Combinatorics for Moments of a Randomly Stopped Quadratic Variation Process

In Particular forJump Processes

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Some High School Algebra

Suppose x_1 and x_2 are random variables and we want to calculate $\mathbb{E}\left[(x_1+x_2)^2\right]$. One way to proceed might be

$$\begin{aligned}\mathbb{E}\left[(x_1+x_2)^2\right] &= \mathbb{E}\left[\left(x_1^2+x_2^2\right)\right]+2\mathbb{E}\left[x_1x_2\right]\\ &= \left(\mathbb{E}\left[x_1^2\right]+\mathbb{E}\left[x_2^2\right]\right)+2\rho\left(\mathbb{E}\left[x_1\right]\mathbb{E}\left[x_2\right]\right)\end{aligned}$$

where ρ, defined as satisfying $\mathbb{E}\left[x_1x_2\right]=\rho\mathbb{E}\left[x_1\right]\mathbb{E}\left[x_2\right]$, can be called a covariation coefficient and

$$\mathbb{E}\left[(x_1+x_2)^2\right]=\left(\mathbb{E}\left[x_1^2\right]+\mathbb{E}\left[x_2^2\right]\right)+2\rho\left\{\frac{\left(\mathbb{E}\left[x_1\right]+\mathbb{E}\left[x_2\right]\right)^2}{2}-\frac{\left(\mathbb{E}\left[x_1\right]^2+\mathbb{E}\left[x_2\right]^2\right)}{2}\right\}$$

The purpose of this paper is to state and prove Theorem 1 below which generalizes this simple example to an arbitrary (possibly random) number of terms $x_1+x_2+...+x_J$ and beyond 2 to an arbitrary moment $\mathbb{E}\left[(x_1+x_2+...)^n\right]$.

Theorem 1 *If either* 1 *or* 2*:*

1. $x_j \geq 0$ almost always for all j, or
2. $$\mathbb{E}\left[\sum^{II}\left|x_{j_{1,1}}\cdots x_{j_{1,i_1}}x^2_{j_{2,1}}\cdots x^2_{j_{2,i_2}}\cdots x^l_{j_{l,1}}\cdots x^l_{j_{l,i_l}}\cdots\right|\right] < \infty,$$

   for all sets of indexed non-negative integers $\left\{i_l : \sum_l l \cdot i_l = n\right\}$ where, for each such $\{i_l\}$, $\sum^{II}$ is taken over all indexed sets of permutations of sets of non-negative integers $\left\{\{j_{l,i} : 1 \leq i \leq i_l\}_l\right\}$ in which no two integers $j_{l,i}$, $j_{l',i'}$ are equal,

   and if all covariation coefficients of all orders among the $\{x_j\}$ are global, not depending upon the specific subscripts j and j' for any two distinct x_j and $x_{j'}$, as specified in the statement of Lemma 7 below then

$$\mathbb{E}\left[\left(\sum_j x_j\right)^n\right] =$$

$$= {\sum}^{I} \frac{n!}{\prod_l l!^{i_l}} \rho_{\{i_l\}} {\sum}^{IV} \prod_m \frac{1}{j_m!} \left[(-1)^{\sum_l i_{l,m}-1} \frac{\left(\sum_l i_{l,m} - 1\right)!}{\prod_l i_{l,m}!} \sum_j \left(\prod_l \mathbb{E}\left[x_j^l\right]^{i_{l,m}} \right) \right]^{j_m}$$

where ${\sum}^{I}$ is taken over all sets of indexed non-negative integers $\left\{ i_l : \sum_l l \cdot i_l = n \right\}$,

for each such $\{i_l\}$ the covariation coefficient $\rho_{\{i_l\}}$ is as defined in Lemma 7 below

and for each such $\{i_l\}$ the ${\sum}^{IV}$ is taken over all sets of indexed non-negative

integers $\left\{ j_m, i_{l,m} : \sum_m j_m \cdot i_{l,m} = i_l \text{ for all } l \right\}$.

For $n = 2$

$$\begin{array}{lcc} \mathbf{l} = & \mathbf{2} & \mathbf{1} \\ i_l = & 1 & 0 \\ & 0 & 2 \end{array} \qquad i_{l,m} = \begin{array}{cccc} \mathbf{2} & \mathbf{1} & \mathbf{j}_m & m \\ \left[\ 1\right. & 0 & 1 & \left.1\ \right] \\ \left[\ 0\right. & 2 & 1 & \left.1\ \right] \\ \left[\ 0\right. & 1 & 2 & \left.1\ \right] \end{array}$$

$$\mathbb{E}\left[\left(\sum_j x_j\right)^2\right] = \left(\sum_j \mathbb{E}\left[x_j^2\right]\right) + 2\rho_{\{0,2\}}\left\{\left(-\frac{1}{2}\sum_j \mathbb{E}\left[x_j\right]^2\right) + \frac{1}{2}\left(\sum_j \mathbb{E}\left[x_j\right]\right)^2\right\}$$

How To Calculate

For $n = 3$

$$
\begin{array}{cccc}
\mathbf{l} = & \mathbf{3} & \mathbf{2} & \mathbf{1} \\
i_l = & 1 & 0 & 0 \\
 & 0 & 1 & 1 \\
 & & & \\
 & & & \\
 & 0 & 0 & 3 \\
\end{array}
\qquad
i_{l,m} =
\begin{array}{ccccc}
\mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{j}_m & m \\
\left[\, 1 \right. & 0 & 0 & 1 & \left. 1 \,\right] \\
\left[\, 0 \right. & 1 & 1 & 1 & \left. 1 \,\right] \\
\lceil\, 0 & 1 & 0 & 1 & 1 \,\rceil \\
\lfloor\, 0 & 0 & 1 & 1 & 2 \,\rfloor \\
\left[\, 0 \right. & 0 & 3 & 1 & \left. 1 \,\right] \\
\lceil\, 0 & 0 & 2 & 1 & 1 \,\rceil \\
\lfloor\, 0 & 0 & 1 & 1 & 2 \,\rfloor \\
\left[\, 0 \right. & 0 & 1 & 3 & \left. 1 \,\right] \\
\end{array}
$$

How To Calculate

$$\mathbb{E}\left[\left(\sum_j x_j\right)^3\right] =$$

$$\begin{aligned}
= & \left(\sum_j \mathbb{E}\left[x_j^3\right]\right) \\
& +3\rho_{\{0,1,1\}}\left[\left(-\sum_j \mathbb{E}\left[x_j^2\right]\mathbb{E}\left[x_j\right]\right) + \left(\sum_j \mathbb{E}\left[x_j^2\right]\right)\left(\sum_j \mathbb{E}\left[x_j\right]\right)\right] \\
& +6\rho_{\{0,0,3\}}\left[\begin{array}{c} \left(\frac{1}{3}\sum_j \mathbb{E}\left[x_j\right]^3\right) + \left(-\frac{1}{2}\sum_j \mathbb{E}\left[x_j\right]^2\right)\left(\sum_j \mathbb{E}\left[x_j\right]\right) \\ +\frac{1}{6}\left(\sum_j \mathbb{E}\left[x_j\right]\right)^3 \end{array}\right]
\end{aligned}$$

What Do the Covariation Coefficients Mean?

$$\begin{aligned} \mathbb{E}\left[x_j^2 x_k\right] &= \rho_{\{0,1,1\}} \mathbb{E}\left[x_j^2\right] \mathbb{E}\left[x_k\right] \text{ for all } j \neq k \\ \mathbb{E}\left[x_i x_j x_k\right] &= \rho_{\{0.0.3\}} \mathbb{E}\left[x_i\right] \mathbb{E}\left[x_j\right] \mathbb{E}\left[x_k\right] \text{ for all i} \neq j \neq k \end{aligned}$$

- This is what was meant by the hypothesis that "covariation coefficients of all orders are global"
- It is satisfied in many examples of interest, including of course independence among the x_j but also situations when any covariance among them is a function solely of covariance between the x_j and a random stopping time, which was the case in our jump process problem that gave rise to this question.

One More Calculation

Example 20 *For* $n = 4$ *the combinatorics jump to fourteen groupi*

and fourteen terms in the expression for $\mathbb{E}\left[\left(\sum_j x_j\right)^4\right]$.

$\mathbf{l} =$	**4**	**3**	**2**	**1**		**4**	**3**	**2**	**1**	$\mathbf{j}_m$	**m**
$i_l =$	1	0	0	0	$i_{l,m} =$	1	0	0	0	1	1
	0	1	0	1		0	1	0	1	1	1
						0	1	0	0	1	1
						0	0	0	1	1	2
	0	0	2	0		0	0	2	0	1	1
						0	0	1	0	2	1
	0	0	1	2		0	0	1	2	1	1
						0	0	1	1	1	1
						0	0	0	1	1	2
						0	0	1	0	1	1
						0	0	0	2	1	2
						0	0	1	0	1	1
						0	0	0	1	2	2
	0	0	0	4		0	0	0	4	1	1
						0	0	0	3	1	1
						0	0	0	1	1	2
						0	0	0	2	2	1
						0	0	0	2	1	1
						0	0	0	1	2	2
						0	0	0	1	4	1

One More Calculation

$$\mathbb{E}\left[\left(\sum_j x_j\right)^4\right] =$$

$$\begin{aligned}
= & \left(\sum_j \mathbb{E}\left[x_j^4\right]\right) \\
& +4\rho_{\{0,1,0,1\}}\left[\left(-\sum_j \mathbb{E}\left[x_j^3\right]\mathbb{E}\left[x_j\right]\right)+\left(\sum_j \mathbb{E}\left[x_j^3\right]\right)\left(\sum_j \mathbb{E}\left[x_j\right]\right)\right] \\
& +6\rho_{\{0,0,2,0\}}\left[\left(-\frac{1}{2}\sum_j \mathbb{E}\left[x_j^2\right]^2\right)+\frac{1}{2}\left(\sum_j \mathbb{E}\left[x_j^2\right]\right)^2\right] \\
& +12\rho_{\{0,0,1,2\}}\left[\begin{array}{c}\left(\sum_j \mathbb{E}\left[x_j^2\right]\mathbb{E}\left[x_j\right]^2\right)+\left(-\sum_j \mathbb{E}\left[x_j^2\right]\mathbb{E}\left[x_j\right]\right)\left(\sum_j \mathbb{E}\left[x_j\right]\right) \\ +\left(\sum_j \mathbb{E}\left[x_j^2\right]\right)\left(-\frac{1}{2}\sum_j \mathbb{E}\left[x_j\right]^2\right)+\left(\sum_j \mathbb{E}\left[x_j^2\right]\right)\frac{1}{2}\left(\sum_j \mathbb{E}\left[x_j\right]\right)^2\end{array}\right] \\
& +24\rho_{\{0,0,0,4\}}\left[\begin{array}{c}\left(-\frac{1}{4}\sum_j \mathbb{E}\left[x_j\right]^4\right)+\left(\frac{1}{3}\sum_j \mathbb{E}\left[x_j\right]^3\right)\left(\sum_j \mathbb{E}\left[x_j\right]\right) \\ +\frac{1}{2}\left(-\frac{1}{2}\sum_j \mathbb{E}\left[x_j\right]^2\right)^2+\left(-\frac{1}{2}\sum_j \mathbb{E}\left[x_j\right]^2\right)\frac{1}{2}\left(\sum_j \mathbb{E}\left[x_j\right]\right)^2 \\ +\frac{1}{24}\left(\sum_j \mathbb{E}\left[x_j\right]\right)^4\end{array}\right]
\end{aligned}$$

Theorem 1 Conclusion

then

$$\mathbb{E}\left[\left(\sum_j x_j\right)^n\right] =$$

$$= \sum^{I} \frac{n!}{\prod_l l!^{i_l}} \rho_{\{i_l\}} \sum^{IV} \prod_m \frac{1}{j_m!} \left[(-1)^{\sum_l i_{l,m}-1} \frac{\left(\sum_l i_{l,m} - 1\right)!}{\prod_l i_{l,m}!} \sum_j \left(\prod_l \mathbb{E}\left[x_j^l\right]^{i_{l,m}} \right) \right]^{j_m}$$

where $\sum^{I}$ is taken over all sets of indexed non-negative integers $\left\{ i_l : \sum_l l \cdot i_l = n \right\}$, for each such $\{i_l\}$ the covariation coefficient $\rho_{\{i_l\}}$ is as defined in Lemma 7 below and for each such $\{i_l\}$ the $\sum^{IV}$ is taken over all sets of indexed non-negative integers $\left\{ j_m, i_{l,m} : \sum_m j_m \cdot i_{l,m} = i_l \text{ for all } l \right\}$.

How Would You See And Prove Such A Thing?

- First, expand the n-th power multinomially, use Fubini to interchange the order of $\mathbb{E}$ and summation, and use the global covariation assumption to get

$$\mathbb{E}\left[\left(\sum_j x_j\right)^n\right] =$$

$$= \sum^{I} \frac{n!}{\prod_l i_l! l!^{i_l}} \rho_{\{i_l\}} \sum^{II} \mathbb{E}\left[x_{j_{1,1}}\right] \cdots \mathbb{E}\left[x_{j_{1,i_1}}\right] \mathbb{E}\left[x^2_{j_{2,1}}\right] \cdots \mathbb{E}\left[x^2_{j_{2,i_2}}\right] \cdots \mathbb{E}\left[x^l_{j_{l,1}}\right] \cdots \mathbb{E}\left[x^l_{j_{l,i_l}}\right] \cdots$$

where in each term of the sum no two subscripts match and all the powers add up to n; a separate term is included in $\sum^{II}$ for each permutation of the subscripts in each group of powers of l; and the multinomial coefficient $\frac{n!}{\prod_l l!^{i_l}}$ needs to be divided further by $\prod_l i_l!$ because that is the number of permutations of the subscripts for which we keep separate terms in the sum.

How Would You See And Prove Such A Thing?

- Next, notice that all of those monomials without matching subscripts also appear, each with coefficient 1, in the expansion of

$$\prod_l \left(\sum_j \mathbb{E}\left[x_j^l\right] \right)^{i_l}$$

- But that expansion also contains monomials with matching subscripts. Each such monomial with matching subscripts also occurs, with coefficient 1, in the expansion of exactly one of the following expressions

$$\prod_m \left[\sum_j \left(\prod_l \mathbb{E}\left[x_j^l\right]^{i_{l,m}} \right) \right]^{j_m}$$

where the $i_{l,m}$ exponents force the matching of coefficients.

Start Counting

- The monomials with matching subscripts that occur with coefficient 1 in

$$\prod_m \left[\sum_j \left(\prod_l \mathbb{E}\left[x_j^l\right]^{i_{l,m}} \right) \right]^{j_m}$$ each occur

$$\frac{1}{\prod_m j_m!} \prod_l \frac{i_l!}{\prod_m i_{l,m}!^{j_m}}$$ times in

$$\prod_l \left(\sum_j \mathbb{E}\left[x_j^l\right] \right)^{i_l}$$

where the $\prod_l$ of multinomial coefficients gets divided by $\prod_m j_m!$ because each permutation of the subscripts is represented separately in the sum.

So Just Substract, Right?

- It seems that we can eliminate monomials with matching subscripts by subtracting

$$\left(\frac{1}{\prod_m j_m!} \prod_l \frac{i_l!}{\prod_m i_{l,m}!^{j_m}} \right) \prod_m \left[\sum_j \left(\prod_l \mathbb{E}\left[x_j^l\right]^{i_{l,m}} \right) \right]^{j_m} \text{ from } \prod_l \left(\sum_j \mathbb{E}\left[x_j^l\right] \right)^{i_l}$$

- Unfortunately, when we subtract to get rid of monomials with k matching subscripts, we also "by accident" subtract a lot of monomials with $k' > k$ matching subscripts. That means that when we get to the step of eliminating monomials with k' matching subscripts we have to add back all the monomials we substracted "by accident" at all stages of eliminating monomials with $k < k'$ matching subscripts. And, of course, this adding and subtracting compounds itself up and down the line.
- What to do?

New Notation

- Let the set of non-negative integers $\{f_k\}$ indexed by $k \geq 2$ stand for a monomial with exactly f_k groups of k matching subscripts for each $k \geq 2$.
- The idea is to assign a coefficient to each such $\{f_k\}$ that will take care of the entire adding and subtracting up and down the line in such fashion that the monomials with matching subscripts are completely eliminated with no "by accident" leftovers.
- An excruciating, error-ridden excursion into hand calculations seemed to suggest that the coefficient should be

$$\prod_k \left[(-1)^{(k-1)} (k-1)!\right]^{f_k}$$

- An intricate proof by induction on $\sum_k f_k (k-1)$ verified that this was the correct coefficient.

That Finishes It

- For each factor m and each exponent l in
$$\prod_m \left[\sum_j \left(\prod_l \mathbb{E}\left[x_j^l\right]^{i_{l,m}} \right) \right]^{j_m}$$
$i_{l,m}$ is the number of matching subscripts in each "no accidents" group of matching subscripts of x_j^l in a single monomial. There are $\sum_l i_{l,m}$ such "no accidents" matched subscripts across all l in a single monomial.
- That makes the correct coefficient to eliminate matching subscripts in each factor m to be $(-1)^{\sum_l i_{l,m}-1} \left(\sum_l i_{l,m} - 1 \right)!$
- That's our theorem, noting that we have a copy of $\prod_l i_l!$ in the numerators of the inside multinomial factors that cancels out the copy that was divided out of the outside multimomial factors.

then

$$\mathbb{E}\left[\left(\sum_j x_j\right)^n\right] =$$

$$= \sum\nolimits^{I} \frac{n!}{\prod_l l!^{i_l}} \rho_{\{i_l\}} \sum\nolimits^{IV} \prod_m \frac{1}{j_m!} \left[(-1)^{\sum_l i_{l,m}-1} \frac{\left(\sum_l i_{l,m} - 1\right)!}{\prod_l i_{l,m}!} \sum_j \left(\prod_l \mathbb{E}\left[x_j^l\right]^{i_{l,m}} \right) \right]^{j_m}$$

where $\sum^{I}$ is taken over all sets of indexed non-negative integers $\left\{ i_l : \sum_l l \cdot i_l = n \right\}$, for each such $\{i_l\}$ the covariation coefficient $\rho_{\{i_l\}}$ is as defined in Lemma 7 below and for each such $\{i_l\}$ the $\sum^{IV}$ is taken over all sets of indexed non-negative integers $\left\{ j_m, i_{l,m} : \sum_m j_m \cdot i_{l,m} = i_l \text{ for all } l \right\}$.