Regime-Switching Interest Rate Models With Randomized Regimes

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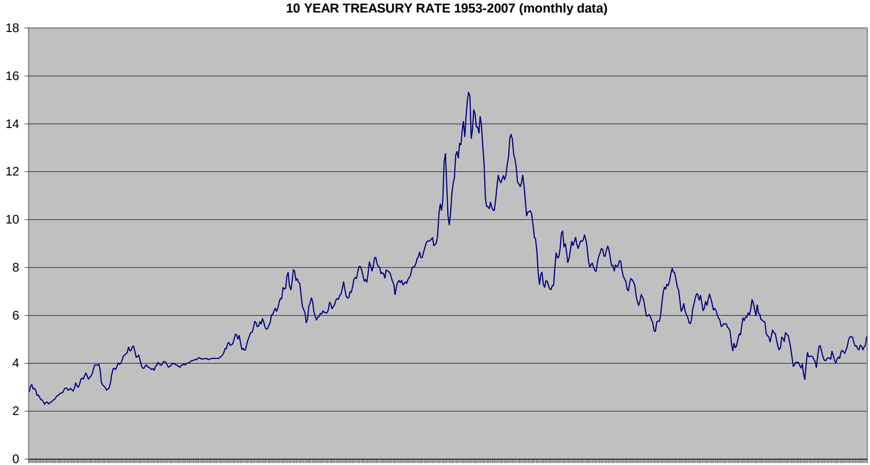
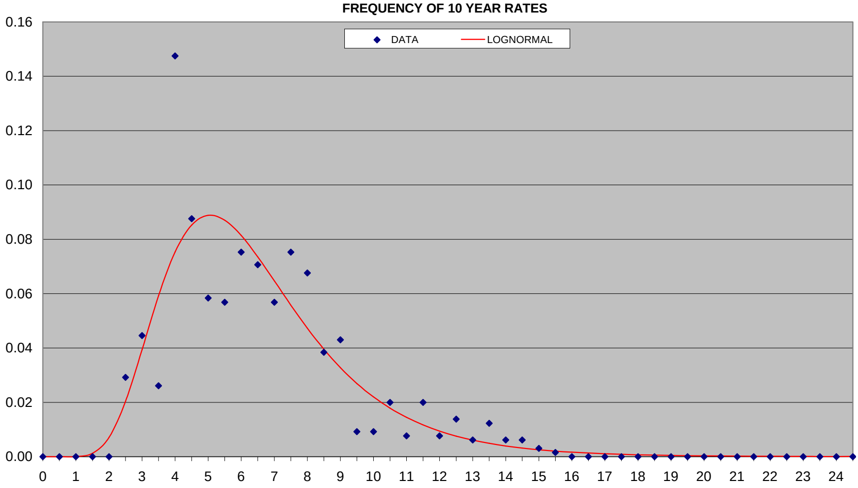
- This year? Numerical examples and extensions

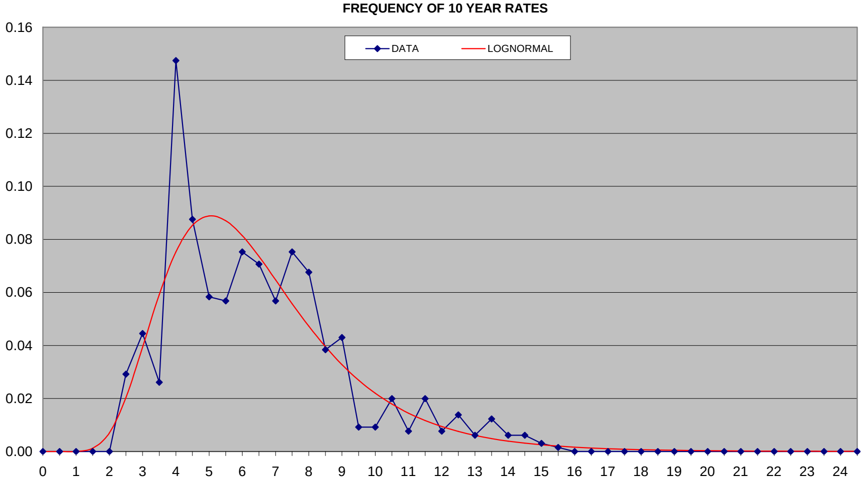
Example: 55 Years of the 10-year Treasury Rate



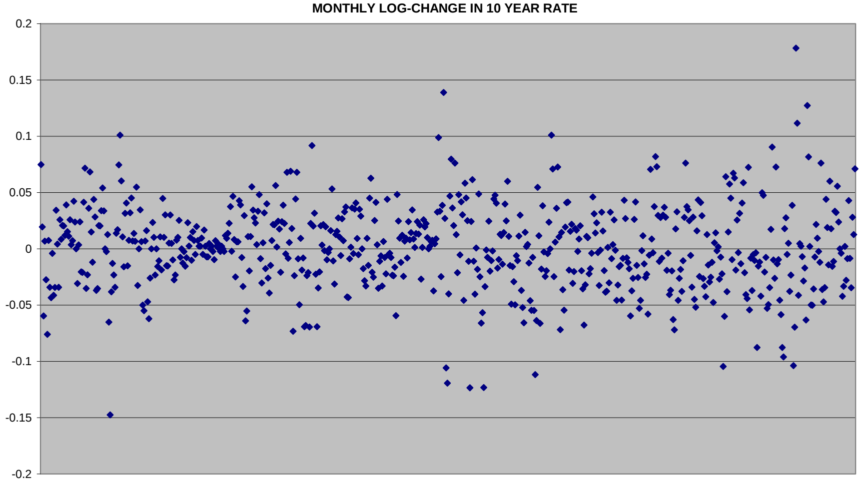
The Distribution of those Interest Rates



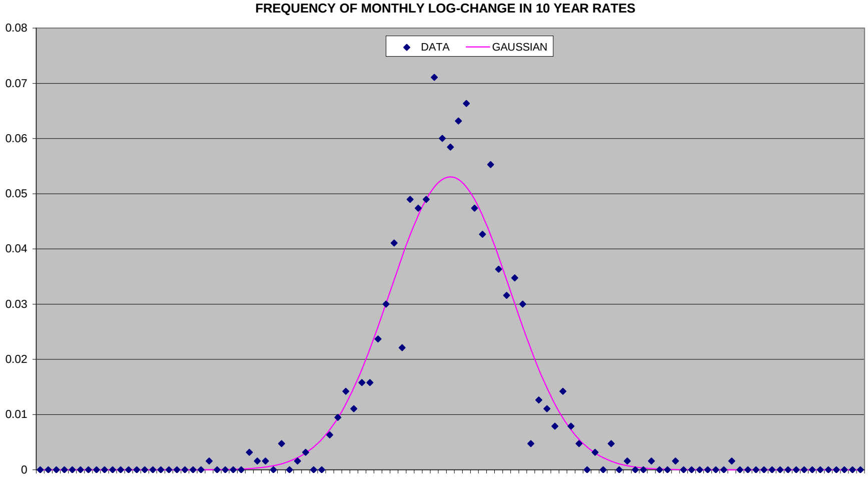
Lognormal 4th Moment Is Just Too High (6th too)



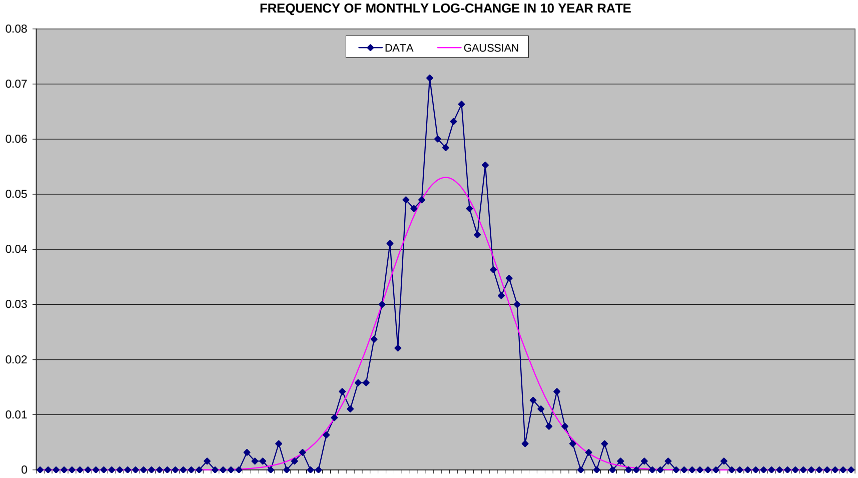
55 Years of Changes in the 10 Year Treasury Rate



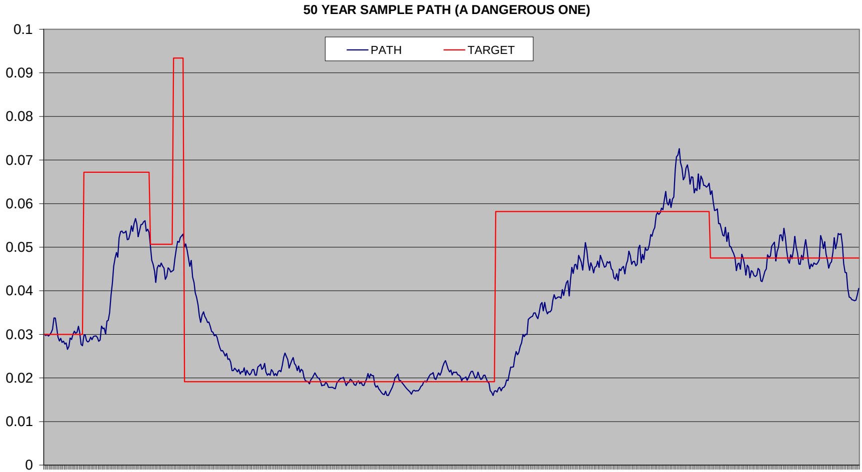
What is the Distribution of Those Changes?



For Rate Changes, Lognormal 4th Moment Too Low



The Fix: Randomize the Reversion Target



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Drift Compensation and Calibration: plain mean-reversion

- It would be intuitive to have:
  $\mathbb{E}\left[r_t\right] = r_0^{(1-F)^t} T_0^{\left[1-(1-F)^t\right]}$

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- It would be intuitive to have:
  $\mathbb{E}\left[r_t\right] = r_0^{(1-F)^t} T_0^{\left[1-(1-F)^t\right]}$
- To find out what drift D_t will ensure it, you can integrate $d\ln(r_t)$:

$$\ln(r_t) = \ln(r_0)\left(1-F\right)^{\frac{t}{dt}dt} + \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\left(1-F\right)^{sdt}$$

$$+\ln(T_0)\left[1-(1-F)^{dt}\right]\sum_{s=1}^{\frac{t}{dt}}(1-F)^{(s-1)dt} \Longleftarrow \text{notice geom. series}$$

$$+dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\left(1-F\right)^{sdt} \text{ which simplifies to:}$$

$$\ln(r_t) = \ln(r_0)\left(1-F\right)^{t} + \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\left(1-F\right)^{sdt}$$

$$+\ln(T_0)\left[1-(1-F)^{t}\right] + dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\left(1-F\right)^{sdt}, \text{ which is}$$

Gaussian.

Drift Compensation and Calibration: plain mean-reversion

- Since $\ln(r_t)$ is Gaussian, $\mathbb{E}\left[r_t\right] = e^{\mu + \frac{1}{2}\sigma^2}$ where the μ and σ^2 are some mess determined by the constants in the expression for $\ln(r_t)$.

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- If you work that mess out and set it equal to $r_0^{(1-F)^t} T_0^{\left[1-(1-F)^t\right]}$, and require that it be true for all t, you can arrive at what the drift compensation function D_t must be to deliver the intuitive $\mathbb{E}\left[r_t\right]$:

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  - $D_t = -\frac{1}{4}\sigma^2 \left[1 + (1-F)^{2t}\right]$ in the continuous case
- There is a similar closed form for the variance of r_t based on $\mathbb{E}\left[r_t^2\right] = e^{2\mu + \frac{1}{2}(2\sigma)^2}$ which can help calibrate the model to historical variance $F = 1 - \left\{1 - \frac{\sigma_{obs}^2 dt}{\ln(V_{obs} + T^2) - \ln(T^2)}\right\}^{\frac{1}{2dt}}$

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- Practical work with the randomized reversion target model all but requires you to know similar closed forms for drift compensation and variance, but now when you integrate no geometric series appears.

Drift Compensation & Calibration: random mean-reversion

- $$\ln(r_t) = \ln(r_0)\left(1-F\right)^t + \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\left(1-F\right)^{sdt}$$
$$+\left[1-\left(1-F\right)^{dt}\right]\sum_{s=1}^{\frac{t}{dt}}\sum_{j=0}^{\infty} \mathbf{1}_{[\mathbf{j},\mathbf{j}+\mathbf{1})}(sdt)\ln\left(\mathbf{T}_j\right)\left(1-F\right)^{t-sdt} \Leftarrow \text{ugly}$$
$$+dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\left(1-F\right)^{sdt}$$

Drift Compensation & Calibration: random mean-reversion

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  $+\left[1-(1-F)^{dt}\right]\sum_{s=1}^{\frac{t}{dt}}\sum_{j=0}^{\infty} \mathbf{1}_{[\mathbf{j},\mathbf{j+1})}(sdt)\ln(\mathbf{T}_j)\,(1-F)^{t-sdt}$ $\Leftarrow$ugly

  $+dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\,(1-F)^{sdt}$, but switch the order of summation:

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$$+\left[1-(1-F)^{dt}\right]\sum_{s=1}^{\frac{t}{dt}}\sum_{j=0}^{\infty}\mathbf{1}_{[\mathbf{j},\mathbf{j+1})}(sdt)\ln(\mathbf{T}_j)\,(1-F)^{t-sdt} \Leftarrow \text{ugly}$$
$$+dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\,(1-F)^{sdt},$$ but switch the order of summation:

- $$\ln(r_t) = \ln(r_0)\,(1-F)^t + \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\,(1-F)^{sdt}$$
$$+\ln(T_0)\left[(1-F)^{(t-\mathbf{t}_1)_+} - (1-F)^t\right]$$
$$+\sum_{j=1}^{\infty}\ln(\mathbf{T}_j)\left[(1-F)^{(t-\mathbf{t}_{j+1})_+} - (1-F)^{(t-\mathbf{t}_j)_+}\right] \Longleftarrow \text{after telescoping}$$
$$+dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\,(1-F)^{sdt},$$ so $r_t \neq$ lognormal, = log-log-gamma?

Condition on the Times When Regimes Switch

- $$\ln(r_t) = \ln(r_0)\,(1-F)^t + \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\,(1-F)^{sdt}$$
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- Conditioning on the $\mathbf{t}_j$ random variables and using a lognormal model (reasonable) for the random targets $\mathbf{T}_j$ (so each $\ln(\mathbf{T}_j)$ is Gaussian) we again have a (messy) Gaussian for the conditional $\ln(r_t)$. Can that help in calculating an unconditioned $\mathbb{E}\left[r_t\right]$ and variance?

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- The answer is "Yes" ... up to an approximate expansion.

Edgeworth Expansion for the Unconditioned Moments

- We expect the tails of $\ln(r_t)$ to be supressed in favor of the shoulders. That suggests that $\mathbb{E}[r_t]$, and higher moments as well, might be approximated efficiently by an Edgeworth expansion for $\ln(r_t)$. It works out to be surprisingly simple:

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where $(2n)? = (2n-1)(2n-3)\cdots(1)$
- Conditional Gaussian ensures that odd higher moments vanish.
- The problem now is to calculate μ, σ^2, and the μ_{2j}

Expected Value Easy

- Remember,

$$\ln(r_t) = \ln(r_0)\,(1-F)^t + \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\,(1-F)^{sdt}$$
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- $\mathbb{E}\left[(1-F)^{(t-\mathbf{t}_1)_+}\right]$ turns out to be a Laplace transform that we can calculuate (later).

Higher Moments Hard

- Remembering that the even central moments of std normal are $(2n)? = (2n-1)(2n-3)\cdots(1)$, the even central moments of $\ln(r_t)$ are $\mathbb{E}\left[\left\{\ln(r_t) - \mathbb{E}\left[\ln(r_t)\right]\right\}^{2n}\right]$
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- The $\{\}^n$ part can be expanded binomially, but that still leaves terms like...

Still Need To Evaluate Terms Like

- ... $\mathbb{E}\left[\left(\sum_{j=1}^{\infty}\mathbf{e}_j^2\right)^m\right]$ where the $\mathbf{e}_j = \left\{(1-F)^{(t-\mathbf{t}_{j+1})_+} - (1-F)^{(t-\mathbf{t}_j)_+}\right\}$ fail to be independent and are each complicated in their own right.

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- Lemma: $\mathbb{E}\left[\mathbf{e}_{j_1}^{2a_1}\cdots\mathbf{e}_{j_k}^{2a_k}\right] = \rho_{a_1,\ldots,a_k}\mathbb{E}\left[\mathbf{e}_{j_1}^{2a_1}\right]\cdots\mathbb{E}\left[\mathbf{e}_{j_k}^{2a_k}\right]$ independent of $\{j_1,\ldots,j_k\}$ for distinct $\{j_1,\ldots,j_k\}$

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- How does that help?

For Example

- $\mathbb{E}\left[\left(\sum_{j=1}^{\infty}\mathbf{e}_j^2\right)^2\right] = \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4 + \sum_{j=1}^{\infty}\mathbf{e}_j^2\left\{\left(\sum_{i=1}^{\infty}\mathbf{e}_i^2\right) - e_j^2\right\}\right]$

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- It gets complicated fast

For m=3

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{j=1}^{\infty}\mathbf{e}_j^2\right)^3\right] =& \mathbb{E}\left[\begin{array}{c}
\sum_{j=1}^{\infty}\mathbf{e}_j^6 + 3\sum_{j=1}^{\infty}\mathbf{e}_j^4\left\{\left(\sum_{i=1}^{\infty}\mathbf{e}_i^2\right) - e_j^2\right\} \\
+\sum_{j=1}^{\infty}\mathbf{e}_j^2\left\{\begin{array}{c}
\left(\sum_{i=1}^{\infty}\mathbf{e}_i^2\left[\left(\sum_{k=1}^{\infty}\mathbf{e}_k^2\right) - e_i^2 - e_j^2\right]\right) \\
-e_j^2\left[\left(\sum_{k=1}^{\infty}\mathbf{e}_k^2\right) - e_j^2\right] + e_j^4
\end{array}\right\}
\end{array}\right] \\
=& \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^6\right] + 3\rho_{2,1}\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4\right]\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right] \\
&- \left(3\rho_{2,1} - \rho_{1,1,1}\right)\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^4\mathbb{E}\left[e_j^2\right]\right] + \rho_{1,1,1}\left\{\left(\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\right)^3\right. \\
&\left. -3\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\mathbb{E}\left[e_j^2\right]\right] + \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\left(\mathbb{E}\left[e_j^2\right]\right)^2\right]\right\}
\end{aligned}
$$

Complicated, but each piece is simpler

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- Some notation: to save ink later let $\nu(x)$ stand for $x^n \prod_{k=1}^{n} \mathbb{E}\left[x^k\right]^{n_k}$ so our expression abbreviates to $\mathbb{E}\left[\sum_{j=1}^{\infty} \nu\left(\mathbf{e}_j\right)\right]$

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- **So** $t = \overline{\mathbf{d}}_1 + \mathbf{d}_2 + ... + \mathbf{d}_{\mathbf{J}-1} + \overline{\mathbf{d}}_{\mathbf{J}}$

The Result

- $$\mathbb{E}\left[\sum_{j=1}^{\infty} \nu\left(\mathbf{e}_j\right)\right] =$$
$$K\left\{\mathbb{E}\left[\nu\left(\left(1-F\right)^{\mathbf{d}\wedge t}\right)\right] - \mathbb{P}\left[\mathbf{d} \geq t\right] \nu\left(\left(1-F\right)^{t}\right)\right\} \frac{\mathbb{E}\left[\nu\left(1-\left(1-F\right)^{\mathbf{d}}\right)\right]}{1-\mathbb{E}\left[\nu\left(\left(1-F\right)^{\mathbf{d}}\right)\right]}$$
$$+\mathbb{E}\left[\nu\left(1-\left(1-F\right)^{\mathbf{d}\wedge t}\right)\right] - \mathbb{P}\left[\mathbf{d} \geq t\right] \nu\left(1-\left(1-F\right)^{t}\right)$$

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- Where $K = 1-\mathbb{E}\left[\left(\mathbb{E}\left[\nu\left(\left(1-F\right)^{\mathbf{d}}\right)\right]\right)^{\mathbf{J}-2} \mid \mathbf{J}>1\right]$
$$= 1-\left(1-G\right)^{t}\frac{\mathbb{E}\left[\left(1-G\right)^{-\overline{\mathbf{d}}\wedge t}\right]-\mathbb{P}\left[\overline{\mathbf{d}}\geq t\right]\left(1-G\right)^{-t}}{\mathbb{E}\left[\left(1-G\right)^{\overline{\mathbf{d}}\wedge t}\right]-\mathbb{P}\left[\overline{\mathbf{d}}\geq t\right]\left(1-G\right)^{t}}$$

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- And G is defined by $\left(1-G\right) = \exp\left\{-\mathcal{L}_{\mathbf{d}}^{-1}\left(\mathbb{E}\left[\nu\left(\left(1-F\right)^{\mathbf{d}}\right)\right]\right)\right\}$, $\mathcal{L}_{\mathbf{d}}$ being the Laplace transform

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$+\mathbb{E}\left[\nu\left(1-(1-F)^{\mathbf{d}\wedge t}\right)\right] - \mathbb{P}\left[\mathbf{d} \geq t\right] \nu\left(1-(1-F)^t\right)$
- Where $K = 1 - \mathbb{E}\left[\left(\mathbb{E}\left[\nu\left((1-F)^{\mathbf{d}}\right)\right]\right)^{\mathbf{J}-2} \mid \mathbf{J} > 1\right]$
$= 1 - (1-G)^t \frac{\mathbb{E}\left[(1-G)^{-\mathbf{d}\wedge t}\right] - \mathbb{P}\left[\mathbf{d} \geq t\right](1-G)^{-t}}{\mathbb{E}\left[(1-G)^{\mathbf{d}\wedge t}\right] - \mathbb{P}\left[\mathbf{d} \geq t\right](1-G)^{t}}$
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$\mathbb{E}\left[(1-G)^{\mathbf{d}}\right] = \mathcal{L}_{\mathbf{d}}\left\{\mathcal{L}_{\mathbf{d}}^{-1}(\mathbb{E}\left[\nu\left((1-F)^{\mathbf{d}}\right)\right]\right\} = \mathbb{E}\left[\nu\left((1-F)^{\mathbf{d}}\right)\right]$

Asymptotically

$$\lim_{t\to\infty} \mathbb{E}\left[\sum_{j=1}^{\infty} \nu\left(\mathbf{e}_j\right)\right] =$$

$$\frac{\mathbb{E}\left[\nu\left((1-F)^{\mathbf{d}}\right)\right]\mathbb{E}\left[\nu\left(1-(1-F)^{\mathbf{d}}\right)\right]}{1-\mathbb{E}\left[\nu\left((1-F)^{\mathbf{d}}\right)\right]} + \mathbb{E}\left[\nu\left(1-(1-F)^{\mathbf{d}}\right)\right]$$

Everything Is Closed Form

- Everything is now of the form $\mathbb{P}\left[\overline{\mathbf{d}} \geq t\right]$ and $\mathbb{E}\left[x^{\mathbf{v}}\right]$ for $\mathbf{v}$ one of the random variables $\mathbf{d}$, $\overline{\mathbf{d}}$ and $\overline{\mathbf{d}} \wedge t$

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- $\mathbb{E}\left[x^{\mathbf{v}}\right] = \mathcal{L}_{\mathbf{v}}\left[-\ln(x)\right]$, where $\mathcal{L}_{\mathbf{v}}$ is the Laplace transform of $\mathbf{v}$
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- $\mathcal{L}_{\mathbf{d}}(x) = (1+\beta x)^{-\alpha}$, $\mathcal{L}_{\mathbf{d}}^{-1}(y) = \frac{1}{\beta}\left(y^{-\frac{1}{\alpha}} - 1\right)$,
  $\mathcal{L}_{\bar{\mathbf{d}}}(x) = \frac{1}{\alpha\beta x}\left[1 - (1+\beta x)^{-\alpha}\right]$, $\mathcal{L}_{\bar{\mathbf{d}} \wedge t}(x) =$
  $\frac{1}{\alpha\beta x}\left\{1 - e^{-xt}\left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right)\right] - (1+\beta x)^{-\alpha}\,\Gamma\left(\alpha; \frac{(1+\beta x)t}{\beta}\right)\right\}$
  $+e^{-xt}\left\{1 - \Gamma\left(\alpha+1; \frac{t}{\beta}\right) - \frac{t}{\alpha\beta}\left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right)\right]\right\}$,
  $\mathbb{P}\left[\bar{\mathbf{d}} \geq t\right] = 1 - \Gamma\left(\alpha+1; \frac{t}{\beta}\right) - \frac{t}{\alpha\beta}\left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right)\right]$

Even Those Uniform Correlation Coefficients

- $\rho_{a,b} = \frac{1}{D} \left\{ \mathbb{E}\left[(1-F)^{2(a-b)\bar{\mathbf{d}} \wedge t} \right] - \mathbb{P}\left[\bar{\mathbf{d}} \geq t\right] (1-F)^{2(a-b)t} \right\}$

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- and $\rho_{a_1,\ldots,a_k} = \rho_{a_1,a_2+\ldots+a_k}\rho_{a_2,\ldots,a_k}$ recursively

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- $\left[\widehat{f_W}(t)\left(\frac{1}{\widehat{\phi}(t)}\right)\right]_{t=0}^{(n)} = \left(\frac{1}{\widehat{\phi}(t)}\right)_{t=0}^{(n)} + \sum_{j=1}^{n} \frac{n!}{j!(n-j)!}\widehat{f_W}^{(j)}(0)\left(\frac{1}{\widehat{\phi}(t)}\right)_{t=0}^{(n-j)}$

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- For something even more flexible, use a different function than ϕ; try logistic, gamma, inverse gamma or inverse logistic

How Do You Get The Moments?

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- Remarkably, $\sum_{k=0}^{m}\frac{(2k)?(2(m-k))?}{(2k)!(2(m-k))!}(-1)^k = 0$ when $m>0$

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- Why? $0 = \left[\widehat{\phi}(t)\left(\frac{1}{\widehat{\phi}(t)}\right)\right]_{t=0}^{(2m)} = \sum_{k=0}^{m} \frac{(2k)?(2(m-k))?}{(2k)!(2(m-k))!}(-1)^k$

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- Finally, $\mathbb{E}\left[\mathbf{r}_t^l\right] =$
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- Conditional on $\mathbf{J} = j' > 1$ the following are each independent sets: $\{\mathbf{J}, \overline{\mathbf{d}}_1\}$, $\{\overline{\mathbf{d}}_1, \mathbf{d}_2, ..., \mathbf{d}_{j'-1}\}$, $\{\mathbf{d}_2, ..., \mathbf{d}_{j'-1}, \overline{\mathbf{d}}_{j'}\}$ and $\{\mathbf{J}, \overline{\mathbf{d}}_{\mathbf{J}}\}$

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- This is enough independence to get a geometric series inside the main expectation $\mathbb{E}\left[\sum_{j=1}^{\infty} \nu\left(\mathbf{e}_j\right)\right]$ and to pull apart the two sides of the correlation expectation for $\rho_{a,b}$, leaving a common term involving $\overline{\mathbf{d}}_1$ and $\overline{\mathbf{d}}_{\mathbf{J}}$ which can be evaluated by writing $\overline{\mathbf{d}}_1 = t - \left(\overline{\mathbf{d}}_{\mathbf{J}} + \mathbf{d}_{\mathbf{J}-1} + ... + \mathbf{d}_2\right)$