE $\mathbf{a}_1^T$, $\mathbf{a}_2^T$, ... ,$\mathbf{a}_n^T$
  - But $a_{n\ n}$ has to satisfy some σ conditions:

$$a_{nn} > -(r_n - r_f) \text{ and for all } i < n$$

$$a_{nn} > -(r_n - r_f) + \frac{(a_{in} + (r_i - r_f))^2}{a_{ii} + (r_i - r_f)}$$

  - That means the choice of $\mathbf{a}_{n-1}^T$ couldn't have been completely free

$$-\frac{1}{w_n}(w_1 a_{n\ 1} + \ \ldots \ + w_{n-1} a_{n\ n-1}) > -(r_n - r_f) \text{ and for all } i < n$$

$$-\frac{1}{w_n}(w_1 a_{n\ 1} + \ \ldots \ + w_{n-1} a_{n\ n-1}) > -(r_n - r_f) + \frac{(a_{in} + (r_i - r_f))^2}{a_{ii} + (r_i - r_f)}$$

$$\begin{aligned} \text{where } a_{n\ 1} &= a_{1\ n} - (r_n - r_1) \\ &\ldots \\ a_{n\ n-1} &= a_{n-1\ n} - (r_n - r_{n-1}). \end{aligned}$$

  - Since all $w_i > 0$, just pick a small enough $a_{n-1\ n}$ (negative if need be)

HOW HARD IS IT TO CHOOSE A?

- WORK BACKWARDS TO CHOOSE $\mathbf{a}_1^T$, $\mathbf{a}_2^T$, ... ,$\mathbf{a}_n^T$
  - One possible problem at $i = n - 1$:

$$-\frac{1}{w_n}\left(w_1 a_{n\ 1} + \ \dots \ + w_{n-1}\left(a_{n-1\ n} - \left(r_n - r_{n-1}\right)\right)\right) >$$

$$> -\left(r_n - r_f\right) + \frac{\left(a_{n-1\ n} + \left(r_{n-1} - r_f\right)\right)^2}{a_{n-1\ n-1} + \left(r_{n-1} - r_f\right)}$$

    so $a_{n-1\ n}$ on both sides, and squared (so made positive) on the right. Does picking $a_{n-1\ n}$ small enough (negative if need be) still work?
  - YES! $\mathbf{a}_{n-1}^T$ is perpendicular to $\mathbf{w}$ with all $w_j > 0$ so picking a small enough $a_{n-1\ n}$ (negative if need be) forces $a_{n-1\ n-1}$ to increase and it turns out (some delicate analysis) to be enough to make the inequality work.

$$\begin{Bmatrix} \dots & \dots & \dots \\ \dots & \uparrow & \downarrow \\ \dots & \downarrow & \uparrow \end{Bmatrix}$$

HOW HARD IS IT TO CHOOSE A?

- THE HARD PART IS DONE
  - Given $\mathbf{a}_1^T$, $\mathbf{a}_2^T$, ... ,$\mathbf{a}_{n-2}^T$ we saw that we can choose $\mathbf{a}_{n-1}^T$, $\mathbf{a}_n^T$ that satisfy the σ conditions.
- GO BY INDUCTION STARTING AT $\mathbf{a}_1^T$
  - For $1 \leq i \leq n-2$, given $\mathbf{a}_1^T$, $\mathbf{a}_2^T$, ... ,$\mathbf{a}_{i-1}^T$ always free to choose a_{ii} big enough to satisfy the σ conditions, then $a_{i\ i+1}$, ... ,$a_{i\ n}$ any values that keep $\mathbf{a}_i^T \cdot \mathbf{w} = 0$
- SOME SPAN RESTRICTIONS ON CHOICES OF $\mathbf{a}_i^T$
  - To ensure that $\mathbf{a}_1^T$, $\mathbf{a}_2^T$, ... ,$\mathbf{a}_i^T$ spans at least an $(i-1)$-dimensional subspace of the $(n-1)$-dimensional hyperplane perpendicular to $\mathbf{w}$ we have to disallow some choices in the induction.
  - The whole set of disallowed matrices $\mathbf{A}$ altogether has dimension $\leq \frac{n(n-1)}{2} - 3$.
  - The whole set of allowed matrices $\mathbf{A}$ has dimension $\frac{n(n-1)}{2}$, still for a fixed choice of $\mathbf{w}$ and $(\mathbf{r} - r_f\mathbf{e})$ and still requiring

$$\mathbf{\Sigma} \cdot \mathbf{w} = (\mathbf{r} - r_f\mathbf{e})$$

BACK TO THE LAZY PEDAGOGUE

- A FEW MORE DEGREES OF FREEDOM
  - We can multiply any $\mathbf{\Sigma} = \mathbf{R}+\mathbf{A}$ developed above by any constant k.
  - We can choose w_1, ... ,w_n subject to $\mathbf{e}^T \cdot \mathbf{w} = 1$ and $w_i > 0$
  - So altogether the space of possible solutions $\mathbf{\Sigma}$ has dimension $\frac{n(n-1)}{2} + 1 + n - 1 = \frac{(n+1)n}{2}$, now with a possible disallowed set of dimension $\leq \frac{(n+1)n}{2} - 3$.
- $A \cdot w = 0$ IS THE MAIN NON-OBVIOUS RESTRICTION
  - The σ restrictions are all visible in $\mathbf{\Sigma}$ and true for any empirical $\mathbf{\Sigma}$
  - Invertibility of $\mathbf{\Sigma}$ is apparent or easy to check, at least for small n.
  - The space of possible $n \times n$ symmetric matrices has dimension $n + (n-1) + ... + 1 = \frac{(n+1)n}{2}$, same as the space of solutions
  - The disallowed set of matrices has measure (probability) 0.
  - But the odds of an $(n-1)$-dimensional hyperplane through the origin being perpendicular to a vector $\mathbf{w}$ with all $w_i > 0$ is $\left(\frac{1}{2}\right)^{n-1}$
  - So solutions seemingly are rare only because of that factor $\left(\frac{1}{2}\right)^{n-1}$

ONE MORE SOURCE OF MEANINGFUL CONSTRAINT

- REMEMBER THAT RETURNS ARE CONSTRAINED
  - In meaningful models
$$\mathbf{w}^T \cdot (\mathbf{r} - r_f \mathbf{e}) = \left(\mathbf{w}^T \cdot \mathbf{r} - r_f\right) = (r_M - r_f) > 0$$
  - If there are any risky assets with expected return $r_i < r_f$ then this inequality cuts off a fraction (call it f) of the otherwise possible $\mathbf{w}$.
  - Now the only possible $\mathbf{w}$ are in the intersection of the set having all $w_i > 0$ with the half-space having $\mathbf{w}^T \cdot (\mathbf{r} - r_f \mathbf{e}) > 0$.
  - This in turn eliminates the same fraction f of the set of otherwise possible matrices $\mathbf{A}$, whose row vectors have to live on the plane perpendicular to $\mathbf{w}$, so possible $\boldsymbol{\Sigma}$ are now rare by a factor $f\left(\frac{1}{2}\right)^{n-1}$.
  - This further militates against the lazy pedagogue's chance of success, since he likes to illustrate negative covariances and negative risk premia.
  - It impairs by the same factor the probability of a random empirical $\boldsymbol{\Sigma}$ to satisfy CAPM, even if it satisfies all the σ constraints and invertibility.
  - If I have understood Phelim Boyle's work, the "almost positive" condition does not yet contemplate this possibility. How should it be modified/generalized to this case?

ALMOST FORGOT - What Is That Matrix A Anyway?

Claim: $\mathbf{A} = \mathbf{Cov}\left[\bar{\mathbf{r}}, \left(\bar{\mathbf{r}}^T - \bar{r}_M \mathbf{e}^T\right)\right]$ where bar means random,

$\bar{\mathbf{r}}$ is the random vector of actual asset returns in the market and

$\bar{r}_M$ is the random actual return on the market as a whole, i.e.

$$\bar{r}_M = \bar{\mathbf{r}}^T \cdot \mathbf{w}$$

Proof: $\mathbf{Cov}\left[\bar{\mathbf{r}}, \bar{\mathbf{r}}^T\right] = \boldsymbol{\Sigma}$ by definition.

$$\begin{aligned}
\mathbf{Cov}\left[\bar{\mathbf{r}}, \bar{r}_{\mathbf{M}} \mathbf{e}^T\right] &= \left\{\mathbf{Cov}\left[\bar{\mathbf{r}}, \bar{r}_M\right] \mathbf{e}^T\right\}, \text{ with } n \text{ identical columns} \\
&= \left\{\mathbf{Cov}\left[\bar{\mathbf{r}}, \left(\bar{\mathbf{r}}^T \cdot \mathbf{w}\right)\right] \mathbf{e}^T\right\} \\
&= \left\{\left(\mathbf{Cov}\left[\bar{\mathbf{r}}, \bar{\mathbf{r}}^T\right] \cdot \mathbf{w}\right) \mathbf{e}^T\right\} \\
&= \left\{(\boldsymbol{\Sigma} \cdot \mathbf{w}) \mathbf{e}^T\right\} = \left\{(\mathbf{r} - r_f \mathbf{e}) \mathbf{e}^T\right\} = \mathbf{R} \text{ so} \\
\mathbf{Cov}\left[\bar{\mathbf{r}}, \left(\bar{\mathbf{r}}^T - \bar{r}_M \mathbf{e}^T\right)\right] &= \boldsymbol{\Sigma} - \mathbf{R} = \mathbf{A}
\end{aligned}$$

THANKS

PHELIM BOYLE

MY STUDENTS THIS SUMMER