

Structure of the CAPM Covariance Matrix

James Bridgeman
University of Connecticut

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- SEE HOW CAPM COVARIANCE MATRIX:
 - ACTUALLY HOLDS THE EXPECTED RETURNS
 - ACTIVELY REFLECTS THE MARKET WEIGHTS
- MOSTLY PEDAGOGY
- A TINY BIT OF LIGHT ON LITERATURE

- MARKET COVARIANCE MATRIX

$$\Sigma = \begin{Bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{Bmatrix} \text{ invertible, } \sigma_{ij} = \sigma_{ji}, \sigma_{ii} > 0, \text{ and } \sigma_{ii}\sigma_{jj} > \sigma_{ij}^2$$

- MARKET WEIGHTS & EXPECTED RETURNS

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix} \text{ market weights, } \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix}, \mathbf{e}^T \cdot \mathbf{w} = 1, w_i > 0,$$

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ \dots \\ r_n \end{pmatrix} \text{ expected returns, } r_f \text{ risk-free rate, } (\mathbf{r} - r_f \mathbf{e}) \text{ market risk premia}$$

CAPM CONCLUSIONS

ASSUMING MARKET IS EFFICIENT (I.E. IS A FRONTIER PORTFOLIO)

- COVARIANCE-RETURNS-WEIGHTS RELATIONSHIP

$$(\mathbf{r} - r_f \mathbf{e}) = \boldsymbol{\Sigma} \cdot \mathbf{w}$$

$$\mathbf{w} = \frac{\boldsymbol{\Sigma}^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \boldsymbol{\Sigma}^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} \quad (\text{denominator ensures } \mathbf{e}^T \cdot \mathbf{w} = 1)$$

- RETURNS-BETAS RELATIONSHIP

$$(\mathbf{r} - r_f \mathbf{e}) = \boldsymbol{\beta} \left(\mathbf{w}^T \cdot \mathbf{r} - r_f \right) = \boldsymbol{\beta} (r_M - r_f)$$

$$\text{where } \boldsymbol{\beta} = \frac{\boldsymbol{\Sigma} \cdot \mathbf{w}}{\mathbf{w}^T \cdot \boldsymbol{\Sigma} \cdot \mathbf{w}}$$

and $r_M = \mathbf{w}^T \cdot \mathbf{r}$ is the expected market return

- EXAM QUESTION

- Be Sure To Emphasize Effect Of Negative Covariance

$$\Sigma = \begin{Bmatrix} .01 & .10 & -.20 \\ .10 & .04 & .25 \\ -.20 & .25 & .09 \end{Bmatrix}, \mathbf{r} = \begin{Bmatrix} .02 \\ .10 \\ .20 \end{Bmatrix}, r_f = .03, \text{ what is } \mathbf{w}?$$

- WAIT UNTIL GRADING TO WORK OUT THE ANSWER
- ANSWER

$$\Sigma^{-1} = \begin{Bmatrix} 4.5 & 4.5 & -2.5 \\ 4.5 & 3.0 & 1.7 \\ -2.5 & 1.7 & .73 \end{Bmatrix}, \mathbf{w} = \frac{\Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} = \begin{Bmatrix} -.27 \\ .80 \\ .47 \end{Bmatrix}, \text{ Rats!}$$

- WHAT WENT WRONG?

- AHA! Forgot Requirement $\sigma_{ii}\sigma_{jj} > \sigma_{ij}^2$. Easy To Fix Next Semester.

ENLIGHTENED PERHAPS, BUT STILL LAZY

- EXAM TIME THE FOLLOWING SEMESTER
 - Still Be Sure To Emphasize Effect Of Negative Covariance

$$\Sigma = \begin{Bmatrix} .010 & .015 & -.028 \\ .015 & .040 & .05 \\ -.028 & .05 & .090 \end{Bmatrix}, \mathbf{r} = \begin{Bmatrix} .02 \\ .10 \\ .20 \end{Bmatrix}, r_f = .03, \text{ what is } \mathbf{w}?$$

- STILL WAIT UNTIL GRADING TO WORK OUT THE ANSWER
- ANSWER

$$\Sigma^{-1} = \begin{Bmatrix} -13 & 33 & -23 \\ 33 & -1.4 & 11 \\ -23 & 11 & -2.1 \end{Bmatrix}, \mathbf{w} = \frac{\Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} = \begin{Bmatrix} -1.92 \\ 2.03 \\ .89 \end{Bmatrix}, \text{ Worse!}$$

- WHAT WENT WRONG THIS TIME?
 - God Knows, Just Avoid The Negative Covariances Next Semester

- THIRD SEMESTER'S A CHARM?

- Let Everything Be Positive, Including All Of The Risk Premia

$$\Sigma = \begin{Bmatrix} .010 & .015 & .028 \\ .015 & .040 & .05 \\ .028 & .05 & .090 \end{Bmatrix}, \mathbf{r} = \begin{Bmatrix} .04 \\ .10 \\ .20 \end{Bmatrix}, r_f = .03, \text{ what is } \mathbf{w}?$$

- CONFIDENTLY WAIT TO WORK OUT THE ANSWER
- ANSWER

$$\Sigma^{-1} = \begin{Bmatrix} 791 & 36 & -266 \\ 36 & 83.5 & -57.6 \\ -266 & -57.6 & 126 \end{Bmatrix}, \mathbf{w} = \frac{\Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} = \begin{Bmatrix} 1.47 \\ .15 \\ -.62 \end{Bmatrix}, \text{ No!}$$

- THERE'S MORE TO THIS THAN MEETS THE EYE!
- WHAT DOES Σ HAVE TO LOOK LIKE TO BE A MARKET?

A FAMILIAR PHENOMENON IN THE LITERATURE

- BEST & GRAUER (1985, 1992)
 - Efficient Markets with all $w_i > 0$ form a segment of the Frontier with length $\rightarrow 0$ as number of assets $n \rightarrow \infty$
- Brennan & Lo (2010)
 - For any given $(\mathbf{r} - r_f \mathbf{e})$, *Impossible* Σ are those with some $w_i < 0$. Then $\mathbb{P}[\Sigma \text{ Impossible}] \nearrow$ geometrically with n for reasonable distribution assumptions on Σ .
- LEDOIT & WOLF (2004, 2013, 2014)
 - Introduce statistical Shrinkage techniques to transform Empirical Σ into a Frontier Portfolio
- LEVY & ROLL (2010)
 - Empirical Σ has "high" \mathbb{P} of being "close" to a Frontier Portfolio, with "close" attained by $\pm \epsilon$ on σ_{ii} and r_i . Note ρ_{ij} can remain fixed.
- BOYLE (2012, 2014)
 - On large class of Σ , Frontier Portfolio equivalent to Σ Almost Positive

IS THERE A SIMPLE-MINDED WAY TO SEE ALL THIS?

- GIVEN \mathbf{w} and $\mathbf{r} - r_f \mathbf{e}$ WITH ALL $w_i > 0$
- WHAT DO Σ THAT SIT ON FRONTIER LOOK LIKE?

$$\mathbf{w} = \frac{\Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} \text{ so}$$

$$\Sigma \cdot \mathbf{w} = \frac{(\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} \text{ and for any } k$$

$$\begin{aligned} (k\Sigma) \cdot \mathbf{w} &= \frac{k(\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} \\ &= \frac{(\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot (k\Sigma)^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} \end{aligned}$$

So it is enough to find all Σ such that

$$\Sigma \cdot \mathbf{w} = (\mathbf{r} - r_f \mathbf{e})$$

and then multiply by any constant.

IS THERE A SIMPLE-MINDED WAY TO SEE ALL THIS?

- EASY TO FIND A MATRIX \mathbf{R} WITH $\mathbf{R} \cdot \mathbf{w} = \mathbf{r} - r_f \mathbf{e}$

$$\text{Just let } \mathbf{R} = \left\{ (\mathbf{r} - r_f \mathbf{e}) \mathbf{e}^T \right\}$$

with n identical column vectors each equal to $(\mathbf{r} - r_f \mathbf{e})$. We can also express \mathbf{R} as n row vectors

$$\mathbf{R} = \begin{Bmatrix} (r_1 - r_f) \mathbf{e}^T \\ (r_2 - r_f) \mathbf{e}^T \\ \dots \\ (r_n - r_f) \mathbf{e}^T \end{Bmatrix}.$$

- UNFORTUNATELY THE CHOICE $\Sigma = \mathbf{R}$ IS NOT INVERTIBLE
 - Rows and columns clearly fail to be independent
- COULD EASILY FAIL TO HAVE $\sigma_{ii} > 0$
- IS UNLIKELY TO HAVE $\sigma_{ij} = \sigma_{ji}$ OR $\sigma_{ii} \sigma_{jj} > \sigma_{ij}^2$ FOR ALL i, j

IS THERE A SIMPLE-MINDED WAY TO SEE ALL THIS?

- BUT MAYBE THERE IS A MATRIX \mathbf{A} WITH $\mathbf{A} \cdot \mathbf{w} = \mathbf{0}$

$$\text{So } (\mathbf{R} + \mathbf{A}) \cdot \mathbf{w} = \mathbf{R} \cdot \mathbf{w} = (r - r_f \mathbf{e})$$

- AND WITH $\Sigma = (\mathbf{R} + \mathbf{A})$ INVERTIBLE AND SATISFYING

$$\sigma_{ij} = \sigma_{ji}, \sigma_{ii} > 0, \text{ and } \sigma_{ii}\sigma_{jj} > \sigma_{ij}^2$$

- VIEW \mathbf{A} AS ROW VECTORS

$$\mathbf{A} = \left\{ \begin{array}{c} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \dots \\ \mathbf{a}_n^T \end{array} \right\}, \text{ with each } \mathbf{a}_i^T = \{a_{i1} \ a_{i2} \ \dots \ a_{in}\}$$

- THE CONDITION FOR $\mathbf{A} \cdot \mathbf{w} = \mathbf{0}$ IS

$$\mathbf{a}_i^T \cdot \mathbf{w} = 0 \text{ for all } i$$

in other words, all n of the row vectors \mathbf{a}_i^T must be in the $(n - 1)$ -dimensional hyperplane through the origin perpendicular to the market weight vector \mathbf{w} .

THERE IS A SIMPLE-MINDED WAY TO UNDERSTAND!

- IF ALL Σ ARISE THIS WAY
- AND IF ALL OF THE OTHER CONDITIONS CAN BE MET
 - (PERHAPS TWO BIG IFs)
- THEN
 - The odds of an $(n - 1)$ -dimensional hyperplane through the origin being perpendicular to a vector \mathbf{w} with all $w_i > 0$ is $\left(\frac{1}{2}\right)^{n-1}$
 - Half the lines through the origin in 2-space are perpendicular to something in first quadrant.
 - A quarter of the planes through the origin in 3-space are perpendicular to something in the first octant.
 - And so on ...
 - The lazy pedagogue had at best 1 chance in 4 to write down a valid Σ
 - even after he respected all of the σ constraints
 - It's immediately clear that the odds to randomly encounter a valid Σ disappear at least exponentially in n , independent of (reasonable) distribution assumptions

SIGMA CONDITIONS ON A EASY TO UNDERSTAND

- $\sigma_{ij} = \sigma_{ji}$ means that

$$\begin{aligned}(r_i - r_f) + a_{ij} &= (r_j - r_f) + a_{ji} \\ \text{so } a_{ij} &= a_{ji} - (r_i - r_j)\end{aligned}$$

and \mathbf{A} can be expressed also as consisting of n column vectors

$$\mathbf{A} = \{ \mathbf{c}_1 \quad \mathbf{c}_2 \quad \dots \quad \mathbf{c}_n \} \text{ where } \mathbf{c}_j = \mathbf{a}_j - (\mathbf{r} - r_j \mathbf{e}).$$

- $\sigma_{ii} > 0$ means that

$$\begin{aligned}(r_i - r_f) + a_{ii} &> 0 \\ \text{so } a_{ii} &> -(r_i - r_f).\end{aligned}$$

- $\sigma_{ii}\sigma_{jj} > \sigma_{ij}^2$ means that

$$\begin{aligned}(a_{ii} + (r_i - r_f))(a_{jj} + (r_j - r_f)) &> (a_{ij} + (r_i - r_f))^2 \\ \text{so } a_{jj} &> -(r_j - r_f) + \frac{(a_{ij} + (r_i - r_f))^2}{a_{ii} + (r_i - r_f)}.\end{aligned}$$

- Looks like an induction might be possible.

INVERTIBILITY CONDITIONS

$$\Sigma = \mathbf{R} + \mathbf{A} = \begin{Bmatrix} (r_1 - r_f) \mathbf{e}^T + \mathbf{a}_1^T \\ (r_2 - r_f) \mathbf{e}^T + \mathbf{a}_2^T \\ \dots \\ (r_n - r_f) \mathbf{e}^T + \mathbf{a}_n^T \end{Bmatrix}$$

IS INVERTIBLE IFF ITS n ROW VECTORS ARE INDEPENDENT

- INDEPENDENCE HOLDS IF AND ONLY IF

- 1 The n row vectors $\mathbf{a}_1^T, \mathbf{a}_2^T, \dots, \mathbf{a}_n^T$ span the $(n-1)$ -dimensional hyperplane perpendicular to the market weight vector \mathbf{w} , and
- 2 $\mathbf{w}^T \cdot (\mathbf{r} - r_f \mathbf{e}) \neq 0$, i.e. the market risk premium vector $(\mathbf{r} - r_f \mathbf{e})$ is not perpendicular to the market weight vector \mathbf{w} .

- PROOF

- 1 $(r_1 - r_f) \mathbf{e}^T, \dots, (r_n - r_f) \mathbf{e}^T$ are collinear so the row vectors of $\mathbf{R} + \mathbf{A}$ cannot span an n -space unless $\mathbf{a}_1^T, \dots, \mathbf{a}_n^T$ span an $(n-1)$ -space, and the $(n-1)$ -dimensional hyperplane perpendicular to the market weight vector \mathbf{w} is the $(n-1)$ -space they are in.

INVERTIBILITY CONDITIONS

- PROOF (continued)

- ① (prior slide)

- ② Requires a slightly fussy proof that essentially follows from the fact that \mathbf{e}^T and \mathbf{w} both have all components positive and $\mathbf{a}_1^T, \dots, \mathbf{a}_n^T$ are all perpendicular to \mathbf{w} .

- CONDITION 2 HOLDS IN ANY REASONABLE MODEL

$$\mathbf{w}^T \cdot (\mathbf{r} - r_f \mathbf{e}) = \left(\mathbf{w}^T \cdot \mathbf{r} - r_f \right) = (r_M - r_f) > 0$$

where $r_M = \mathbf{w}^T \cdot \mathbf{r}$ is the expected return on the risky market.

- SO $\Sigma = \mathbf{R} + \mathbf{A}$ IS INVERTIBLE IN A REASONABLE MODEL IF AND ONLY IF

- ① The n row vectors $\mathbf{a}_1^T, \mathbf{a}_2^T, \dots, \mathbf{a}_n^T$ span the $(n - 1)$ -dimensional hyperplane perpendicular to the market weight vector \mathbf{w}

HOW HARD IS IT TO CHOOSE A?

- WORK BACKWARDS TO CHOOSE $\mathbf{a}_1^T, \mathbf{a}_2^T, \dots, \mathbf{a}_n^T$
 - Suppose you already have chosen $\mathbf{a}_1^T, \mathbf{a}_2^T, \dots, \mathbf{a}_{n-1}^T$.
 - Then there is no choice about what \mathbf{a}_n^T must be. By symmetry:

$$a_{n1} = a_{1n} - (r_n - r_1)$$

$$a_{n2} = a_{2n} - (r_n - r_2)$$

...

$$a_{n,n-1} = a_{n-1,n} - (r_n - r_{n-1}).$$

Since $\mathbf{a}_n^T \cdot \mathbf{w} = 0$ the choice for a_{nn} also is fixed. The requirement is

$$w_1 a_{n1} + \dots + w_{n-1} a_{n,n-1} + w_n a_{nn} = 0$$

$$\text{so } a_{nn} = -\frac{1}{w_n} (w_1 a_{n1} + \dots + w_{n-1} a_{n,n-1}),$$

which finishes the complete determination of \mathbf{a}_n^T .

HOW HARD IS IT TO CHOOSE A?

- WORK BACKWARDS TO CHOOSE $\mathbf{a}_1^T, \mathbf{a}_2^T, \dots, \mathbf{a}_n^T$
 - But a_{nn} has to satisfy some σ conditions:

$$a_{nn} > -(r_n - r_f) \text{ and for all } i < n$$

$$a_{nn} > -(r_n - r_f) + \frac{(a_{in} + (r_i - r_f))^2}{a_{ii} + (r_i - r_f)}$$

- That means the choice of \mathbf{a}_{n-1}^T couldn't have been completely free

$$-\frac{1}{w_n} (w_1 a_{n1} + \dots + w_{n-1} a_{n,n-1}) > -(r_n - r_f) \text{ and for all } i < n$$

$$-\frac{1}{w_n} (w_1 a_{n1} + \dots + w_{n-1} a_{n,n-1}) > -(r_n - r_f) + \frac{(a_{in} + (r_i - r_f))^2}{a_{ii} + (r_i - r_f)}$$

$$\text{where } a_{n1} = a_{1n} - (r_n - r_1)$$

...

$$a_{n,n-1} = a_{n-1,n} - (r_n - r_{n-1}).$$

- Since all $w_i > 0$, just pick a small enough $a_{n-1,n}$ (negative if need be)

HOW HARD IS IT TO CHOOSE A?

- WORK BACKWARDS TO CHOOSE $\mathbf{a}_1^T, \mathbf{a}_2^T, \dots, \mathbf{a}_n^T$

- One possible problem at $i = n - 1$:

$$-\frac{1}{w_n} (w_1 a_{n-1} + \dots + w_{n-1} (a_{n-1} - (r_n - r_{n-1}))) >$$
$$> -(r_n - r_f) + \frac{(a_{n-1} + (r_{n-1} - r_f))^2}{a_{n-1} + (r_{n-1} - r_f)}$$

so a_{n-1} on both sides, and squared (so made positive) on the right. Does picking a_{n-1} small enough (negative if need be) still work?

- YES! \mathbf{a}_{n-1}^T is perpendicular to \mathbf{w} with all $w_j > 0$ so picking a small enough a_{n-1} (negative if need be) forces a_{n-1} to increase and it turns out (some delicate analysis) to be enough to make the inequality work.

$$\left\{ \begin{array}{ccc} \dots & \dots & \dots \\ \dots & \uparrow & \downarrow \\ \dots & \downarrow & \uparrow \end{array} \right\}$$

HOW HARD IS IT TO CHOOSE A?

- THE HARD PART IS DONE

- Given $\mathbf{a}_1^T, \mathbf{a}_2^T, \dots, \mathbf{a}_{n-2}^T$ we saw that we can choose $\mathbf{a}_{n-1}^T, \mathbf{a}_n^T$ that satisfy the σ conditions.

- GO BY INDUCTION STARTING AT \mathbf{a}_1^T

- For $1 \leq i \leq n-2$, given $\mathbf{a}_1^T, \mathbf{a}_2^T, \dots, \mathbf{a}_{i-1}^T$ always free to choose a_{ij} big enough to satisfy the σ conditions, then a_{i+1}, \dots, a_n any values that keep $\mathbf{a}_i^T \cdot \mathbf{w} = 0$

- SOME SPAN RESTRICTIONS ON CHOICES OF \mathbf{a}_i^T

- To ensure that $\mathbf{a}_1^T, \mathbf{a}_2^T, \dots, \mathbf{a}_i^T$ spans at least an $(i-1)$ -dimensional subspace of the $(n-1)$ -dimensional hyperplane perpendicular to \mathbf{w} we have to disallow some choices in the induction.
- The whole set of disallowed matrices \mathbf{A} altogether has dimension $\leq \frac{n(n-1)}{2} - 3$.
- The whole set of allowed matrices \mathbf{A} has dimension $\frac{n(n-1)}{2}$, still for a fixed choice of \mathbf{w} and $(\mathbf{r} - r_f \mathbf{e})$ and still requiring

$$\Sigma \cdot \mathbf{w} = (\mathbf{r} - r_f \mathbf{e})$$

BACK TO THE LAZY PEDAGOGUE

- A FEW MORE DEGREES OF FREEDOM

- We can multiply any $\Sigma = \mathbf{R} + \mathbf{A}$ developed above by any constant k .
- We can choose w_1, \dots, w_n subject to $\mathbf{e}^T \cdot \mathbf{w} = 1$ and $w_i > 0$
- So altogether the space of possible solutions Σ has dimension $\frac{n(n-1)}{2} + 1 + n - 1 = \frac{(n+1)n}{2}$, now with a possible disallowed set of dimension $\leq \frac{(n+1)n}{2} - 3$.

- $A \cdot w = 0$ IS THE MAIN NON-OBVIOUS RESTRICTION

- The σ restrictions are all visible in Σ and true for any empirical Σ
- Invertibility of Σ is apparent or easy to check, at least for small n .
- The space of possible $n \times n$ symmetric matrices has dimension $n + (n-1) + \dots + 1 = \frac{(n+1)n}{2}$, same as the space of solutions
- The disallowed set of matrices has measure (probability) 0.
- But the odds of an $(n-1)$ -dimensional hyperplane through the origin being perpendicular to a vector \mathbf{w} with all $w_i > 0$ is $\left(\frac{1}{2}\right)^{n-1}$
- So solutions seemingly are rare only because of that factor $\left(\frac{1}{2}\right)^{n-1}$

ONE MORE SOURCE OF MEANINGFUL CONSTRAINT

• REMEMBER THAT RETURNS ARE CONSTRAINED

- In meaningful models

$$\mathbf{w}^T \cdot (\mathbf{r} - r_f \mathbf{e}) = \left(\mathbf{w}^T \cdot \mathbf{r} - r_f \right) = (r_M - r_f) > 0$$

- If there are any risky assets with expected return $r_i < r_f$ then this inequality cuts off a fraction (call it f) of the otherwise possible \mathbf{w} .
- Now the only possible \mathbf{w} are in the intersection of the set having all $w_i > 0$ with the half-space having $\mathbf{w}^T \cdot (\mathbf{r} - r_f \mathbf{e}) > 0$.
- This in turn eliminates the same fraction f of the set of otherwise possible matrices \mathbf{A} , whose row vectors have to live on the plane perpendicular to \mathbf{w} , so possible Σ are now rare by a factor $f \left(\frac{1}{2} \right)^{n-1}$.
- This further militates against the lazy pedagogue's chance of success, since he likes to illustrate negative covariances and negative risk premia.
- It impairs by the same factor the probability of a random empirical Σ to satisfy CAPM, even if it satisfies all the σ constraints and invertibility.
- If I have understood Phelim Boyle's work, the "almost positive" condition does not yet contemplate this possibility. How should it be modified/generalized to this case?

ALMOST FORGOT - What Is That Matrix A Anyway?

Claim: $\mathbf{A} = \mathbf{Cov} \left(\bar{\mathbf{r}}, \left(\bar{\mathbf{r}}^T - \bar{r}_M \mathbf{e}^T \right) \right)$ where bar means random,

$\bar{\mathbf{r}}$ is the random vector of actual asset returns in the market and \bar{r}_M is the random actual return on the market as a whole, i.e.

$$\bar{r}_M = \bar{\mathbf{r}}^T \cdot \mathbf{w}$$

Proof: $\mathbf{Cov} \left(\bar{\mathbf{r}}, \bar{\mathbf{r}}^T \right) = \Sigma$ by definition.

$$\begin{aligned} \mathbf{Cov} \left(\bar{\mathbf{r}}, \bar{r}_M \mathbf{e}^T \right) &= \left\{ \mathbf{Cov} \left(\bar{\mathbf{r}}, \bar{r}_M \right) \mathbf{e}^T \right\}, \text{ with } n \text{ identical columns} \\ &= \left\{ \mathbf{Cov} \left(\bar{\mathbf{r}}, \left(\bar{\mathbf{r}}^T \cdot \mathbf{w} \right) \right) \mathbf{e}^T \right\} \\ &= \left\{ \left(\mathbf{Cov} \left(\bar{\mathbf{r}}, \bar{\mathbf{r}}^T \right) \cdot \mathbf{w} \right) \mathbf{e}^T \right\} \\ &= \left\{ \left(\Sigma \cdot \mathbf{w} \right) \mathbf{e}^T \right\} = \left\{ \left(\mathbf{r} - r_f \mathbf{e} \right) \mathbf{e}^T \right\} = \mathbf{R} \text{ so} \end{aligned}$$

$$\mathbf{Cov} \left(\bar{\mathbf{r}}, \left(\bar{\mathbf{r}}^T - \bar{r}_M \mathbf{e}^T \right) \right) = \Sigma - \mathbf{R} = \mathbf{A}$$

THANKS

PHELIM BOYLE

MY STUDENTS THIS SUMMER