Informal Outline of Risk Neutral Pricing In Continuous Time

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October 22, 2015

These notes are informal and intended to outline rapidly the risk neutral pricing model in continuous time. Neither the definitions nor the proofs herein are rigorous, but they are accurate reflections of the underlying rigorous models. The intention is to get students through the big picture in a fairly short time, preparatory to their more rigorous journies. Here is a summary view of the contents.

I: THE GAUSSIAN DIFFUSION $\mathbf{S}(\mathbf{t})$

$\mathbf{dS}(\mathbf{u}) = \boldsymbol{\alpha}(\mathbf{u}, \mathbf{S}(\mathbf{u}))\,\mathbf{du} + \boldsymbol{\sigma}(\mathbf{u}, \mathbf{S}(\mathbf{u}))\,\mathbf{dB}(\mathbf{u})$

(Also called: The Risky Asset, The Random Process, The Underlying Process, The Gaussian Process.)

$$
\begin{aligned}
& S(t) \text{ is a Random Variable for each } t \text{ defined by} \\
S(t) &= S(0) + \int_0^t dS(u) \text{ where for each } u < t \\
dS(u) &= \alpha(u, S(u))\,du + \sigma(u, S(u))\,dB(u)\,, \text{ where for all } u,\ \sigma(u, S(u)) \neq 0 \text{ and} \\
dB(u) &= \text{independent mean 0 variance } du \text{ normal Random Variables for each } u \\
\text{With } B(0) &= 0,\ dB(u) \text{ is called a Brownian Motion.}
\end{aligned}
$$

$$
\begin{aligned}
\text{Integrating, } B(t) &= \int_0^t dB(u) \text{ is a normal mean 0 variance } t \text{ Random Variable} \\
\text{Integrating, } S(t) &= S(0) + \int_0^t \alpha(u, S(u))\,du + \int_0^t \sigma(u, S(u))\,dB(u) \text{ where} \\
(1)\ \alpha(u, S(u)) &= \text{a (possibly) Random Variable called "drift" for each } u < t \\
\text{so } \int_0^t \alpha(u, S(u))\,du &= \text{a (possibly) Random Variable "accumulated drift", with mean} \\
\mathbb{E}\left[\int_0^t \alpha(u, S(u))\,du\right] &= \int_0^t \mathbb{E}[\alpha(u, S(u))]\,du \text{ and variance} \\
\mathbb{V}\left[\int_0^t \alpha(u, S(u))\,du\right] &= \int_0^t \int_0^t co\mathbb{V}[\alpha(u, S(u)), \alpha(v, S(v))]\,dvdu, \text{ or equivalently} \\
&= 2\int_0^t \int_u^t co\mathbb{V}[\alpha(u, S(u)), \alpha(v, S(v))]\,dvdu, \text{ or equivalently} \\
&= 2\int_0^t \{(t-u)\,\mathbb{V}[\alpha(u, S(u))] + \\
&\quad \int_u^t co\mathbb{V}[\alpha(v, S(v)) - \alpha(u, S(u)), \alpha(u, S(u))]\,dv\}\,du
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \sigma(u,S(u)) &= \text{a (possibly) Random Variable called "volatility" for each } u<t \\
\text{so} \int_0^t \sigma(u,S(u))\,dB(u) &= \text{ a (possibly) Random Variable "accumulated volatility",} \\
& \quad \text{with mean 0 and variance} \\
\mathbb{V}\left[\int_0^t \sigma(u,S(u))\,dB(u)\right] &= \int_0^t \mathbb{E}\left[\sigma^2(u,S(u))\right] du
\end{aligned}
$$

$$
\begin{aligned}
\text{So (3) } S(t) &= S(0)+\int_0^t \alpha(u,S(u))\,du+\int_0^t \sigma(u,S(u))\,dB(u) \ \text{ a Random Variable with mean} \\
\mathbb{E}[S(t)] &= S(0)+\int_0^t \mathbb{E}[\alpha(u,S(u))]\,du = \text{"start value" plus "expected drift", and variance} \\
\mathbb{V}[S(t)] &= \int_0^t \left\{\mathbb{E}\left[\sigma^2(u,S(u))\right]+2(t-u)\,\mathbb{V}[\alpha(u,S(u))]\right\} du \\
& \quad +2\int_0^t co\mathbb{V}\left[\int_u^t \{\alpha(v,S(v))-\alpha(u,S(u))\}\,dv, \alpha(u,S(u))\,du+\sigma(u,S(u)\,dB(u))\right]
\end{aligned}
$$

It is a lot simpler if $\alpha(u,S(u))$ is non-random, in which case

$$
\begin{aligned}
S(t) &= S(0)+\int_0^t \alpha(u,S(u))\,du+\int_0^t \sigma(u,S(u))\,dB(u) \ \text{ a Random Variable with mean} \\
\mathbb{E}[S(t)] &= S(0)+\int_0^t \alpha(u,S(u))\,du = \text{"start value" plus "accumulated drift", and variance} \\
\mathbb{V}[S(t)] &= \int_0^t \mathbb{E}\left[\sigma^2(u,S(u))\right] du
\end{aligned}
$$

It is even simpler if the diffusion is something called a local Martingale (we might just say Martingale later on, but we're always talking about local Martingales), by which is meant

$$\begin{aligned}
\alpha\left(u, S\left(u\right)\right) &= 0 \text{ for all } u \text{ so} \\
S\left(t\right) &= S\left(0\right) + \int_0^t \sigma\left(u, S\left(u\right)\right) dB\left(u\right) \text{ a Random Variable with mean} \\
\mathbb{E}\left[S\left(t\right)\right] &= S\left(0\right) \text{ and variance} \\
\mathbb{V}\left[S\left(t\right)\right] &= \int_0^t \mathbb{E}\left[\sigma^2\left(u, S\left(u\right)\right)\right] du
\end{aligned}$$

In fact, in this case

$$\mathbb{E}\left[S\left(t\right) \mid S\left(v\right)\right] = S\left(v\right) \text{ for all } v \leq t$$

II: FUNCTIONS OF A DIFFUSION $\mathbf{F}\left(\mathbf{t}, \mathbf{X}\left(\mathbf{t}\right)\right)$

If $X\left(t\right)$ is a diffusion (such as $S\left(t\right)$ above) and $F(u, x)$ is a differentiable function of two variables then the Itô Rule (Itô Lemma) shows the structure of a new diffusion $F\left(t, X\left(t\right)\right)$ by the formula:

$$dF\left(u, X\left(u\right)\right) = \frac{\partial F}{\partial u}\left(u, X\left(u\right)\right) du + \frac{\partial F}{\partial x}\left(u, X\left(u\right)\right) dX\left(u\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\left(u, X\left(u\right)\right) dX\left(u\right) dX\left(u\right)$$

When working out the expression $dX\left(u\right) dX\left(u\right)$ the rules are:

$$\begin{aligned}
du du &= 0 \\
du dB\left(u\right) &= 0 \\
dB\left(u\right) dB\left(u\right) &= du \\
\text{so, for example, } dS\left(u\right) dS\left(u\right) &= \sigma^2\left(u, S\left(u\right)\right) du
\end{aligned}$$

(With these rules, you might notice that the Itô Lemma is just a second order Taylor's expansion).

The same general principle applies to multivariate functions $G\left(u, x, y\right)$ giving multivariate diffusions $G\left(t, X\left(t\right), Y\left(t\right)\right)$.

In particular the function $G\left(u, x, y\right) = xy$ can be used to get the Itô Rule for the product of two diffusions:

$$\begin{aligned}
d\left(X\left(u\right) Y\left(u\right)\right) &= \frac{\partial\left(xy\right)}{\partial x}\left(u, X\left(u\right), Y\left(u\right)\right) dX\left(u\right) + \frac{\partial\left(xy\right)}{\partial y}\left(u, X\left(u\right), Y\left(u\right)\right) dY\left(u\right) \\
&\quad + \frac{1}{2}\frac{\partial^2\left(xy\right)}{\partial x^2}\left(u, X\left(u\right), Y\left(u\right)\right) dX\left(u\right) dX\left(u\right) \\
&\quad + \frac{1}{2}\frac{\partial^2\left(xy\right)}{\partial y^2}\left(u, X\left(u\right), Y\left(u\right)\right) dY\left(u\right) dY\left(u\right) \\
&\quad + \frac{\partial^2\left(xy\right)}{\partial x \partial y}\left(u, X\left(u\right), Y\left(u\right)\right) dX\left(u\right) dY\left(u\right) \\
&= Y\left(u\right) dX\left(u\right) + X\left(u\right) dY\left(u\right) + dX\left(u\right) dY\left(u\right)
\end{aligned}$$

III: GEOMETRICAL GAUSSIAN DIFFUSION
$\mathbf{dS\,(u) = \boldsymbol{\alpha}\,(u)\,S\,(u)\,du + \boldsymbol{\sigma}\,(u)\,S\,(u)\,dB\,(u)}$

Let the diffusion take the form

$$\begin{aligned}
\alpha\,(u, S\,(u)) &= \alpha\,(u)\,S\,(u) \text{ where } \alpha\,(u) \text{ is a (possibly) Random Variable for each } u \\
\sigma\,(u, S\,(u)) &= \sigma\,(u)\,S\,(u) \text{ where } \sigma\,(u) \neq 0 \text{ is a (possibly) Random Variable for each } u \\
\text{Then } d\ln S\,(u) &= \frac{\partial \ln s}{\partial s}\,(u, S\,(u))\,dS\,(u) + \frac{1}{2}\frac{\partial^2 \ln s}{\partial s^2}\,(u, S\,(u))\,dS\,(u)\,dS\,(u) \\
&= \frac{1}{S\,(u)}\,(\alpha\,(u)\,S\,(u)\,du + \sigma\,(u)\,S\,(u)\,dB\,(u)) \\
&\quad -\frac{1}{2}\frac{1}{S^2\,(u)}\,(\alpha\,(u)\,S\,(u)\,du + \sigma\,(u)\,S\,(u)\,dB\,(u))\,(\alpha\,(u)\,S\,(u)\,du + \sigma\,(u)\,S\,(u)\,dB\,(u)) \\
&= \alpha\,(u)\,du + \sigma\,(u)\,dB\,(u) - \frac{1}{2}\sigma^2\,(u)\,du \\
&= \left(\alpha\,(u) - \frac{1}{2}\sigma^2\,(u)\right) du + \sigma\,(u)\,dB\,(u)
\end{aligned}$$

Integrating both sides

$$\begin{aligned}
\ln S\,(t) - \ln S\,(0) &= \int_0^t \left(\alpha\,(u) - \frac{1}{2}\sigma^2\,(u)\right) du + \int_0^t \sigma\,(u)\,dB\,(u) \\
\text{so } S\,(t) &= S\,(0)\,e^{\left[\int_0^t \left(\alpha(u) - \frac{1}{2}\sigma^2(u)\right)du + \int_0^t \sigma(u)dB(u)\right]} = S\,(0)\,e^{\Sigma(t)} \\
\text{where } \Sigma\,(t) &= \int_0^t \mu(u)du + \int_0^t \sigma\,(u)\,dB\,(u) \\
\text{with } \mu(u) &= \alpha\,(u) - \frac{1}{2}\sigma^2\,(u) \text{ called the drift coefficient} \\
&\quad \text{and } \sigma\,(u) \text{ the volatility coefficient.}
\end{aligned}$$

If $\Sigma\,(t)$ is a Normal Random Variable, for example if $\alpha\,(u)$ is non-random (or random normal) and $\sigma\,(u)$ is non-random, then $S\,(t)$ is a Lognormal Random Variable. This is a commonly used model.

Whether or not $\Sigma\,(t)$ is a Normal Random Variable we can always verify our starting point:

$$\begin{aligned}
dS\,(u) &= S\,(0)\left(\frac{\partial}{\partial \Sigma}e^{\Sigma(u)}d\Sigma\,(u) + \frac{1}{2}\frac{\partial^2}{\partial \Sigma^2}e^{\Sigma(u)}d\Sigma\,(u)\,d\Sigma\,(u)\right) \\
&= S\,(0)\,e^{\Sigma(u)}\left(\left(\alpha\,(u) - \frac{1}{2}\sigma^2\,(u)\right) du + \sigma\,(u)\,dB\,(u) + \frac{1}{2}\sigma^2\,(u)\,du\right) \\
&= S\,(u)\,(\alpha\,(u)\,du + \sigma\,(u)\,dB\,(u)) = \alpha\,(u)\,S\,(u)\,du + \sigma\,(u)\,S\,(u)\,dB\,(u)
\end{aligned}$$

Note that the volatility coefficient $\sigma\,(u)$ generates drift $\frac{1}{2}\sigma^2\,(u)\,du$.

Looked at another way, if the drift coefficient $\mu(u) = -\frac{1}{2}\sigma^2(u)$ then the drift $\alpha(u)\,du = 0$.

IV. THE RISK FREE ACCUMULATION $\mathbf{e^{\int_0^t r(u)du}}$

(also called a Numeraire)

$e^{\int_0^t r(u)du}$ can be a random variable if the instantaneous risk-free rate $r(u)$ at time u is random, or it can be deterministic if $r(u)$ is deterministic, but in either case

$$
\begin{aligned}
d\left(e^{\int_0^u r(v)dv}\right) &= e^{\int_0^u r(v)dv} r(u)\,du + \frac{1}{2} e^{\int_0^u r(v)dv} (r(u)\,du)(r(u)\,du) \\
&= e^{\int_0^u r(v)dv} r(u)\,du
\end{aligned}
$$

has no $dB(u)$ component so $e^{\int_0^u r(v)dv}$ has what is called Bounded Variation, meaning it is a diffusion with what is called "zero volatility " and all the randomness is in the $r(u)$ and $r(v)$ randomness, if any. Note also that

$$d\left(e^{-\int_0^u r(v)dv}\right) = -e^{\int_0^u r(v)dv} r(u)\,du.$$

$e^{-\int_0^t r(u)du}$ can be used to discount values from t back to 0.

V. A PORTFOLIO $\mathbf{V(t, S(t))}$

(also called a Portfolio Process or a Portfolio Strategy or a Self-Financing Portfolio, Portfolio Process, or Portfolio Strategy)

A Portfolio is defined by two conditions:

1. it is a combination of a Risky Asset and a Risk-Free Accumulation
2. no deposits or withdrawals occur after inception (also called the "Self-Financing" condition).

Condition **1.** is a purely arithmetical condition:

$$V(t,S(t)) = \Delta(t,S(t))\,S(t) + \left[\frac{V(t,S(t)) - \Delta(t,S(t))\,S(t)}{e^{\int_0^t r(u)du}}\right] e^{\int_0^t r(u)du}$$

where $\Delta(t,S(t)))$ describes how much risky asset $S(t)$ is owned at time t (note that how much is owned can be random, including varying with the random value $S(t)$), and then

$$\left[\frac{V(t,S(t)) - \Delta(t,S(t)))\,S(t)}{e^{\int_0^t r(u)du}}\right]$$

automatically describes how much risk-free accumulation $e^{\int_0^t r(u)du}$ is owned at time t.

Condition **2.**, called the "Self-Financing condition" or Self-Financing Portfolio Dynamics, is what makes this a financial model:

$$dV(u,S(u)) = \Delta(u,S(u))\,dS(u) + \left[\frac{V(u,S(u)) - \Delta(u,S(u))\,S(u)}{e^{\int_0^u r(v)dv}}\right] d\left(e^{\int_0^u r(v)dv}\right).$$

The portfolio value only changes according to the change in the value of the risky asset owned and the change in the value of the risk-free accumulation owned. Note that there has been no use of the Itô Rule up to now. So far it is just arithmetic and finance. Now use the Itô Rule to calculate (as in **IV**)

$$\begin{aligned} d\left(e^{\int_0^u r(v)dv}\right) &= e^{\int_0^u r(v)dv}\, r(u)\,du, \text{ so that the Portfolio Dynamics are} \\ dV(u,S(u)) &= \Delta(u,S(u))\,dS(u) + \left[V(u,S(u)) - \Delta(u,S(u))\,S(u)\right] r(u)\,du, \end{aligned}$$

which is another way of expressing Condition **2.**, the Self-Financing Portfolio Dynamics.

$$\begin{aligned} \text{Since } dS(u) &= \alpha(u,S(u))\,du + \sigma(u,S(u))\,dB(u) \ \text{ we have} \\ dV(u,S(u)) &= \Delta(u,S(u))\left[\alpha(u,S(u))\,du + \sigma(u,S(u))\,dB(u)\right] \\ &\quad + \left[V(u,S(u)) - \Delta(u,S(u))\,S(u)\right] r(u)\,du \\ &= \left\{\Delta(u,S(u))\,\alpha(u,S(u)) + \left[V(u,S(u)) - \Delta(u,S(u))\,S(u)\right] r(u)\right\} du \\ &\quad + \Delta(u,S(u))\,\sigma(u,S(u))\,dB(u) \end{aligned}$$

The volatility of the Portfolio is the volatility of the Risky Asset it contains; the drift of the portfolio is the drift of the Risky Asset it contains plus the return on the Risk-Free Accumulation it contains.

VI. A REPLICATING PORTFOLIO $\mathbf{V}\left(\mathbf{t},\mathbf{S}\left(\mathbf{t}\right)\right)$ FOR A PAYOFF
(also called a Hedge, a Hedge Portfolio, a Hedge Strategy, or a Hedge Process)

Given a Payoff Value $PayOff_T\left(S\left(T\right)\right)$ (also called a "Claim") at a terminal time T that is a function of the value of the Risky Asset $S\left(T\right)$ at that time, a Replicating Portfolio for the Claim is a portfolio whose value $V\left(T,S\left(T\right)\right)$ at time T is the Payoff Value

$$V\left(T,S\left(T\right)\right)=PayOff_T\left(S\left(T\right)\right)$$

If there is such a Replicating Portfolio, we say that the Claim is Replicable. Note that both the Portfolio Value at T and the Payoff Value at T are random variables, so the Portfolio is a Replicating Portfolio if its random value at T exactly reproduces the random value of the Payoff at T no matter what that turns out to be.

A market model is called Arbitrage-Free (or No-Arbitrage, or No Free Lunch) if any two portfolios whose values at time $t<T$ are identical cannot then go on, with probability strictly greater than 0%, to have one of them with random value exceeding the random value of the other at time T, while with 100% probability that random value is never less than the random value of the other at time T.

A model that is not Arbitrage-Free would be useless for financial valuation because it would allow a non-0% probability of creating value out of nothing (this is called a Free Lunch with No Risk) by borrowing one portfolio at time t and simultaneously selling it and buying the other at time t for the same price with a probability greater than 0% of realizing a profit at time T together with 0% probability of realizing a loss at time T: Returning the borrowed portfolio at time T would be achieved at no risk and greater than 0% probability of profit by selling the bought portfolio for a price (with probability 100%) no lower than the price of the borrowed portfolio and with greater than 0% probability a price higher than the price of the borrowed portfolio.

If a Replicating Portfolio exists for a random Payoff Value (or Claim) at time T in an Arbitrage-Free market model, then the value at any time $t<T$ of owning the right to that random Payoff at T is precisely the value $V\left(t,S\left(t\right)\right)$ at t of the Replicating Portfolio.

This is called the Law of One Price for a Replicable Claim in an Arbitrage-Free market model. If one owns the Replicating Portfolio one cannot fail to reach (or better, one has 100% probability of reaching) a Portfolio Value at T exactly equal to the Payoff Value at T, and no other Replicating Portfolio can offer a better value because the market model is Arbitrage-Free.

A market model is called a Complete Market if all Claims are Replicable.

VII. BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION (In Dimension One)

Supppose the market is *one-dimensional*, meaning that there is only one Risky Asset $S(t)$ (or more accurately that there is only one Brownian Motion $dB(u)$ driving all risky assets in the market). Also assume that the functions α and σ that define the dynamics $dS(u) = \alpha(u, S(u))\,du + \sigma(u, S(u))\,dB(u)$ are reasonably nice functions.

In a one-dimensional market a Replicating Portfolio exists for a given Payoff Value (i.e. a given Claim is Replicable) if and only if the Black-Scholes Partial Differential Equation (shown below) has a solution with terminal value equal to the Payoff Value.

Proof: For any function $V(t,s)$ to which the Itô Rule applies:

$$\begin{aligned} dV(u, S(u)) &= \frac{\partial V}{\partial u} du + \frac{\partial V}{\partial s} dS(u) + \frac{1}{2}\frac{\partial^2 V}{\partial s^2} dS(u)\,dS(u) \\ &= \left[\frac{\partial V}{\partial u} + \frac{\partial V}{\partial s}\alpha(u, S(u)) + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2(u, S(u))\right] du \\ &\quad + \frac{\partial V}{\partial s}\sigma(u, S(u))\,dB(u) \end{aligned}$$

Therefore, $V(t, S(t))$ will be a Portfolio if and only if there is a $\Delta(u, S(u))$ such that

1.

$$\Delta(u, S(u))\,\sigma(u, S(u))\,dB(u) = \frac{\partial V}{\partial s}(u, S(u))\,\sigma(u, S(u))\,dB(u) \text{ and}$$

2.

$$\begin{aligned} &\{\Delta(u, S(u))\,\alpha(u, S(u)) + [V(u, S(u)) - \Delta(u, S(u))\,S(u)]\,r(u)\}\,du \\ = &\left[\frac{\partial V}{\partial u}(u, S(u)) + \frac{\partial V}{\partial s}(u, S(u))\,\alpha(u, S(u)) + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(u, S(u))\,\sigma^2(u, S(u))\right] du \end{aligned}$$

In order to satisfy **1**. it is sufficient that

$$\Delta(u,s) = \frac{\partial V}{\partial s}(u,s)$$

where this $\Delta(u,s)$ is called the Delta or the Hedge Ratio. Since we are assuming $\sigma(u, S(u)) \neq 0$ it is also necessary for Delta to take this value in order for $V(t, S(t))$ to be a Portfolio. Putting this value for $\Delta(u,s)$ into **2.**

$$\left\{\frac{\partial V}{\partial s}(u,s)\,\alpha(u,s) + \left[V(u,s) - \frac{\partial V}{\partial s}(u,s)\,s\right] r(u)\right\} du$$

$$= \left[\frac{\partial V}{\partial u}(u,s) + \frac{\partial V}{\partial s}(u,s)\,\alpha(u,s) + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(u,s)\,\sigma^2(u,s)\right] du$$

$$\left[V(u,s) - \frac{\partial V}{\partial s}(u,s)\,s\right] r(u) = \frac{\partial V}{\partial u}(u,s) + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(u,s)\,\sigma^2(u,s)$$

and rearranging gives what is called the Black-Scholes Partial Differential Equation

$$V(u,s)\,r(u) = \frac{\partial V}{\partial u}(u,s) + \frac{\partial V}{\partial s}(u,s)\,s r(u) + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(u,s)\,\sigma^2(u,s)$$

If this equation has a solution $V(t,s)$ for which $V(T,s) = PayOff_T(s)$ for all s, then $V(t, S(t))$ is a Replicating Portfolio, demonstrating that the Claim is Replicable. Conversely, every Replicating Portfolio for a Replicable Claim provides a solution to the equation for which $V(T, S(T)) = PayOff_T(S(T))$. For a Replicable Claim in an Arbitrage-Free market model this solution $V(t,s)$, inserting the value $s = S(t)$, will be the value at time t for owning the right to the Payoff Value at T.

Note that if the Risk-Free Accumulation rate $r(u)$ is random, then a solution $V(t,s)$ must be random as well and care will have to be taken in interpretting this result. (What does it mean for a random $V(t,s)$ where $s = S(t)$ to be the value for owning the right to the Payoff?)

If $r(u)$ is non-random, however, then in an Arbitrage-Free market model the Black-Scholes Partial Differential Equation gives a perfectly non-random way of finding the value of a random future Payoff Value prior to the payoff time.

VIII. MARTINGALE MEASURE $\mathbb{Q}$

(also called Risk-Neutral Measure, Risk-Neutral Probability, Equivalent Martingale Measure, MAKE BELIEVE MEASURE)

A probability measure $\mathbb{Q}$ such that for all $0 \leq v \leq t$,

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^t r(u)du} S(t)\,|S(v)\right] = e^{-\int_0^v r(u)du} S(v)$$

is called a local Martingale Measure (provided that $S(t)$ satisfies certain technical integrability conditions that we aren't going to discuss.)

It is a probability measure for which the present value today of the random future value of the risky asset at any point in time is a local martingale.

(WARNING: it is a vast oversimplification here to write the expected value on the left to be conditional only on the value $S(v)$. In a rigorous presentation it would be conditional on all randomness in the Brownian motion $dB(y)$ for all $y \leq v$. A technical concept from probability theory called a sigma-algebra would be used to implement this idea.)

A result called the local Martingale Representation Theorem (assuming $e^{-\int_0^t r(u)du} S(t)$ satisfies the technical conditions) says that such a $\mathbb{Q}$ exists if and only if there is a random process $\Gamma(t)$ with the property that

$$d\left(e^{-\int_0^u r(v)dv} S(u)\right) = \Gamma(u)\, dB_{\mathbb{Q}}(u)$$

where $dB_{\mathbb{Q}}(u)$ is Brownian motion *using the* $\mathbb{Q}$ *probability* to define the necessary normal distributions. After integrating both sides from v to t

$$e^{-\int_0^t r(u)du} S(t) - e^{-\int_0^v r(u)du} S(v) = \int_v^t \Gamma(u)\, dB_{\mathbb{Q}}(u)$$

leading to (oversimplifying the conditional expectation)

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^t r(u)du} S(t)\,|S(v)\right] - e^{-\int_0^v r(u)du} S(v) &= \mathbb{E}_{\mathbb{Q}}\left[\int_v^t \Gamma(u)\, dB_{\mathbb{Q}}(u)\,|S(v)\right] \\ &= 0 \text{ because } \mathbb{E}_{\mathbb{Q}}\left[dB_{\mathbb{Q}}\right] = 0 \end{aligned}$$

A local Martingale Measure $\mathbb{Q}$ is an artificial construct that does not purport to be the actual probability of anything observable in the financial world! Rather, it is a tool for specifying calculations of present values that would otherwise be much harder to describe.

IX. GOOD NEWS (In Dimension One) -LOCAL MARTINGALE MEASURE Q EXISTS & IS UNIQUE

Supppose the market is *one-dimensional*, meaning that there is only one Risky Asset $S(t)$ (or more accurately that there is only one Brownian Motion $dB(u)$ in the Physical Probability Measure $\mathbb{P}$ driving all risky assets in the market).

In a one-dimensional market, a local Martingale Measure $\mathbb{Q}$ always exists and it is unique.

Proof: To see whether a local Martingale Measure $\mathbb{Q}$ can exist, use the Itô Rule to calculate

$$d\left(e^{-\int_0^u r(v)dv} S(u)\right)$$

and see whether we can find the required random process $\Gamma(t)$ described in Section **VIII** with

$$d\left(e^{-\int_0^u r(v)dv} S(u)\right) = \Gamma(u)\, dB_{\mathbb{Q}}(u)$$

The Itô Rule gives

$$d\left(e^{-\int_0^u r(v)dv} S(u)\right) = d\left(e^{-\int_0^u r(v)dv}\right) S(u) + e^{-\int_0^u r(v)dv} dS(u) + 0$$

where the 0 comes from the fact that $du dS(u) = 0$. Continuing,

$$d\left(e^{-\int_0^u r(v)dv} S(u)\right) =$$

$$\begin{aligned}
&= -e^{-\int_0^u r(v)dv} r(u)\, du S(u) + e^{-\int_0^u r(v)dv} \left[\alpha(u, S(u))\, du + \sigma(u, S(u))\, dB(u)\right] \\
&= e^{-\int_0^u r(v)dv} \sigma(u, S(u)) \left[\frac{\alpha(u, S(u)) - r(u) S(u)}{\sigma(u, S(u))} du + dB(u)\right]
\end{aligned}$$

once again using $\sigma(u, S(u)) \neq 0$. So take

$$\Gamma(u) = e^{-\int_0^u r(v)dv} \sigma(u, S(u))$$

and ask whether it is possible that there is a probability measure $\mathbb{Q}$ with

$$dB_{\mathbb{Q}}(u) = \frac{\alpha(u, S(u)) - r(u) S(u)}{\sigma(u, S(u))} du + dB(u)$$

so that we would have

$$d\left(e^{-\int_0^u r(v)dv} S(u)\right) = \Gamma(u)\, dB_{\mathbb{Q}}(u)$$

as required by the local Martingale Representation Theorem?

It turns out that a result called Girsanov's Theorem gives the answer *yes!*

In particular, Girsanov's Theorem says that we can define an artificial probability for any set of outcomes $\mathcal{A}(t)$ observable at or prior to t by

$$\mathbb{Q}[\mathcal{A}(t)] = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}(t)\,\mathbf{1}_{\mathcal{A}(t)}\right]$$

where $\mathbf{1}_{\mathcal{A}(t)}$ is the indicator function for the set of outcomes $\mathcal{A}(t)$ (i.e. $\mathbf{1}_{\mathcal{A}(t)} = 1$ when the outcome is in the set $\mathcal{A}(t)$, and $= 0$ otherwise), $\mathbb{E}_{\mathbb{P}}$ is the expected value using the actual observable probability $\mathbb{P}$ (called Physical Probability or Real-World Probability) and $\frac{d\mathbb{Q}}{d\mathbb{P}}(t)$ is the process (called Radon-Nikodym Derivative Process) defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(t) = e^{-\int_0^t \frac{1}{2}\left(\frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))}\right)^2 du - \int_0^t \frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))} dB(u)}.$$

Furthermore, when we define that artificial probability measure $\mathbb{Q}$ the Girsanov Theorem says that Brownian motion under $\mathbb{Q}$ is precisely

$$dB_{\mathbb{Q}}(u) = \frac{\alpha(u,S(u)) - r(u)S(u)}{\sigma(u,S(u))}du + dB(u)$$

just as we wanted.

In this set-up, $\frac{\alpha(u,S(u))-r(u)S(u)}{\sigma(u,S(u))}$ is called the Market Price of Risk Process. Since it determines both the definition of $\mathbb{Q}$ and the relation of $dB_{\mathbb{Q}}(u)$ to $dB(u)$, which is unique, it implies that $\mathbb{Q}$ is unique.

NOTICE: In most financial modeling the Geometric Gaussian Diffusion of section **III** is used, where $\alpha(u,S(u)) = \alpha(u)S(u)$ and $\sigma(u,S(u)) = \sigma(u)S(u)$. In that case, the Market Price of Risk Process simplifies to $\frac{\alpha(u)-r(u)}{\sigma(u)}$. Remember that from the beginning (section **I**) we have been assuming that $\sigma(u,S(u)) \neq 0$ so in the Geometric Gaussian Diffusion $\sigma(u) \neq 0$.

Given that Girsanov's Theorem guarantees (at least in the one-dimensional case) that such local Martingale Measure probabilities $\mathbb{Q}$ must a exist, *a large part of Financial Engineering consists of coming up with practically useful forms of probability distributions for $\mathbb{Q}$ that allow for sensible calculations of present values today, without ever worrying about what the supposedly observable underlying future probability $\mathbb{P}$ might actually be.*

One reason for this is that local Martingales are much easier to work with than general random processes. *AN EVEN MORE IMPORTANT REASON IS ...*

X. THE PV OF ANY PORTFOLIO IS A LOCAL $\mathbb{Q}$-MARTINGALE

If $V(t,S(t))$ is a Portfolio (as defined in **V.**) then for all $0 \leq v \leq t$ and for a local Martingale Measure $\mathbb{Q}$ as defined in **VIII**,

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^t r(u)du} V\left(t, S(t)\right) | S\left(v\right)\right] = e^{-\int_0^v r(u)du} V\left(v, S(v)\right).$$

In other words, the same property that defines $\mathbb{Q}$ in terms of $S(t)$ actually holds true for all Portfolios: the present value today of the random future value of the portfolio at any point in time is a martingale in the $\mathbb{Q}$ probability measure.

(REMEMBER: We are vastly oversimplifying when we write the expectation to be conditional on just $S\left(v\right)$. See section **VIII**.)

To **prove** this start with the Itô Formula:

$$\begin{aligned}
& d\left(e^{-\int_0^u r(v)dv} V\left(u, S(u)\right)\right) \\
= \; & d\left(e^{-\int_0^u r(v)dv}\right) V\left(u, S(u)\right) + e^{-\int_0^u r(v)dv} dV\left(u, S(u)\right) + d\left(e^{-\int_0^u r(v)dv}\right) dV\left(u, S(u)\right)
\end{aligned}$$

Now use the Itô Formula on the first term and the Portfolio Dynamics from section **V** on the second term:

$$\begin{aligned}
& d\left(e^{-\int_0^u r(v)dv} V\left(u, S(u)\right)\right) \\
= \; & -e^{-\int_0^u r(v)dv} r\left(u\right) du V\left(u, S(u)\right) + e^{-\int_0^u r(v)dv} \left\{\Delta\left(u, S\left(u\right)\right) dS\left(u\right) \right. \\
& \left. + \left[V\left(u, S\left(u\right)\right) - \Delta\left(u, S\left(u\right)\right) S\left(u\right)\right] r\left(u\right) du\right\} + 0
\end{aligned}$$

where the 0 for the third term comes from $dudu = 0$ and $dudB\left(u\right) = 0$. Now substitute the definition of $V(u,S(u))$ from section **V** and simplify

$$
\begin{aligned}
& d\left(e^{-\int_0^u r(v)dv} V(u,S(u))\right) \\
= & -e^{-\int_0^u r(v)dv} r(u)\,du \left\{\Delta(u,S(u))\,S(u) + \left[\frac{V(u,S(u)) - \Delta(u,S(u))\,S(u)}{e^{\int_0^u r(v)dv}}\right] e^{\int_0^u r(v)dv}\right\} \\
& + e^{-\int_0^u r(v)dv} \left\{\Delta(u,S(u))\,dS(u) + \left[V(u,S(u)) - \Delta(u,S(u))\,S(u)\right] r(u)\,du\right\} \\
= & \ \Delta(u,S(u))\,e^{-\int_0^u r(v)dv} \left\{-S(u)\,r(u)\,du + dS(u)\right\}
\end{aligned}
$$

Finally, notice that in the last expression

$$
\begin{aligned}
e^{-\int_0^u r(v)dv} \left\{-S(u)\,r(u)\,du + dS(u)\right\} &= d\left(e^{-\int_0^u r(v)dv} S(u)\right) \text{ by Itô and} \\
&= \Gamma(u)\,dB_{\mathbb{Q}}(u) \text{ from section } \mathbf{VIII}
\end{aligned}
$$

So

$$
d\left(e^{-\int_0^u r(v)dv} V(u,S(u))\right) = \Delta(u,S(u))\,\Gamma(u)\,dB_{\mathbb{Q}}(u)
$$

Integrate both sides and take $\mathbb{Q}$ expected values conditional on $S(v)$ (oversimplifying again)

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^t r(u)du} V(t,S(t))\,|S(v)\right] - e^{-\int_0^v r(u)du} V(v,S(v)) \\
= & \ \mathbb{E}_{\mathbb{Q}}\left[\int_v^t \Delta(u,S(u))\,\Gamma(u)\,dB_{\mathbb{Q}}(u)\,|S(v)\right] = 0
\end{aligned}
$$

$$
\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^t r(u)du} V(t,S(t))\,|S(v)\right] = e^{-\int_0^v r(u)du} V(v,S(v))
$$

which is what we were trying to prove. In effect, $\Delta(u,S(u))\,\Gamma(u)$ provides the local Martingale Representation, as in section **VIII**, for $e^{-\int_0^u r(v)dv} V(u,S(u))$.

XI. LOCAL MARTINGALE MEASURE IMPLIES NO-ARBITRAGE

If a local Martingale Measure $\mathbb{Q}$ exists then the market model is Arbitrage-Free

Proof: Suppose $V(t, S(t))$ and $U(t, S(t))$ are two Portfolios such that $V(T, S(T)) \geq U(T, S(T))$ with probability 100% and $V(T, S(T)) > U(T, S(T))$ with probability greater than 0%. For the market model to be Arbitrage-Free, we need to to be able to prove that $V(t, S(t)) \neq U(t, S(t))$, with probability greater than 0%, for all $t \leq T$.

To do this, define a new Portfolio $W(t, S(t)) = V(t, S(t)) - U(t, S(t))$. By section **X**, the present value $e^{-\int_0^t r(u)du} W(t, S(t))$ of this new Portfolio is a local $\mathbb{Q}$-Martingale, so for all $t \leq T$, again oversimplifying,

$$e^{-\int_0^t r(u)du} W(t, S(t)) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(u)du} W(T, S(T)) \mid S(t)\right]$$

But for this new Portfolio $W(T, S(T)) = V(T, S(T)) - U(T, S(T)) > 0$ with probability greater than 0% and $W(T, S(T)) \geq 0$ with 100% probability so, with probability greater than 0%,

$$\begin{aligned}
e^{-\int_0^t r(u)du} W(t, S(t)) &> 0 \text{ meaning that} \\
W(t, S(t)) &> 0 \text{ so} \\
V(t, S(t)) - U(t, S(t)) &> 0 \text{ and} \\
V(t, S(t)) &\neq U(t, S(t)),
\end{aligned}$$

all with probability greater than 0%, for all $t \leq T$ as required. So the market model is Arbitrage-Free.

Looking at the converse, for a market model to be Arbitrage Free in general does not quite imply the existence of a local Martingale Measure, but see section **XV**(1) for something very close to that.

XII. MARTINGALE VALUE FOR A REPLICABLE CLAIM

(also called Risk-Neutral Value or Risk-Neutral Price or Martingale Price)

If $\mathbb{Q}$ is a local Martingale Measure and if a Replicable Claim with payoff at time T equal to $Payoff_T(S(T))$ has Replicating Portfolio $V(t, S(t))$, then at all times $t \leq T$ the value at t for owning the Claim (the right to the payoff at T) is the Martingale Value

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} Payoff_T(S(T)) \mid S(t)\right],$$

i.e. the expected value using the local Martingale Measure of the present value at t of the Payoff Value, conditional on the value of $S(t)$ (again, oversimplifying the conditional value.)

The **proof** of this is easy:

$$e^{-\int_0^t r(u)du} V(t, S(t)) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(u)du} V(T, S(T)) \,|S(t)\right]$$

because the present value of any Portfolio is a local $\mathbb{Q}$ martingale by section **X**. But then multiply both sides by $e^{\int_0^t r(u)du}$

$$V(t, S(t)) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} V(T, S(T)) \,|S(t)\right]$$

and use the assumption that $V(t, S(t))$ is a Replicating Portfolio for the Claim, so $V(T, S(T)) = Payoff_T(S(T))$, so

$$V(t, S(t)) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} Payoff_T(S(T)) \,|S(t)\right]$$

By section **XI** the market model is Arbitrage-Free, so as in section **VI** (using the Law of One Price) this Replicating Portfolio value must be the value at t for owning the right to the payoff, the Replicable Claim, at T. Owning the portfolio at t has 100% probability to give us the payoff $V(T, S(T)) = Payoff_T(S(T))$, and in an Arbitrage-Free market model no other Replicating Portfolio can offer a higher value at t with probability greater than 0%.

XIII. MORE GOOD NEWS (In Dimension One) - MARKET IS COMPLETE – ALL CLAIMS ARE REPLICABLE - PV OF EVERY CLAIM IS THE MARTINGALE VALUE

Supppose the market is *one-dimensional*, meaning that there is only one Risky Asset $S(t)$ (or more accurately that there is only one Brownian Motion $dB(u)$ in the Physical Probability Measure $\mathbb{P}$ driving all risky assets in the market).

In this one-dimensional case, a Replicating Portfolio exists for any payoff $Payoff_T(S(T))$, i.e. in a one-dimensional market model all claims are Replicable .

(A market model with this property that Replicating Portfolios always exists, that all claims are Replicable, is called a Complete Market. So we are saying that a one-dimensional market model is always Complete.)

As a **corollary** to the Completeness of a one-dimensional market model we can conclude that in a one-dimensional market model the Martingale Value gives the present value at any time $t < T$ for owning any Claim, the right to any $Payoff_T(S(T))$ at T.

This conclusion follows from the facts (a) that a one dimensional market model has a unique local Martingale Measure $\mathbb{Q}$ (section **IX**) and (b) that, given Completeness, the right to any $Payoff_T(S(T))$ at T is Replicable and the Martingale Value gives the value at any time $t < T$ for the right to $Payoff_T(S(T))$ at T (section **XII)**.

Proof of Completeness, that a Replicating Portfolio exists for any $Payoff_T(S(T))$: If we assume that we have already proved the 2nd Fundamental Theorem of Asset Pricing (section **XV**(2)) then the fact that in dimension one we have a unique $\mathbb{Q}$ (section **IX**) gives us Completeness, i.e. that a Replicating Portfolio exists for any $Payoff_T(S(T))$. But since we only sketch the general proof of **XV**(2), for any dimension, here is a more precise proof of Completeness in this one dimensional case:

For *any* probability measure $\mathbb{Q}$ (whether local Martingale Measure or not) we can define, just as a function for now, and again oversimplfying the conditional expectation,

$$\begin{aligned} V(t,S(t)) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\displaystyle\int_t^T r(u)du} Payoff_T(S(T)) \mid S(t)\right] \text{ so, of course,} \\ V(T,S(T)) &= Payoff_T(S(T)) \end{aligned}$$

By section **V**, this function $V(t,S(t))$ will be a Portfolio (hence, a Replicating Portfolio) if we can find a process $\Delta(t,S(t))$ such that

$$dV(u,S(u)) = \Delta(u,S(u))\,dS(u) + [V(u,S(u)) - \Delta(u,S(u))\,S(u)]\,r(u)\,du$$

To find such a process $\Delta(u,S(u))$, start with the definition of $V(t,S(t))$ and write (oversimplifying)

$$e^{-\displaystyle\int_0^t r(u)du} V(t,S(t)) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\displaystyle\int_0^T r(u)du} Payoff_T(S(T)) \mid S(t)\right]. \qquad (*)$$

For any $t \le v \le T$ the recursive property of conditional expectations now gives

$$e^{-\displaystyle\int_0^t r(u)du} V(t,S(t)) =$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(u)du} Payoff_T\left(S\left(T\right)\right) \mid S\left(v\right)\right] \mid S\left(t\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^v r(u)du} V\left(v, S\left(v\right)\right) \mid S\left(t\right)\right] \text{ by } (*) \text{ with } v \text{ replacing } t.
\end{aligned}
$$

This last equation means that $e^{-\int_0^t r(u)du} V\left(t, S\left(t\right)\right)$ is a local $\mathbb{Q}$-Martingale. (Note that we could not use section **X** to prove this because we don't yet know whether $V\left(t, S\left(t\right)\right)$ is a portfolio.)

Now, by the local Martingale Representation Theorem stated in Section **VIII** the fact that $e^{-\int_0^t r(u)du} V\left(t, S\left(t\right)\right)$ is a local $\mathbb{Q}$-Martingale implies that there is some random process $\Gamma\left(t\right)$ (not the same one as in sections **VIII** and **IX**) with

$$
\begin{aligned}
d\left(e^{-\int_0^u r(v)dv} V\left(u, S\left(u\right)\right)\right) &= \Gamma\left(u\right) dB_{\mathbb{Q}}\left(u\right) \text{ so using Itô} \\
e^{-\int_0^u r(v)dv}\left\{-r\left(u\right) du V\left(u, S\left(u\right)\right) + dV\left(u, S\left(u\right)\right)\right\} &= \Gamma\left(u\right) dB_{\mathbb{Q}}\left(u\right) \text{ and}
\end{aligned}
$$

$$
dV\left(u, S\left(u\right)\right) = e^{\int_0^u r(v)dv} \Gamma\left(u\right) dB_{\mathbb{Q}}\left(u\right) + r\left(u\right) V\left(u, S\left(u\right)\right) du \qquad (**)
$$

Now if we assume that $\mathbb{Q}$ is a local Martingale Measure then by section **IX** we can write

$$
\begin{aligned}
dB_{\mathbb{Q}}\left(u\right) &= \frac{\alpha\left(u, S\left(u\right)\right) - r\left(u\right) S\left(u\right)}{\sigma\left(u, S\left(u\right)\right)} du + dB\left(u\right) \text{ so} \\
dB\left(u\right) &= dB_{\mathbb{Q}}\left(u\right) - \frac{\alpha\left(u, S\left(u\right)\right) - r\left(u\right) S\left(u\right)}{\sigma\left(u, S\left(u\right)\right)} du \text{ and then} \\
dS\left(u\right) &= \alpha\left(u, S\left(u\right)\right) du + \sigma\left(u, S\left(u\right)\right) dB\left(u\right) \\
&= r\left(u\right) S\left(u\right) du + \sigma\left(u, S\left(u\right)\right) dB_{\mathbb{Q}}\left(u\right) \text{ and finally} \\
\sigma\left(u, S\left(u\right)\right) dB_{\mathbb{Q}}\left(u\right) &= dS\left(u\right) - r\left(u\right) S\left(u\right) du \qquad (***)
\end{aligned}
$$

Now we can make the definition of $\Delta\left(t, S\left(t\right)\right)$:

$$
\begin{aligned}
\Delta\left(t, S\left(t\right)\right) &= e^{\int_0^t r(u)du} \frac{\Gamma\left(t\right)}{\sigma\left(t, S\left(t\right)\right)} \text{ and in } (**) \text{ this gives} \\
dV\left(u, S\left(u\right)\right) &= \Delta\left(u, S\left(u\right)\right) \sigma\left(u, S\left(u\right)\right) dB_{\mathbb{Q}}\left(u\right) + r\left(u\right) V\left(u, S\left(u\right)\right) du
\end{aligned}
$$

So by $(***)$

$$\begin{aligned} dV(u,S(u)) &= \Delta(u,S(u))\,dS(u) - \Delta(u,S(u))\,r(u)\,S(u)\,du + r(u)\,V(u,S(u))\,du \\ &= \Delta(u,S(u))\,dS(u) + [V(u,S(u)) - \Delta(u,S(u))\,S(u)]\,r(u)\,du \end{aligned}$$

which is exactly what is required by section **V** to make our function $V(t,S(t))$ actually be a Portfolio (hence, a Replicating Portfolio.)

To sum up: In a one dimensional model, there is a local Martingale Measure (by section **IX**, which also says it is unique), the existence of that local Martingale Measure implies that the market model is Arbitrage-Free (by section **XI**) and that in this one-dimensional Complete Market a Replicating Portfolio exists for any future Payoff (by this section, **XIII**), which in turn guarantees (by section **XII**) that the Martingale Value (unique in this case) is the correct value at any $t < T$ for any future Payoff at T.

XIV. MORE COMPLICATED MODEL IN DIMENSION > 1

If more than one independent Brownian Motions drives the market model, label them $dB_1(u)$, ... , $dB_d(u)$ and call it a d-dimensional market model. Use vector notation

$$d\mathbf{B}(u) = \begin{Bmatrix} dB_1(u) \\ dB_2(u) \\ \ldots \\ dB_d(u) \end{Bmatrix}$$

In such a model, the d independent Brownian Motions will drive n (not necessarily independent) Risky Assets $S_1(t)$, ... , $S_n(t)$. Again use vector notation

$$\mathbf{S}(t) = \begin{Bmatrix} S_1(t) \\ S_2(t) \\ \ldots \\ S_n(t) \end{Bmatrix}$$

The dynamics of this model are

$$d\mathbf{S}(u) = \boldsymbol{\alpha}(u,\mathbf{S}(u))\,du + \boldsymbol{\sigma}(u,\mathbf{S}(u)) \bullet d\mathbf{B}(u)$$

where $\boldsymbol{\alpha}(u,\mathbf{S}(u))$ is an n-dimensional Drift Vector process

$$\boldsymbol{\alpha}(u,\mathbf{S}(u)) = \begin{Bmatrix} \alpha_1(u,\mathbf{S}(u)) \\ \alpha_2(u,\mathbf{S}(u)) \\ \ldots \\ \alpha_n(u,\mathbf{S}(u)) \end{Bmatrix}$$

and $\boldsymbol{\sigma}(u,\mathbf{S}(u))$ is an $n \times d$-dimensional Volatility Matrix process

$$\boldsymbol{\sigma}(u,\mathbf{S}(u)) = \begin{Bmatrix} \sigma_{11}(u,\mathbf{S}(u)) & \sigma_{12}(u,\mathbf{S}(u)) & ... & \sigma_{1d}(u,\mathbf{S}(u)) \\ \sigma_{21}(u,\mathbf{S}(u)) & \sigma_{22}(u,\mathbf{S}(u)) & ... & \sigma_{2d}(u,\mathbf{S}(u)) \\ ... & ... & ... & ... \\ \sigma_{n1}(u,\mathbf{S}(u)) & \sigma_{n2}(u,\mathbf{S}(u)) & ... & \sigma_{nd}(u,\mathbf{S}(u)) \end{Bmatrix}$$

Note that the Volatility Matrix $\boldsymbol{\sigma}(u,\mathbf{S}(u))$ encodes the covariance among the components of $d\mathbf{S}(u)$. They are not necessarily independent, and if $n > d$ they cannot be independent.

A Portfolio in this market model is determined by an n-dimensional row vector process

$$\boldsymbol{\Delta}(t,S(t)) = \begin{Bmatrix} \Delta_1(t,\mathbf{S}(t)) & \Delta_2(t,\mathbf{S}(t)) & ... & \Delta_n(t,\mathbf{S}(t)) \end{Bmatrix}$$

giving the Portfolio composition

$$V(t,\mathbf{S}(t)) = \boldsymbol{\Delta}(t,S(t)) \bullet \mathbf{S}(t) + \left[\frac{V(t,\mathbf{S}(t)) - \boldsymbol{\Delta}(t,S(t)) \bullet \mathbf{S}(t)}{e^{\int_0^t r(u)du}} \right] e^{\int_0^t r(u)du}$$

together with Portfolio Dynamics

$$dV(u,\mathbf{S}(u)) = \boldsymbol{\Delta}(u,S(u)) \bullet d\mathbf{S}(u) + [V(t,\mathbf{S}(t)) - \boldsymbol{\Delta}(t,S(t)) \bullet \mathbf{S}(t)]\, r(u)\, du$$

XV. MORE COMPLICATED PROPERTIES IN DIMENSION >1

In dimension > 1 the results of sections **I-VI**, **VIII** and **X-XII** remain valid after appropriate restatement into vector-matrix terms along the lines of the formulation in section **XIV**.

The results of sections **VII**, **IX** and **XIII**, however, do not always remain valid in dimension > 1.

We cannot always conclude **VII** that a Replicating Portfolio exists for a given Payoff if and only if there is a solution to the Black-Scholes Partial Differential equation in $(n+1)$ variables (1 time variable and n risky asset variables) with terminal values equal to the given Payoff values.

We cannot conclude **IX** that a local Martingale Measure always exists and, even if one does exist, we cannot conclude automatically that it is unique.

We cannot conclude **XIII** that the market model is always a Complete Market. As a consequence we cannot always be sure that the present value of a future Payoff is given by a Martingale Value or, even if a Martingale Value for the Payoff exists, we cannot conclude automatically that it is a unique value (as a present value needs to be).

The problem is that to prove the results in sections **VII**, **IX** and **XIII** at some step in the derivation we had to divide by $\sigma(u, S(u))$ (which was assumed $\neq 0$). In the vector-matrix formulation for dimension > 1 in section **XIV**, however, $\boldsymbol{\sigma}(u, \mathbf{S}(u))$ is now an $n \times d$ matrix. We can't just divide both sides of an equation by such a matrix.

In order to cancel out a matrix multiplication by $\boldsymbol{\sigma}(u, \mathbf{S}(u))$ on both sides of an equation we need the matrix $\boldsymbol{\sigma}(u, \mathbf{S}(u))$ to be of full rank, in this case rank d. That means that $\boldsymbol{\sigma}(u, \mathbf{S}(u))$ needs to contain a $d \times d$ submatrix that is invertible so that vector equality after multiplication by $\boldsymbol{\sigma}(u, \mathbf{S}(u))$ implies vector equality before multiplication. So now we always assume that the matrix $\boldsymbol{\sigma}(u, \mathbf{S}(u))$ is of full rank d for all u.

The problem goes deeper, so that we cannot even restore fully the results of **IX**, and **XIII** by just assuming that the matrix $\boldsymbol{\sigma}(u, \mathbf{S}(u))$ is of full rank d. The problem is that $\boldsymbol{\sigma}(u, \mathbf{S}(u))$ can be of full rank at each time u while having different rows constituting the required $d \times d$ invertible matrix at different times. This prevents picking the same d Risky Assets (out of the n Risky Assets in the model in total) at different times u to put in a Portfolio carrying the relevant information about the market model across time.

Rather than try to restrict market models to the case where it is possible to specify exactly how this matrix rank problem can be resolved, it is more common to identify more general properties of a market model that capture the required conclusions precisely. We'll state these here without proof:

1. **First Fundamental Theorem of Asset Pricing**

   A local Martingale Measure exists for a Market Model if and only if the model contains No Free Lunch with Vanishing Risk.

   A Free Lunch with Vanishing Risk is a sequence of pairs of Portfolios $V_n(t, \mathbf{S}(t))$ and $U_n(t, \mathbf{S}(t))$ such that

$$
\begin{aligned}
\mathbb{P}\left[\lim_{n\to\infty}\{V_n(T, \mathbf{S}(T)) - U_n(T, \mathbf{S}(T))\} \geq 0\right] &= 1 \text{ and} \\
\mathbb{P}\left[\lim_{n\to\infty}\{V_n(T, \mathbf{S}(T)) - U_n(T, \mathbf{S}(T))\} > 0\right] &> 0\text{, but for each } n \\
\mathbb{P}\left[|V_n(t, \mathbf{S}(t)) - U_n(t, \mathbf{S}(t))| \neq 0\right] &= 0 \text{ for some } t < T \\
\text{and where also } V_n(u, \mathbf{S}(u)) - U_n(u, \mathbf{S}(u)) &\geq C \text{ for } t < u < T
\end{aligned}
$$

   for all n for some constant C independent of n, where $\lim_{n\to\infty}\{V_n(T, \mathbf{S}(T)) - U_n(T, \mathbf{S}(T))\}$ is the random variable whose maximum difference (in absolute value) from the random variable $V_n(T, \mathbf{S}(T)) - U_n(T, \mathbf{S}(T))$ at each given n converges to 0 as $n \to \infty$. The existence of a local Martingale Measure is equivalent to there being no such sequences of pairs of Portfolios possible. The proof is delicate. See Delbaen and Schachermayer, The Mathematics of Arbitrage. This result is consistent with **XI**, the existence of a local

Martingale Measure implies No Arbitrage, because No Free Lunch with Vanishing Risk implies, as a simple consequence, No Arbitrage. (Just let $V_n(t, \mathbf{S}(t))$ and $U_n(t, \mathbf{S}(t))$ be the Arbitrage $V(t, \mathbf{S}(t))$ and $U(t, \mathbf{S}(t))$ for all n.)

2. **Second Fundamental Theorem of Asset Pricing**

   Assuming that a local Martingale Measure exists, then that local Martingale Measure is unique if and only if the market model is a Complete Market.

   When Replicating Portfolios exist (a Complete Market) in the presence of a local Martingale Measure then all Replicating Portfolios have the same value at any given time (because in the presence of a local Martingale Measure there is No Arbitrage) and both the Replicating Portfolio value and the Martingale Value give the present value for any Payoff, so there cannot be different Martingale Values. Conversely, when the local Martingale Measure is unique the unique Martingale Value can be used to define a Replicating Portfolio for any Payoff, using a careful analysis of how the properties of the full rank matrix $\boldsymbol{\sigma}(u, \mathbf{S}(u))$ across different times u relate to uniqueness of the local Martingale Measure.

3. **Black-Scholes Partial Differential Equation**

   If there is a solution $V(t, \mathbf{s})$ to the Black-Scholes Partial Differential equation in $(n+1)$ variables (1 time variable and n risky asset variables) with terminal values equal to the given Payoff values of some Claiim then a Replicating Portfolio exists with

   $$\boldsymbol{\Delta}(t, S(t))) = \mathrm{grad}_{\mathbf{s}} V(t, \mathbf{s}) \mid_{t, \mathbf{S}(t)} ,$$

   the vector of partial derivatives of the solution $V(t, \mathbf{s})$ with respect to the n variables $s_1, \ldots, s_n$ evaluated at the point $t, \mathbf{S}(t)$, making the Claim a Replicable one.

   Conversely, however, the existence of a Replicating Portfolio with the given Payoff values of some Replicable Claim does not guarantee a solution $V(t, \mathbf{s})$ to the Black-Scholes Partial Differential equation in $(n+1)$ variables (1 time variable and n risky asset variables) with terminal values equal to the given Payoff values of the Replicable Claim, not even in the case that the matrix $\boldsymbol{\sigma}(u, \mathbf{S}(u))$ is of full rank, d. The best that can be guaranteed is that the portfolio $\boldsymbol{\Delta}(t, S(t)))$ vector will determine d of the n components in the gradient vector $\mathrm{grad}_{\mathbf{s}} V(t, \mathbf{s}) \mid_{t, \mathbf{S}(t)}$ for each $t, \mathbf{S}(t)$ and it might not be the same d components at different times t, with no guarantee that they will integrate together to a solution of the multivariate Black-Scholes Partial Differential equation over all components at all times.