Toward Formal Dualities in Asset-Liability Modeling

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INTRODUCTION - RISK MODELING

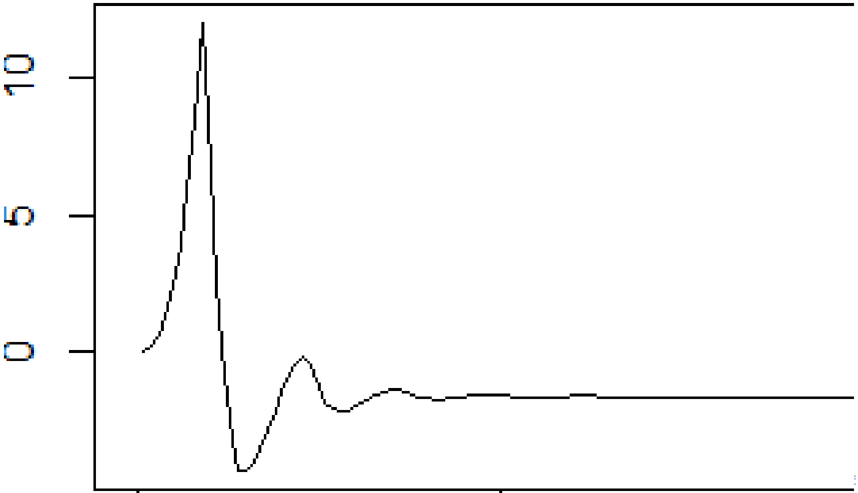
- Often, we model risk with the same models that we use for pricing, planning and forecasting
- For risk modeling we just use extreme inputs, or look at tails of random outputs, or use statistical extreme value theory
- But we fine-tune pricing, planning and forecasting models to work on normal inputs; this easily can foreclose really modeling risk in a holistic sense
- Maybe risk needs radically different models – still relatable to the normal models in some way, perhaps a formal duality.

MODELING ASSET-LIABILITY INTEREST RATE RISK

- Traditionally we model known cash flows and take present values - a balance sheet view
  - Ignore future cash unless implied by balance sheet
  - Test future interest rates' effect on present values:
    - duration/convexity etc.
    - stochastic future interest rates
    - risk-neutral calibrations to market values
- A radically different model could start with going concern assumptions - an income view
  - Look at all normalized on-going future cash flow
  - Test future interest rates' effect on future spreads
- Strictly a work-in-progress: what tools would give a dual model to the balance sheet?

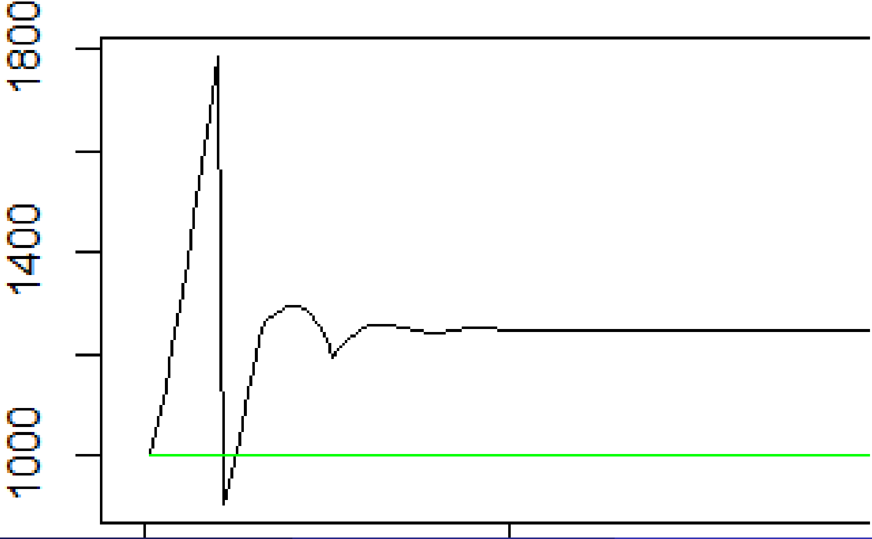
A SIMPLE GOING CONCERN WITH A-L MISMATCH

Take in a steady stream of 10-year bullet liabilities and invest steadily in 15 year ladder asset maturities. The following spreads result if interest rates increase steadily:

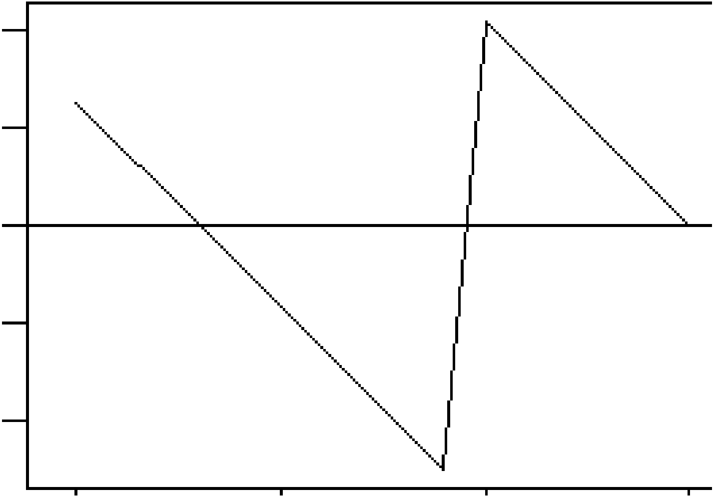


Linear Input Gave Oscillating Output - Why?

Maturity mismatch creates investment mismatch in going concern

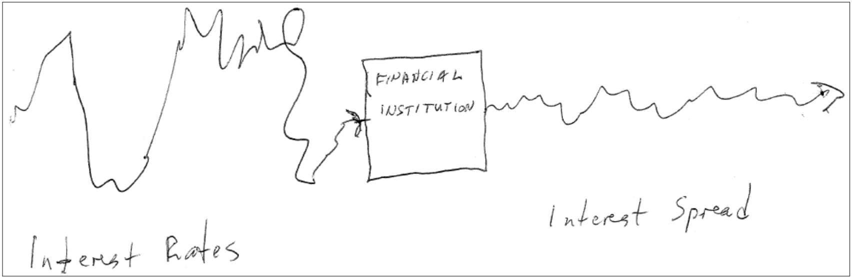


And looking backwards net survivors at each rate oscillates, too



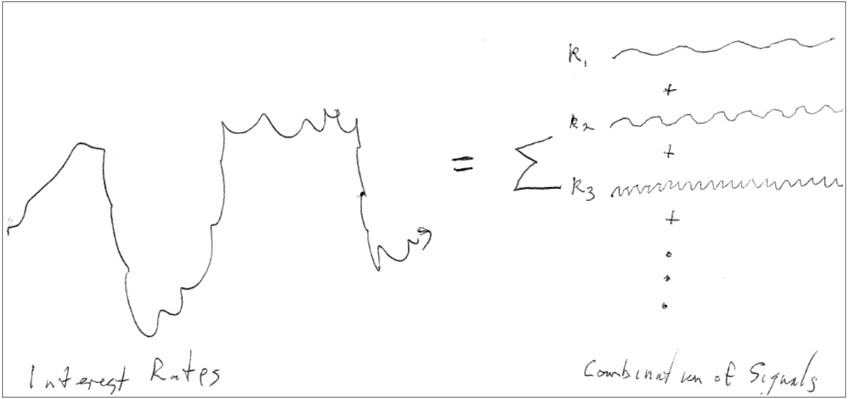
THIS CRIES OUT FOR FOURIER ANALYSIS

Think of a financial instituion as a receiver of a stream of interest rates that modulates them into an output stream of interest spreads (gain/loss)

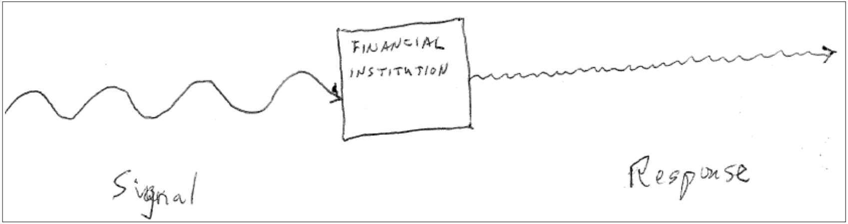


THIS CRIES OUT FOR FOURIER ANALYSIS

The interest rate stream consists of component signals, each with its own strength (and phase)

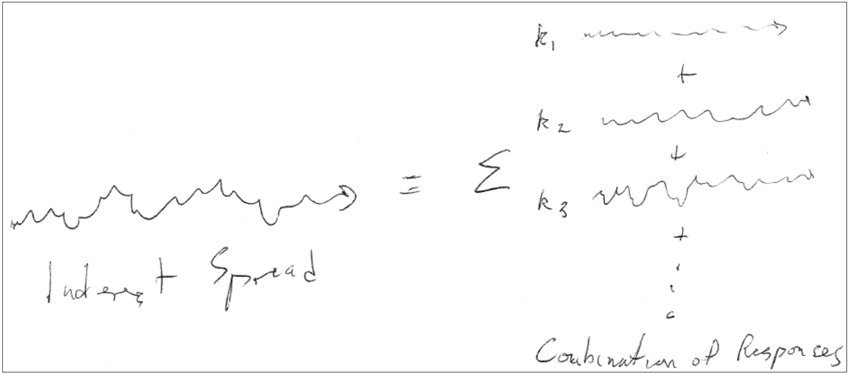


Suppose we know the response of the financial institution to each component signal

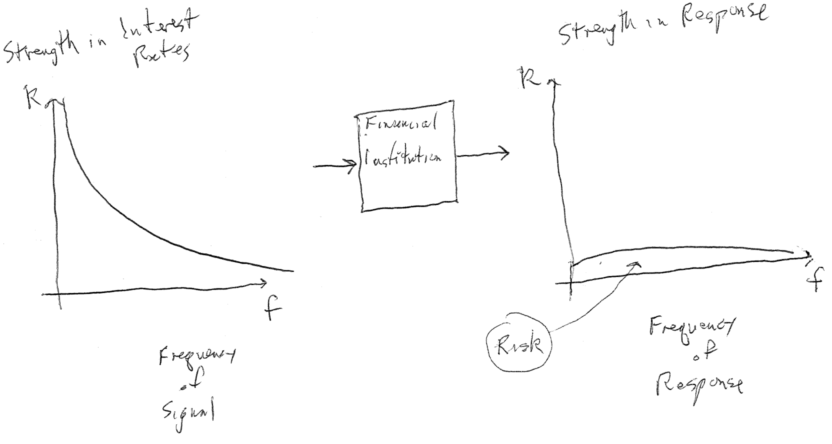


THIS CRIES OUT FOR FOURIER ANALYSIS

Then we can reconstruct the total response (the spread) to the original interest rate stream using the same strengths and phases



FOURIER ANALYSIS JUST CODIFIES THIS



- Strengths are given by |FT|, the modulus of the Fourier Transform
- Interest Rate Risk is the <u>area</u>

THIS IS THE DUAL VIEW OF INTEREST RATE RISK

- It looks at the institutional response to the entire spectrum of interest rate volatility
  - Dual to duration, etc. which looks only at the lowest frequency component(s)
- It looks at the going-concern interest rate spead (income statement)
  - Dual to the balance-sheet view of traditional immunization
  - Like the duality between position and momentum in physics
- Area under the spectrum is the proper risk measure
  - If random phases align against you the whole area contributes to your woe

(Or at least not directly from them)

WHAT WE NEED IS

- A model of the external interest rate spectrum
  - As an abstract random phenomenon, not just past x years or a closed time series
- A model of the modulation process
  - Unique to each financial institution
  - Easily applicable to all possible external signals, not just the usual ones
  - Including going-concern strategy, not just current balance sheet

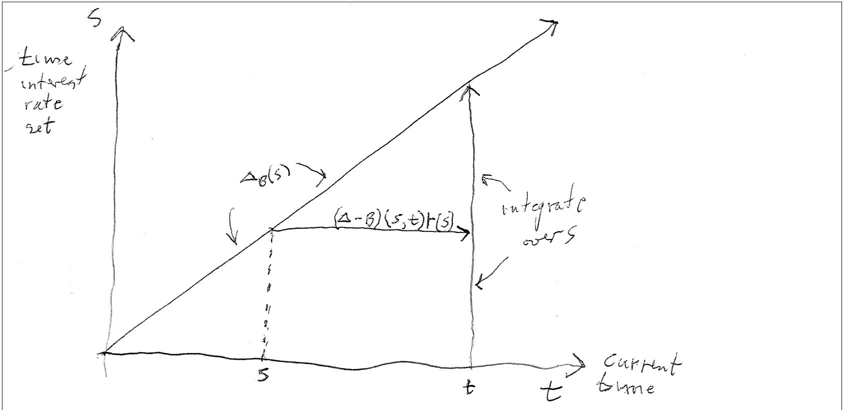
START WITH THE MODULATION PROCESS

Let $r(s) =$ the interest rate at time s
$\Delta_B(s) =$ new Liabilities taken on at time s
(Assume $\Delta_B(s)$ takes a simple going-concern form)
$B(s,t) =$ Liabilities matured out of $r(s)$ by time t
$b(s,t) = \frac{\partial}{\partial t} B(s,t)$ the rate of Liabilities maturing out of $r(s)$ at time t
$\Delta(s) = 1$ for $s \geq 0$ and $= 0$ for $s < 0$
$(\Delta - B)(s,t) =$ Liabilities still owed $r(s)$ at time $t =$ survival function of $B(s,t)$ viewed as a cdf
These functions model the marketing side of a business stratey

This gives a crude going-concern model of interest requirements on the Liabilities

START WITH THE MODULATION PROCESS

Interest requirements on the Liabilities (going concern)



$\Delta_B(s)$ is given

$$\text{Rate of Liability Interest} = \frac{\int_0^t r(s)(\Delta - B)(s,t)\Delta_B(s)\,ds}{\int_0^t (\Delta - B)(s,t)\Delta_B(s)\,ds} = \frac{[(\Delta - B) * (r\Delta_B)](t)}{[(\Delta - B) * \Delta_B](t)}$$

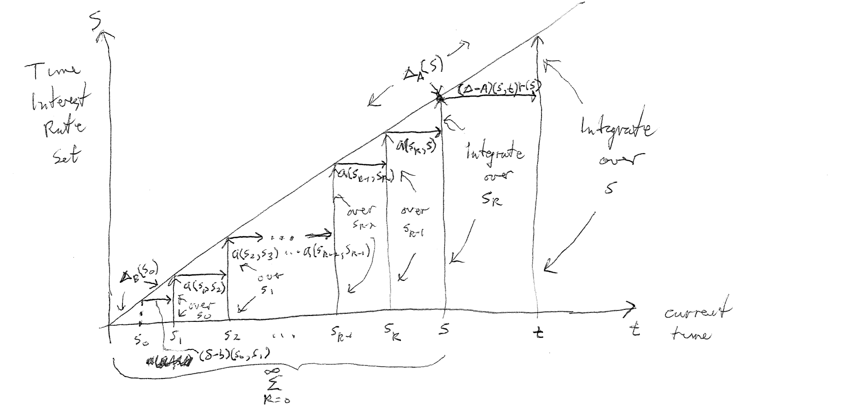
That's a generalization of the usual definition of convolution and it won't be commutative

But $a^{*k} = a * a * ... * a$ k-times makes sense and we will use it.
When we need it, $\delta =$ Dirac delta function (impulse at 0)
In particular, $a^{*(0)} = \delta$

If $\Delta_A(s) =$ new Assets taken on at time s then $\Delta_A(s)$ will be a function of everything else in the model
$A(s,t) =$ Assets matured out of $r(s)$ by time t
$a(s,t) = \frac{\partial}{\partial t} A(s,t)$ the rate of Assets maturing out of $r(s)$ at time t
$(\Delta - A)(s,t) =$ Assets still earning $r(s)$ at time $t =$ survival function of $A(s,t)$ viewed as a cdf.
These functions model the investing side of a business strategy.

Leads to a crude going-concern model of interest earned on the Assets

Interest generated by the Assets (going concern)



$$\Delta_A(s) = \sum_{k=0}^{\infty} \int_0^s a(s_k,s) \int_0^{s_k} a(s_{k-1},s_k) \int_0^{s_{k-1}} \cdots a(s_2,s_3) \int_0^{s_2} a(s_1,s_2) \int_0^{s_1} (\delta-b)(s_0,s_1)\, \Delta_B(s_0)\, ds_0\, ds_1 \cdots ds_{k-2}\, ds_{k-1}\, ds_k$$

$$\text{Rate of Asset Interest} = \frac{\int_0^t r(s)(\Delta-A)(s,t)\Delta_A(s)\,ds}{\int_0^t (\Delta-A)(s,t)\Delta_A(s)\,ds} = \frac{[(\Delta-A)*(r\Delta_A)](t)}{[(\Delta-A)*\Delta_A](t)}$$

$$\Delta_A(s) = \left[\left(\sum_{k=0}^{\infty} a^{*k}\right) * (\delta-b) * \Delta_B\right](s)$$

START WITH THE MODULATION PROCESS

Subtracting the Liability interest from the Asset interest yields the going concern interest rate spread $s\,(t)$ at time t

$$s\,(t) = \frac{\left[(\Delta - A) * \left(r\left[\left(\sum_{k=0}^{\infty} a^{*k}\right) * (\delta - b) * \Delta_B\right]\right)\right](t)}{\left[(\Delta - A) * \left(\left(\sum_{k=0}^{\infty} a^{*k}\right) * (\delta - b) * \Delta_B\right)\right](t)} - \frac{[(\Delta - B) * (r\Delta_B)](t)}{[(\Delta - B) * \Delta_B](t)}$$

where the denominators are equal (a good test of your convolution algebra)

At this point I don't know how to progress any further without assuming a homogeneous business strategy, ie. $B(s,t) = B(t-s)$, $A(s,t) = A(t-s)$, etc. for all s and t Among other things this makes the convolutions commutative.

Some useful facts are $(\Delta - A) * \left(\sum_{k=0}^{\infty} a^{*k} \right) = \Delta$ and $\lim_{t\to\infty} \left(\sum_{k=0}^{\infty} a^{*k} \right)(t) = \frac{1}{\mu_A}$ where μ_A is the mean of A considered as a cdf.

Also, those survival functions $(\Delta - A)$ and $(\Delta - B)$ involved in convolutions (= integrals) suggests that some more means are lurking in these formulas.

CONTINUING WITH THE MODULATION PROCESS

All of this can be pushed to a formula for the going concern interest spread (call it s). If we assume a level stream of new Liabilities the formula for the spread is

$$
\begin{aligned}
s &= \frac{\mu_B}{\mu_A}\left[(\Delta - A) - (\Delta - B)\right] * r \\
&\quad - (\Delta - A) * \left\{ \left[\left(\sum_{k=0}^{\infty} a^{*k} \right) * \left(\frac{\mu_B}{\mu_A}(\Delta - A) - (\Delta - B) \right) \right] r \right\}
\end{aligned}
$$

Amazingly, the messy term is a transient that goes to 0 as the homogenous going-concern reaches steady-state. For a stable growing level of new Liabilities a similar formula obtains, involving well-defined distortions of the A, B, a, and b functions and corresponding μ factors in the formula.

The permanent steady-state term is made-to-order for a Fourier Transform, which takes $*$ to multiplication.

CONCLUSION FOR THE MODULATION PROCESS

For each frequency f the Fourier transform of the steady-state going-concern spread in the case of stable growth of new Liabilities at rate g is $FT[s](f) = \frac{1}{\ln(1+g)+2\pi if}\left(\frac{1-FT[a](f)}{\mu_A} - \frac{1-FT[b](f)}{\mu_B}\right) FT[r](f)$ where the distorted versions of the functions and means must be used if the assumed growth g is not 0.

In other words $\frac{1}{\ln(1+g)+2\pi if}\left(\frac{1-FT[a](f)}{\mu_A} - \frac{1-FT[b](f)}{\mu_B}\right)$ represents how the financial institution modulates the external interest rate frequency strengths $FT[r](f)$ into interest spread frequency responses $FT[s](f)$.

The going-concern business strategy is reflected in the $FT[a](f)$, $FT[b](f)$, μ_A, and μ_B terms, all of which are computable by reasonable formulas

CONCLUSION FOR THE MODULATION PROCESS

Again, $\frac{1}{\ln(1+g)+2\pi i f}\left(\frac{1-FT[a](f)}{\mu_A}-\frac{1-FT[b](f)}{\mu_B}\right)$ represents how the financial institution modulates the external interest rate frequency strengths $FT\left[r\right]\left(f\right)$ into interest spread frequency responses $FT\left[s\right]\left(f\right)$.

The factor $\frac{1}{\ln(1+g)+2\pi i f}$ in the modulation already teaches an important lesson for risk management: a stable, well-managed level of growth is a very effective risk-control mechanism.

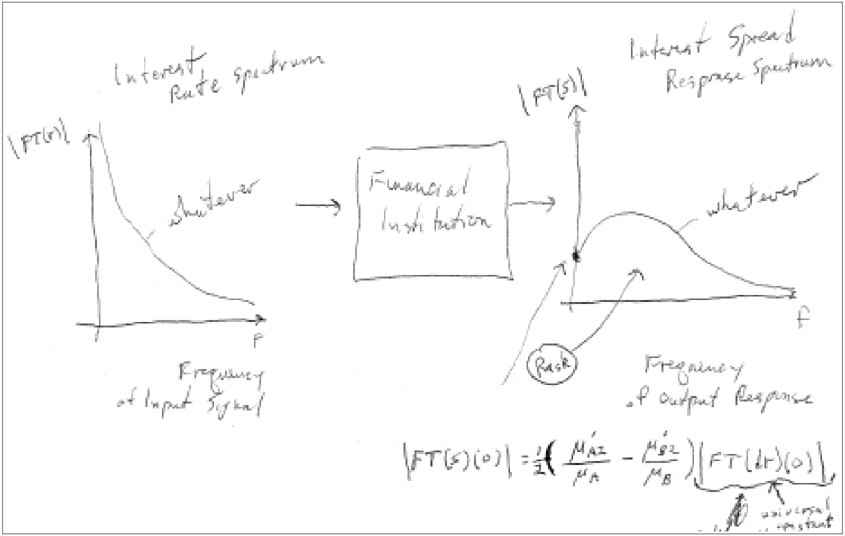
The larger g is, the less the vulnerable the institution is to external rate volatility, with the greatest relative protection coming at lower frequency (small f) components. (Of course, unstable or poorly managed growth creates its own problems.)

CONCLUSION FOR THE MODULATION PROCESS

There is another interesting result at $f = 0$:
$|FT[s](0)| = \frac{1}{2}\left|\left(\frac{\mu'_{A2}}{\mu_A} - \frac{\mu'_{B2}}{\mu_B}\right) FT[dr](0)\right|$ where μ'_{A2} and μ'_{B2} are 2nd raw moments (distorted if there's growth) and $|FT[dr](0)|$ is a universal constant related to the external interest rate spectrum.

The expression $\frac{1}{2}\frac{\mu'_{A2}}{\mu_A}$ is the mean of the equilibrium distribution corresponding to the (distorted) Asset maturity schedule A considered as a cdf. Same thing for $\frac{1}{2}\frac{\mu'_{B2}}{\mu_B}$. So the difference between the means of the equilibrium distributions controls the frequency $f = 0$ part of the modulation process. It should be no surprise that these equilibrium distribution means can be formally related (a duality) with the traditional duration concept. All of the risk area beyond $f = 0$ still remains, however, untouched by either "duration" concept.

CONCLUSION FOR THE MODULATION PROCESS



NEXT COMES THE EXTERNAL RATE SPECTRUM

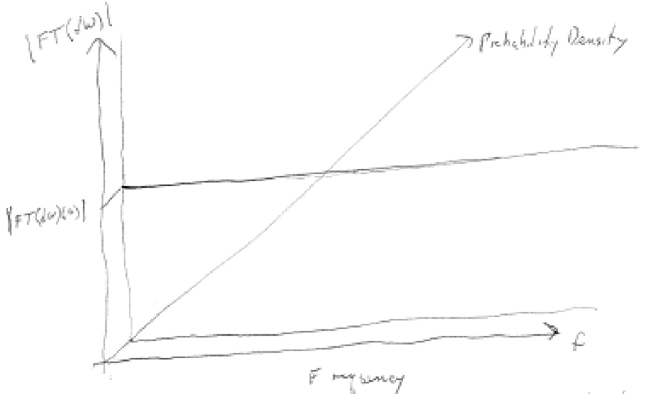
One could do lots of things, but let's assume we'll need a random process and let's further assume that Brownian motion will drive whatever process we end up using

If $dW(t)$ is the random Brownian increment at time t we can derive its Fourier Transform at any frequency f:

$|FT[dw](f)| = |FT[dw](0)|$ and the phase of $FT[dw](f)$ is totally random in f where $FT[dw](0)$ is a random real number, fixed for all time, and unknowable. All of the Fourier frequencies are equally represented and random walk comes from randomized phase relationships.

This is a little like renormalization in physics, it sounds strange but it works since everything we can observe will just be relative to this unknowable thing. (Remember, we promised that the risk model will be very different from our usual models!)

SPECTRUM OF THE BROWNIAN INCREMENT



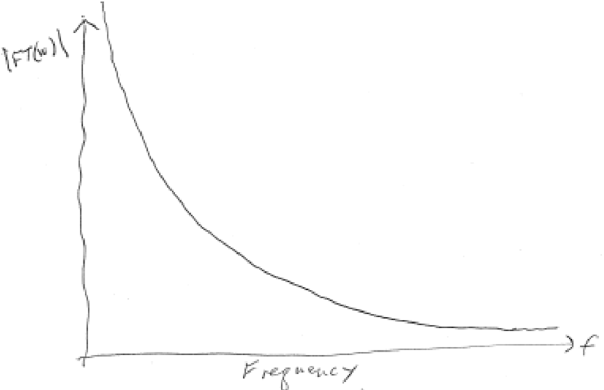
- dw(t) has all Fourier frequencies <u>equally</u> represented at one <u>random</u> value
- Its phases are completely randomized

MOVING TOWARD THE RATE SPECTRUM

Next, we can derive the Fourier Transform of the Brownian Motion $W(t)$ itself by integrating its increment $\mathrm{d}W(t)$. Here, it helps to know that for any differential $dF(t)$ we have $\int_{-\infty}^{t} dF(s) = [\Delta * dF](t)$ and that $FT[\Delta](f) = \frac{1}{2\pi i f}\mathbf{1}_{f\neq 0} + \frac{1}{2}\delta(f)$.

This leads pretty quickly to:

$|FT[W](f)| = \frac{|FT[dW](0)|}{2\pi f}\mathbf{1}_{f\neq 0} + \frac{1}{2}|FT[dW](0)|\,\delta(f)$ where the phase is totally randomized (that's what makes the walk a random one) and where $FT[dw](0)$ is that unknowable real number at the base of the model



- $|FT(w)(f)| = \frac{|FT(dw)(0)|}{2\pi f} \cdot 1_{f \neq 0} + \frac{1}{2}|FT(dw)(0)|\,\delta(f)$
- Remember: $|FT(dW)(0)|$ is a fixed (random) value
- Phases of $FT(w)(f)$ are totally randomized

Make the external interest rate $r(t)$ a Mean-reverting Geometric Brownian Motion.

$$r(t) = \left\{ -\ln(1-F)\left[\ln T - \ln r(t)\right] - \frac{1}{4}\sigma^2\left[1 + \mathbf{1}_{F=0}\right]\right\} r(t)\,dt + \sigma r(t)\,dW(t)$$

For $F = 0$ there is no mean-reversion; the drift compensation creates $E[r(t)] = T$ in the steady state.

With quite a bit of work we can get to:

SPECTRUM FOR EXTERNAL INTEREST RATE

$|FT[r](f)| = \frac{\sigma|[FT(r)*FT(dW)](f)|\mathbf{1}_{f\neq 0}+2\pi fT\delta(f)}{\left|2\pi if-\left\{\frac{1}{2}\sigma^2+Q(f)\sigma[FT(r)*FT(dW)](f)\right\}\mathbf{1}_{F\neq 0}\right|}$ with totally randomized phase.

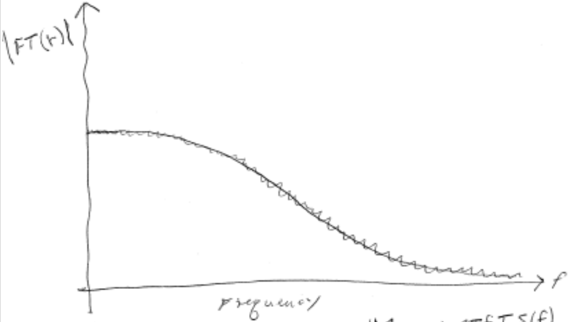
This might look as if we aren't done yet because we have $FT[r](f)$ on both sides of the equation (it is part of those convolutions.)

It turns out, however, that the dependence of the convolution on f just randomizes the phase and introduces tiny fluctuations in modulus.

$[FT(r)*FT(dW)](f)$ is essentially a random weighted average of $|FT[r](h)|$ over all h, with the randomness almost all in the phases. Whatever its actual value, it provides a common base for the relative contribution each separate frequncy f makes to the spectrum.

Note that when $F=0$ this looks a lot like Brownian Motion (goes to ∞ when $f=0$) but mean reversion ($F\neq 0$) sets a maximum on the spectrum at $f=0$

THE EXTERNAL RATE SPECTRUM (F not 0)



- $|FT(r)(f)| = \dfrac{|\sigma[FT(r)*FT(dW)](f)|\, 1_{f\neq 0} + 2\pi f T \delta(f)}{|2\pi i f - \frac{1}{2}\sigma^2 - Q(f)\sigma[FT(r)*FT(dW)](f)|}$
- $[FT(r)*FT(dW)](f)$ is a "nearly" fixed random variable
  (dependence on f is a totally random phase and tiny fluctuations)
- $Q(f)$ is a "nearly" fixed random variable
  (dependence on f is a phase making $Q(f)[FT(r)*FT(dW)](f)$ real, and tiny fluctuations)

THAT'S OUR DUAL MODEL FOR INTEREST RISK

So the interest rate spread in our going-concern has a risk spectrum of:
$|FT[s](f)| = \left| \frac{1}{\ln(1+g)+2\pi if} \left(\frac{1-FT[a](f)}{\mu_A} - \frac{1-FT[b](f)}{\mu_B} \right) FT[r](f) \right|$ where the phase is totally randomized, where the distorted versions of the functions and means must be used if the assumed growth g is not 0, and where

$|FT[r](f)| = \frac{\sigma|[FT(r)*FT(dW)](f)|\mathbf{1}_{f\neq 0}+2\pi fT\delta(f)}{\left|2\pi if-\left\{\frac{1}{2}\sigma^2+Q(f)\sigma[FT(r)*FT(dW)](f)\right\}\mathbf{1}_{F\neq 0}\right|}$ is the external rate spectrum.

The randomization of the phases makes this approach completely impractical for modeling time-specific values, so it is useless for planning or forecasting. But it models risk for us in a completely natural way. (And requires no 10,000 scenarios or Latin-hypercubes.)

References

- For the modulation process
  - My paper at the 1998 International Congress of Actuaries

link to it at bottom of
http://www.math.uconn.edu/~bridgeman/Papers_and_Presentations/index.
But really, anyone who has programmed an ALM model has done this, whether they know it or not

- For the Fourier Analysis - any good text; I like
  - Rudin's Real and Complex Analysis
  - Brigham's Fast Fourier Transform
  - Meikle's A New Twist To Fourier Transforms
- For the application to random walk - you need to be careful; I used an actual-infinitesimals approach following the ideas in
  - Robinson's Non-Standard Analysis
  - But he didn't apply it to random walk ... I did and am confident I got it right