Illustrations of a Regime-Switching Stochastic Interest Rate Model With Randomized Regimes

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Introduction

- Work in progress

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- Two years ago:

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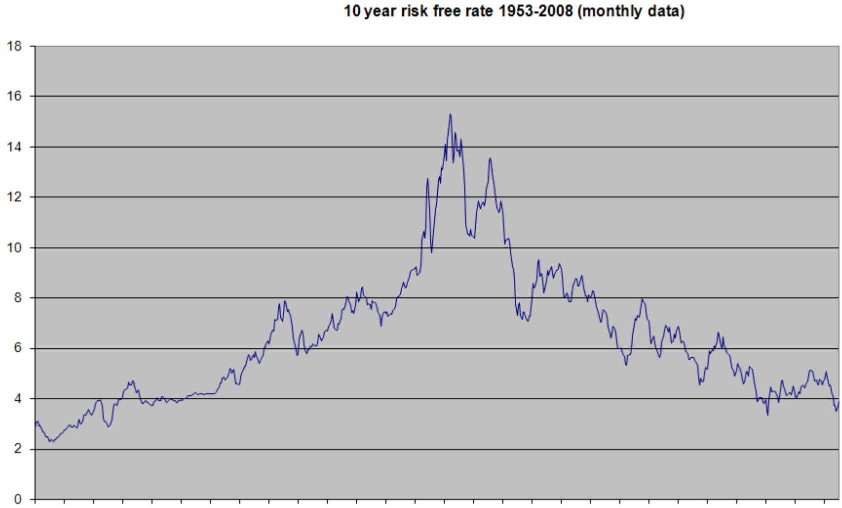
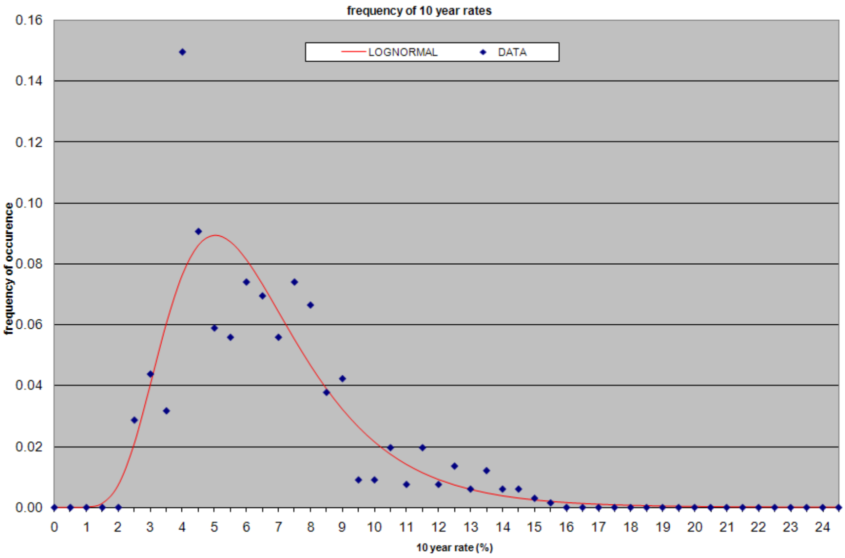
  - Numerical examples

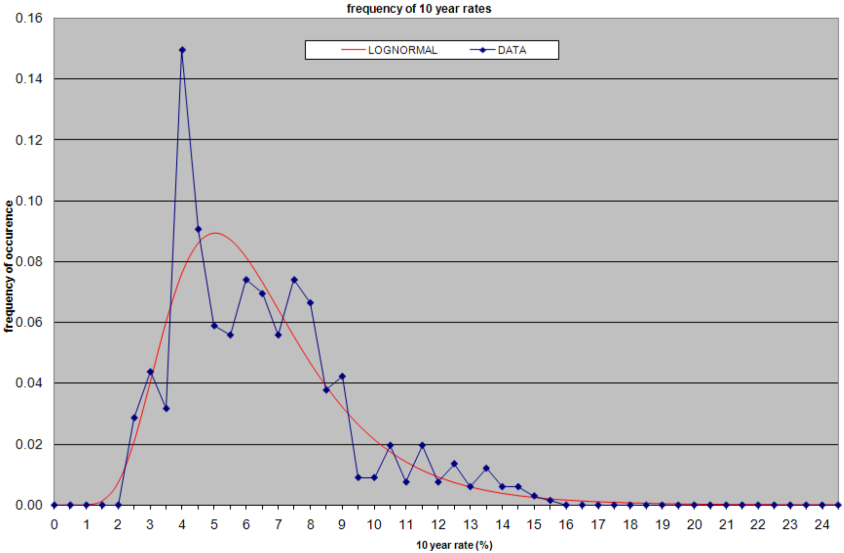
Example: 56 Years of the 10-year Treasury Rate



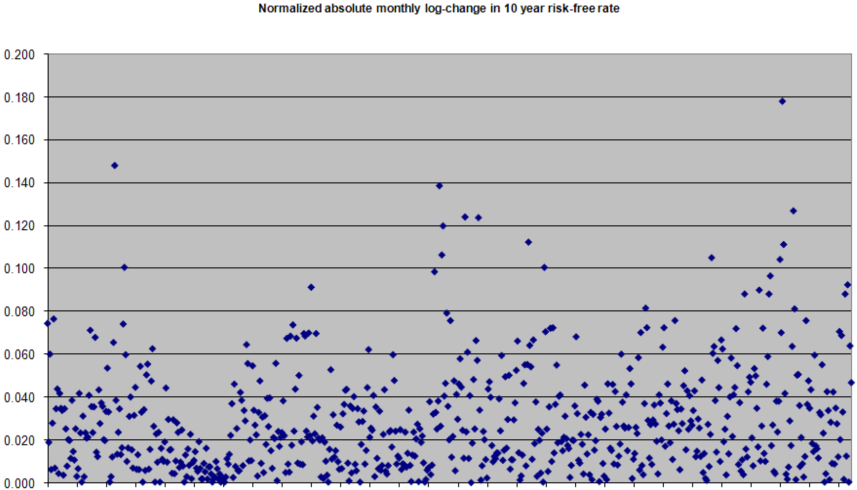
The Distribution of those Interest Rates



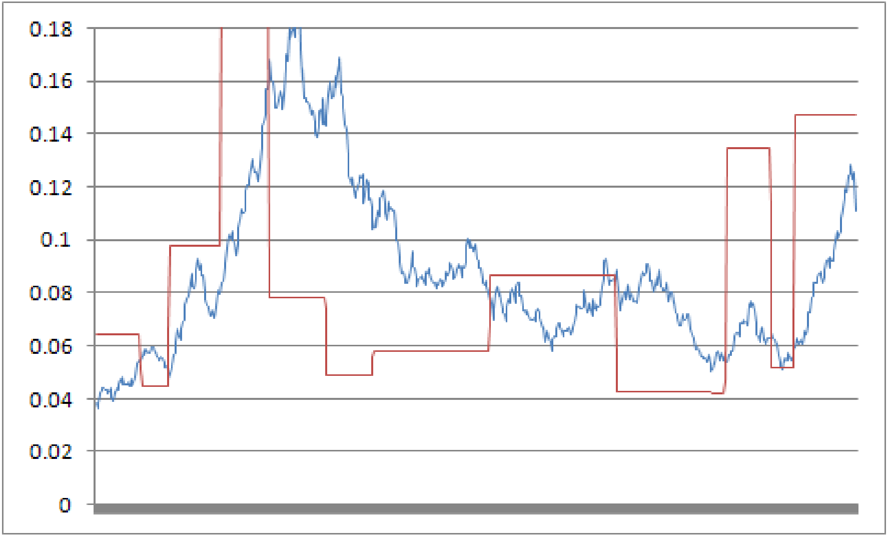
Lognormal 4th Moment Is Just Too High (6th too)



56 Years of Changes in the 10 Year Treasury Rate



The Fix: Randomize the Reversion Target



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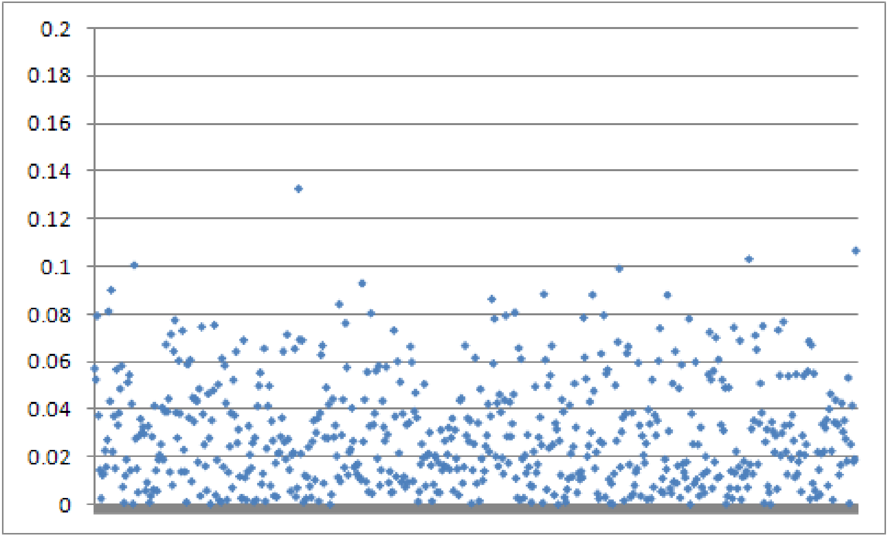

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  - **(**T_0 **is a fixed target value for the first regime.)**
  - $\mathbf{T}_t = \mathbf{T}_k^{\frac{1-(1-F)^{dt\wedge(t-\mathbf{t_k})}}{1-(1-F)^{dt}}} \prod_{j=k'}^{k-1} \mathbf{T}_j^{\frac{(1-F)^{dt\wedge(t-\mathbf{t_{j+1}})}-(1-F)^{dt\wedge(t-\mathbf{t_j})}}{1-(1-F)^{dt}}}$, where the product is over all $\mathbf{t_j}$ that fall in the interval $[t-dt, t)$ plus the immediate prior $\mathbf{t_j}$ (see corrections to last year's paper at www.math.uconn.edu/~bridgeman)

Drift Compensation: the ordinary mean-reversion case

- It would be intuitive to have:
  $\mathbb{E}\left[r_t\right] = r_0^{(1-F)^t} T_0^{\left[1-(1-F)^t\right]}$

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- It would be intuitive to have:
  $\mathbb{E}\left[r_t\right] = r_0^{(1-F)^t} T_0^{\left[1-(1-F)^t\right]}$
- To find out what drift D_t will ensure it, you can integrate $d\ln(r_t)$:

$$\ln(r_t) = \ln(r_0)\left(1-F\right)^{\frac{t}{dt}dt} + \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\left(1-F\right)^{sdt}$$

$$+\ln(T_0)\left[1-(1-F)^{dt}\right]\sum_{s=1}^{\frac{t}{dt}}\left(1-F\right)^{(s-1)dt} \Longleftarrow \text{notice geom. series}$$

$$+dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\left(1-F\right)^{sdt} \text{ which simplifies to:}$$

$$\ln(r_t) = \ln(r_0)\left(1-F\right)^{t} + \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\left(1-F\right)^{sdt}$$

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Gaussian.

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- Since $\ln(r_t)$ is Gaussian, $\mathbb{E}[r_t] = e^{\mu + \frac{1}{2}\sigma^2}$ where the μ and σ^2 are some mess determined by the constants in the expression for $\ln(r_t)$.

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- If you work that mess out and set it equal to $r_0^{(1-F)^t} T_0^{\left[1-(1-F)^t\right]}$, and require that it be true for all t, you can arrive at what the drift compensation function D_t must be to deliver the intuitive $\mathbb{E}\left[r_t\right]$:

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- There is a similar closed form for the variance of r_t
- Practical work with the randomized reversion target model all but requires you to know similar closed forms for drift compensation (can't do physical measure Monte Carlo without drift compensation), but now when you integrate, no convenient geometric series appears. Similarly for variance, which you need to calibrate a Monte Carlo.

Drift Compensation: with randomized reversion target

- In last year's paper I showed that

$$
\begin{aligned}
\ln(\mathbf{r}_t) \quad = \quad & \ln(r_0)\,(1-F)^t + \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\,(1-F)^{sdt} \\
& + \ln(T_0)\left[(1-F)^{(t-\mathbf{t}_1)_+} - (1-F)^t\right] \\
& + \sum_{j=1}^{\infty} \ln(\mathbf{T}_j)\left[(1-F)^{(t-\mathbf{t}_{j+1})_+} - (1-F)^{(t-\mathbf{t}_j)_+}\right] \\
& + dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\,(1-F)^{sdt} \qquad (1.3)
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&\quad + \ln(T_0)\left[(1-F)^{(t-\mathbf{t}_1)_+} - (1-F)^t\right] \\
&\quad + \sum_{j=1}^{\infty} \ln(\mathbf{T}_j)\left[(1-F)^{(t-\mathbf{t}_{j+1})_+} - (1-F)^{(t-\mathbf{t}_j)_+}\right] \\
&\quad + dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\,(1-F)^{sdt} \qquad (1.3)
\end{aligned}
$$

- If you condition on $\{\mathbf{t}_j\}$ this is just Gaussian with parameters some mess determined by the coefficients of the $\mathbf{N}_{t-(s-1)dt}$ and the $\ln(\mathbf{T}_j)$.

Drift Compensation: with randomized reversion target

- In last year's paper I showed that

$$\begin{aligned}\ln(\mathbf{r}_t) \quad = \quad & \ln(r_0)\,(1-F)^t + \sigma\sqrt{dt}\sum_{s=1}^{\frac{t}{dt}} \mathbf{N}_{t-(s-1)dt}\,(1-F)^{sdt} \\ & + \ln(T_0)\left[(1-F)^{(t-\mathbf{t}_1)_+} - (1-F)^t\right] \\ & + \sum_{j=1}^{\infty} \ln(\mathbf{T}_j)\left[(1-F)^{(t-\mathbf{t}_{j+1})_+} - (1-F)^{(t-\mathbf{t}_j)_+}\right] \\ & + dt\sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt}\,(1-F)^{sdt} \qquad (1.3)\end{aligned}$$

- If you condition on $\{\mathbf{t}_j\}$ this is just Gaussian with parameters some mess determined by the coefficients of the $\mathbf{N}_{t-(s-1)dt}$ and the $\ln(\mathbf{T}_j)$.
- So conditional on $\{\mathbf{t}_j\}$ you can calculate the moments of $\mathbf{r}_t$ using knowledge of the moments of a lognormal.

Drift Compensation: with randomized reversion target

- But expressions involving $\ln(\mathbf{T}_j)\left[(1-F)^{(t-\mathbf{t}_{j+1})_+}-(1-F)^{(t-\mathbf{t}_j)_+}\right]$ make it a nightmare to find the unconditioned moments of $\mathbf{r}_t$.

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- So we derived an approximation series based on the old Edgeworth approximation:

$$\mathbb{E}\left[(\mathbf{r}_t)^l\right] =$$

$$= e^{l\mu+\frac{1}{2}(l\sigma)^2}\left\{1+\lim_{N\to\infty}\sum_{j=2}^{N}\frac{l^{2j}}{(2j)!}\left[\mu_{2j}-(2j)?\sigma^{2j}\right]\cdot\right.$$
$$\left.\cdot\sum_{n=0}^{N-j}\frac{(-1)^n(2n)?}{(2n)!}(l\sigma)^{2n}\right\}$$

where $(2n)? = (2n-1)(2n-3)\cdots 1$ and μ, σ^2, and μ_{2j} are the mean, variance and higher central moments of $\ln(\mathbf{r}_t)$.

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  where $(2n)?=(2n-1)(2n-3)\cdots 1$ and μ, σ^2, and μ_{2j} are the mean, variance and higher central moments of $\ln(\mathbf{r}_t)$.
- The first approximation (beyond the lognormal) is $\mathbb{E}\left[(\mathbf{r}_t)^l\right]\approx e^{l\mu+\frac{1}{2}(l\sigma)^2}\left\{1+\frac{l^4}{4!}\left[\mu_4-3\sigma^4\right]\right\}$

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$$\left.+\frac{3}{4!}(l\sigma)^4\right)+\frac{l^6}{6!}\left[\mu_6-15\sigma^6\right]\left(1-\frac{1}{2!}(l\sigma)^2\right)$$
$$\left.+\frac{l^8}{8!}\left[\mu_8-105\sigma^6\right]\right\}$$

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&+ \left[\ln(T_0) - \left(\mu_T + \frac{1}{2}\sigma_T^2\right)\right](1-F)^t\,\mathbb{E}\left[(1-F)^{-t\wedge\mathbf{d}}\right] \\
&+ \mu_T + \frac{1}{2}\sigma_T^2\left[1 - \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\right] \\
&- \frac{1}{2}\sigma^2 dt\,(1-F)^{2dt}\,\frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}}, \text{ where} \qquad \text{(e.3)}
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&\quad + \mu_T + \frac{1}{2}\sigma_T^2\left[1 - \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]\right] \\
&\quad - \frac{1}{2}\sigma^2 dt\left(1-F\right)^{2dt}\frac{1-\left(1-F\right)^{2t}}{1-\left(1-F\right)^{2dt}}, \text{ where}
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- μ_T and σ_T are the parameters for the lognormal $\{T_j\}$.
- $\bar{\mathbf{d}}$ follows the equilibrium distribution of our *gamma*(α, β) interarrival times for regime-switches and $\mathbb{E}\left[\left(1-F\right)^{-t\wedge\bar{\mathbf{d}}}\right] = \mathcal{L}_{\bar{\mathbf{d}}\wedge t}\left(\ln\left(1-F\right)\right)$ is a calculable Laplace transform.

Drift Compensation: with randomized reversion target

- $\mathbf{e}_j = \left\{ (1-F)^{(t-\mathbf{t}_{j+1})_+} - (1-F)^{(t-\mathbf{t}_j)_+} \right\}$, and a delicate evaluation of $\mathbb{E}\left[\sum_{j=1}^{\infty} \mathbf{e}_j^2\right]$ was the main burden of last year's paper (and corrections.)

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-
$$\mathbb{E}\left[\left\{\ln(\mathbf{r}_t) - \mathbb{E}\left[\ln(\mathbf{r}_t)\right]\right\}^2\right] =$$

$$\begin{aligned} = \quad & \sigma^2 dt\,(1-F)^{2dt}\,\frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}} + \sigma_T^2 \mathbb{E}\left[\sum_{j=1}^{\infty} \mathbf{e}_j^2\right] \\ & + (\ln(T_0) - \mu_T)^2 (1-F)^{2t} \left\{\mathbb{E}\left[(1-F)^{-2\mathbf{t_1}\wedge t}\right] \right. \\ & \left. - \left(\mathbb{E}\left[(1-F)^{-\mathbf{t_1}\wedge t}\right]\right)^2\right\} \qquad (2.6.c) \end{aligned}$$

is the variance of $\ln(\mathbf{r}_t)$

Drift Compensation: with randomized reversion target

- This allows calibration of the drift compensation term as

$$
\begin{aligned}
D_t \quad = \quad & -\frac{1}{2}\sigma^2 \frac{(1-F)^{dt}}{1+(1-F)^{dt}} \left(1+(1-F)^{2t-dt}\right) \\
& +\frac{1}{2}\sigma_T^2 \frac{1}{dt\,(1-F)^{dt}} \left\{ 1-(1-F)^{dt} \right. \\
& -(1-F)^{t}\left(\mathbb{E}\left[(1-F)^{-t\wedge\overline{\mathbf{d}}}\right]-\mathbb{E}\left[(1-F)^{-(t-dt)\wedge\overline{\mathbf{d}}}\right]\right) \\
& \left. -\left(\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]_t-(1-F)^{dt}\,\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^2\right]_{t-dt}\right)\right\}
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- Each μ_{2j} requires evaluation of terms such as $\mathbb{E}\left[\left(\sum_{j=1}^{\infty} \mathbf{e}_j^2\right)^m\right]$
- These in turn can be analyzed into terms of the form $\mathbb{E}\left[\sum_{j=1}^{\infty} \mathbf{e}_j^{2n} \prod_{k=1}^{n} \left(\mathbb{E}\left[\mathbf{e}_j^{2k}\right]\right)^{n_k}\right]$, where $\sum_{k=1}^{n} k n_k \leq m - n$.

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- These in turn can be analyzed into terms of the form $\mathbb{E}\left[\sum_{j=1}^{\infty}\mathbf{e}_j^{2n}\prod_{k=1}^{n}\left(\mathbb{E}\left[\mathbf{e}_j^{2k}\right]\right)^{n_k}\right]$, where $\sum_{k=1}^{n} kn_k \leq m-n$.
- Let $\nu(x) = x^{2n}\prod_{k=1}^{n}\left(\mathbb{E}\left[\mathbf{x}^{2k}\right]\right)^{n_k}$ for any x. As $t \to \infty$ $\mathbb{E}\left[\sum_{j=1}^{\infty}\nu(\mathbf{e}_j)\right]$ turns out to equal

$$\frac{\mathbb{E}\left[\nu\left((1-F)^{\bar{\mathbf{d}}}\right)\right]\mathbb{E}\left[\nu\left(1-(1-F)^{\mathbf{d}}\right)\right]}{1-\mathbb{E}\left[\nu\left((1-F)^{\mathbf{d}}\right)\right]}+\mathbb{E}\left[\nu\left(1-(1-F)^{\bar{\mathbf{d}}}\right)\right]$$

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- Using these, and the Edgeworth-based approximation, we put together formulae for the skewness, kurtosis, and 6-thosis of the modeled interest rate $\mathbf{r}_t$.
- Then we played games with EXCEL Solver to find parameters F, σ_T, α, and β to reproduce asymptotically the historical variance, and kurtosis of $\mathbf{r}_t$ as well as the historical volatility (the standard deviation of $\ln(\mathbf{r}_t) - \ln(\mathbf{r}_{t-dt})$.) We set μ_T so that $\mathbb{E}\left[T_j\right]$ equals the historical mean of $\mathbf{r}_t$ which, together with the drift compensation, assured that asymptotically the model would reproduce the historical mean of $\mathbf{r}_t$.

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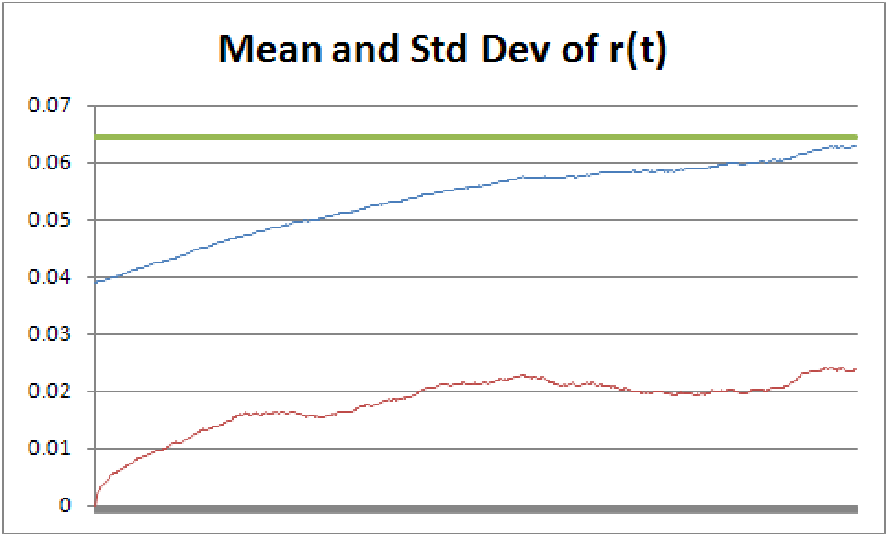
- Looking at results, it seems more reasonable to use parameters that put the historical kurtosis within a sampling error, rather than forcing equality.

With No Regime Switching, Just Match Historical Mean and Variance: F=.0528 (N=662)



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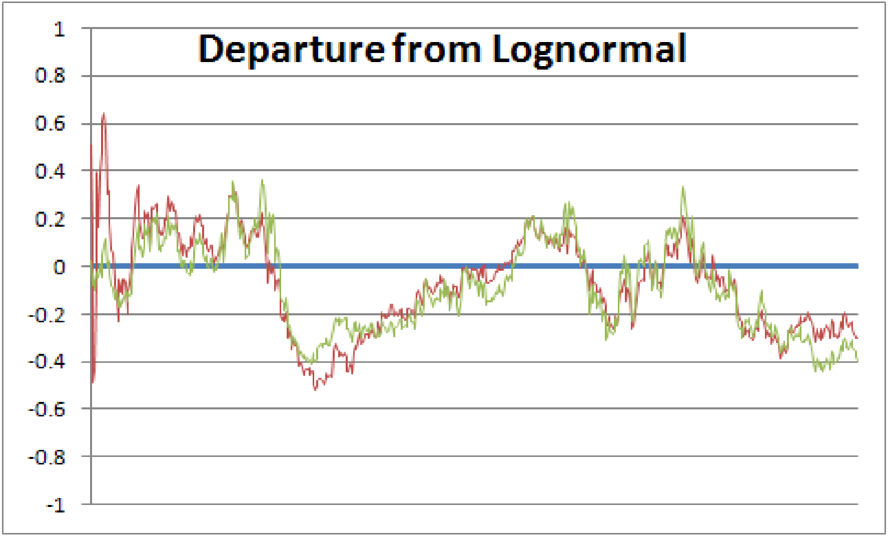

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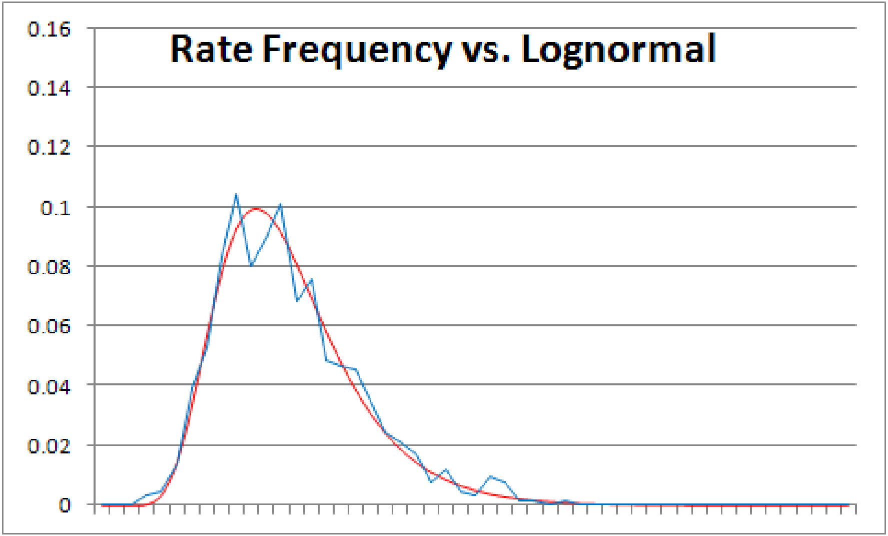

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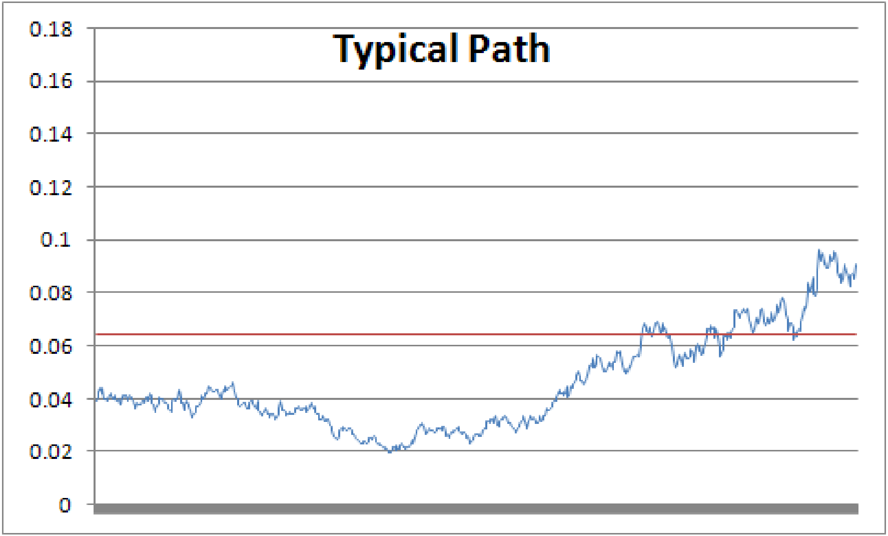

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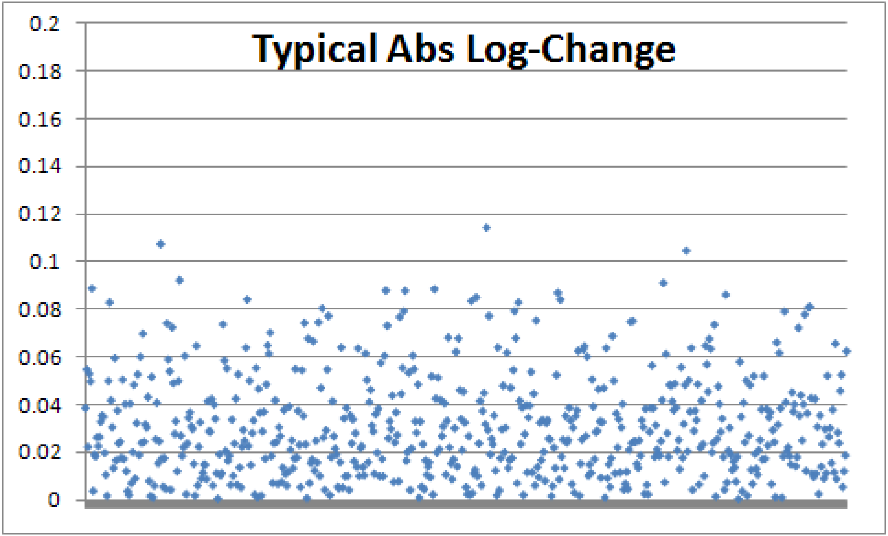

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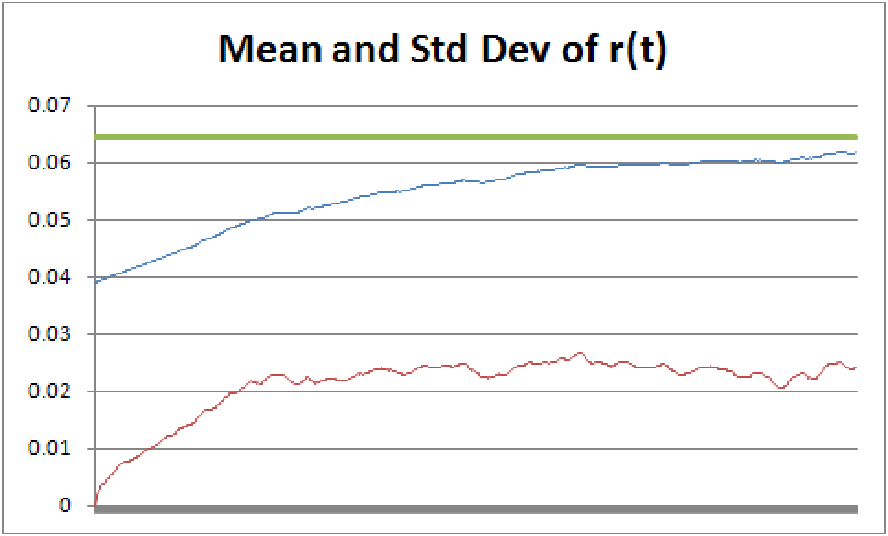

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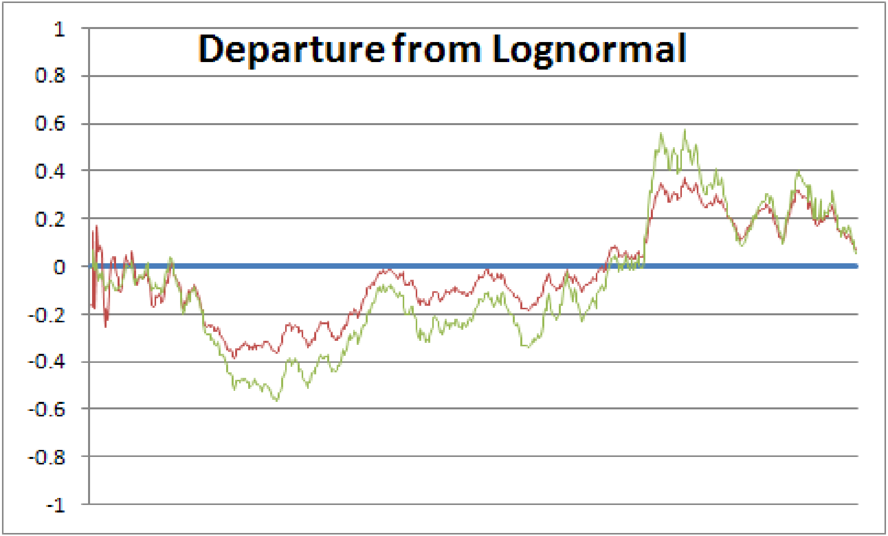

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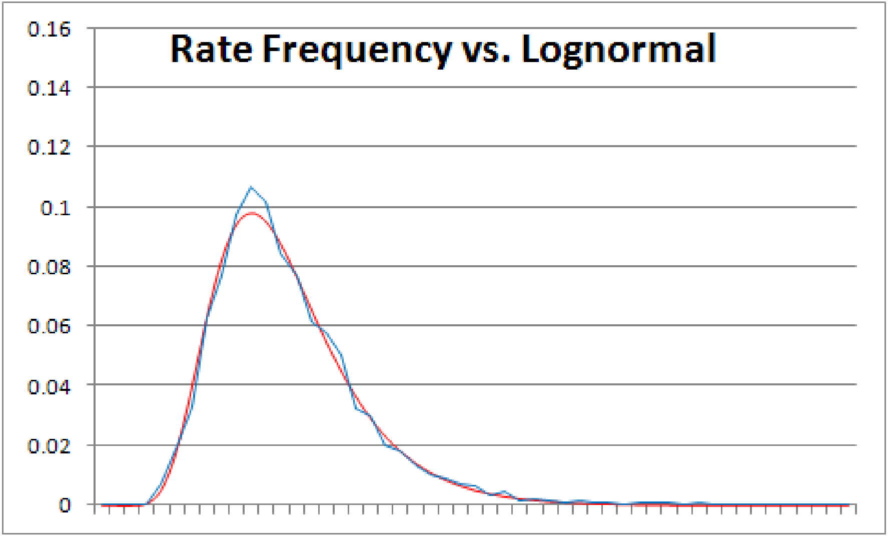
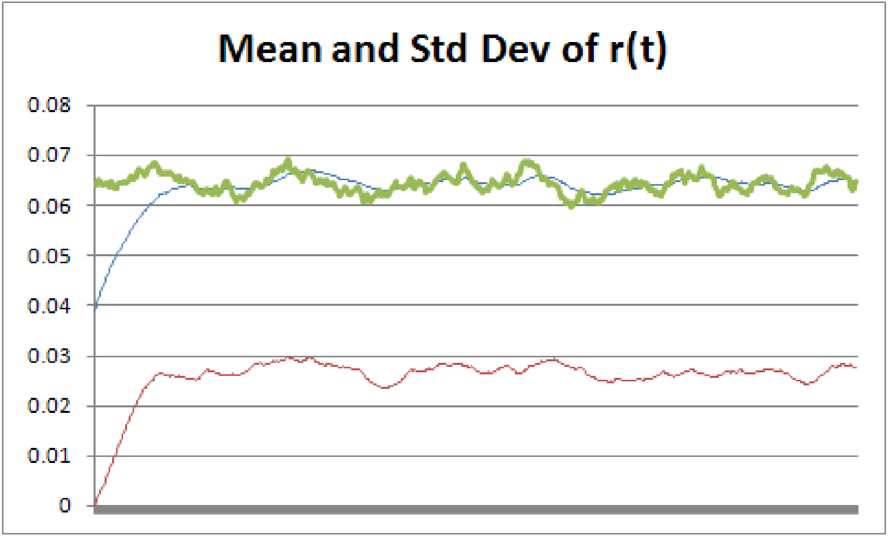


Match historical kurtosis:

$F = .3993/\alpha = 3/\beta = .5/\sigma_T = .6512$ (N=662)



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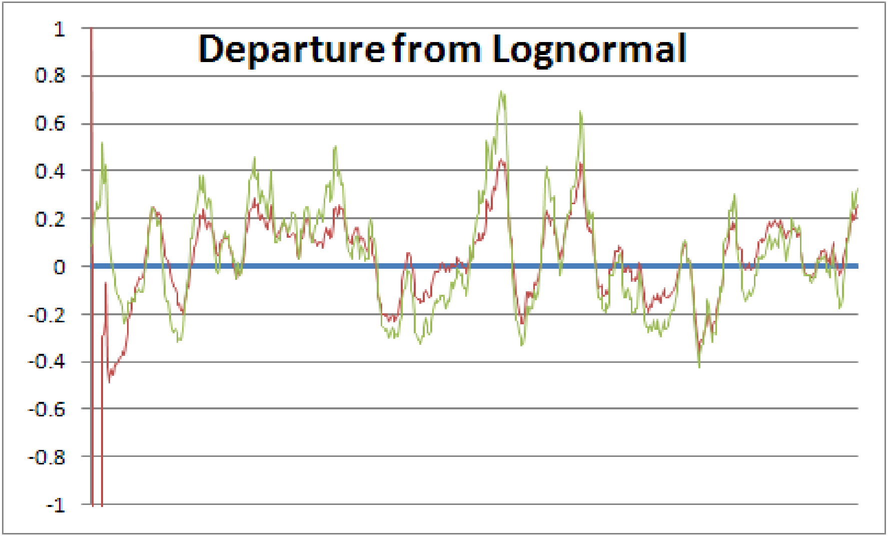

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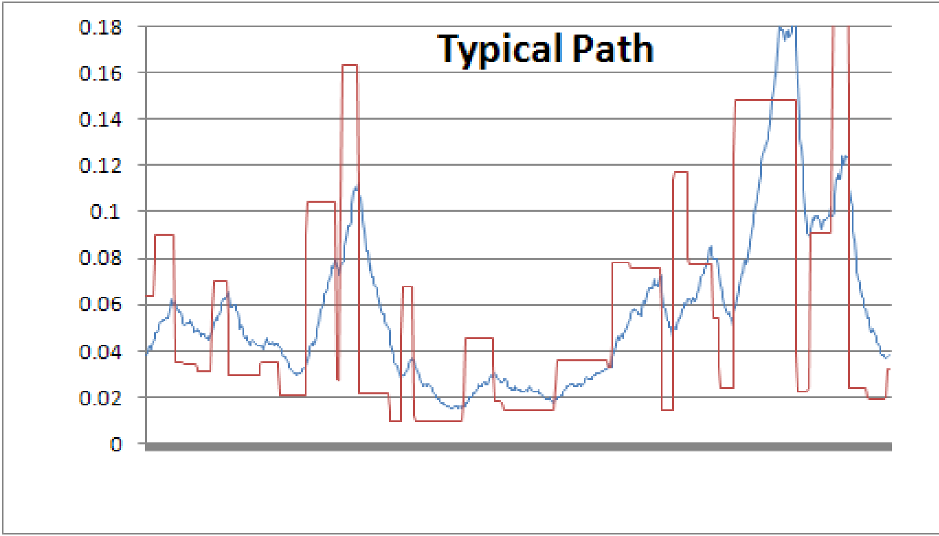

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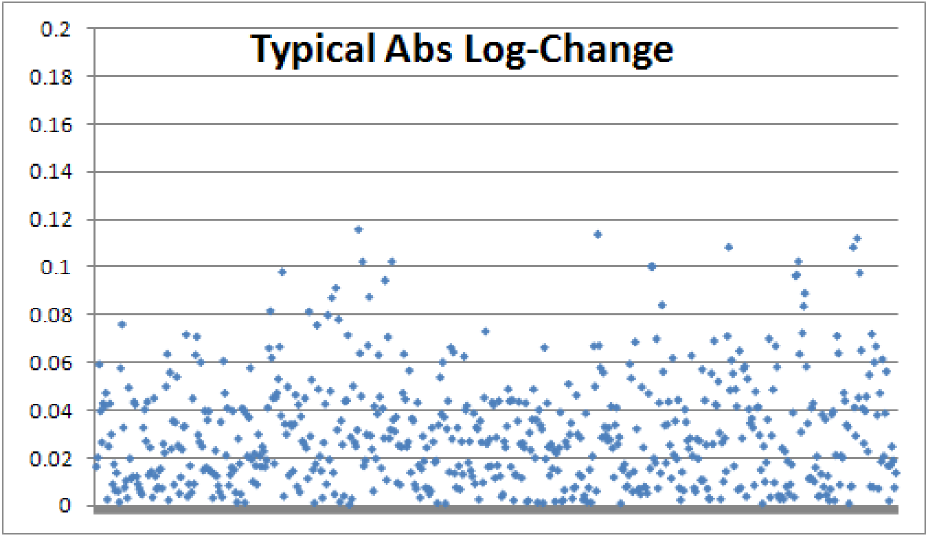

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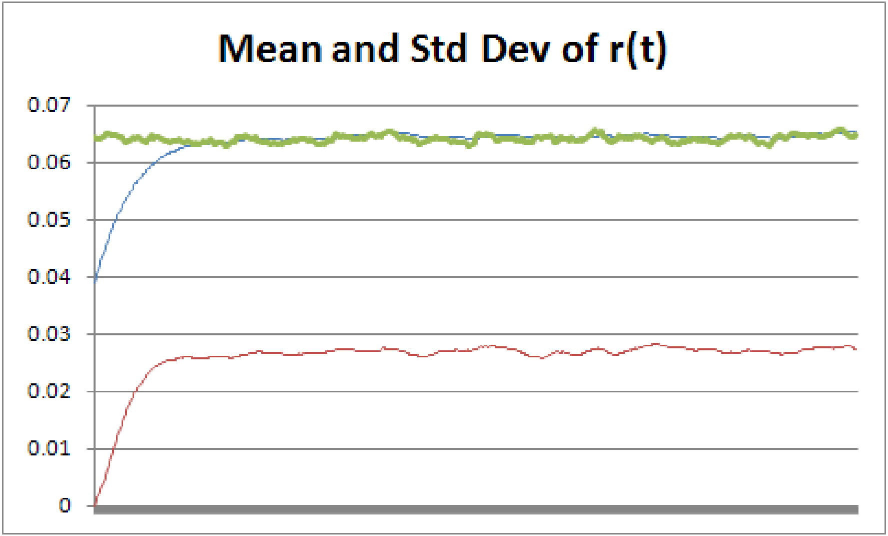

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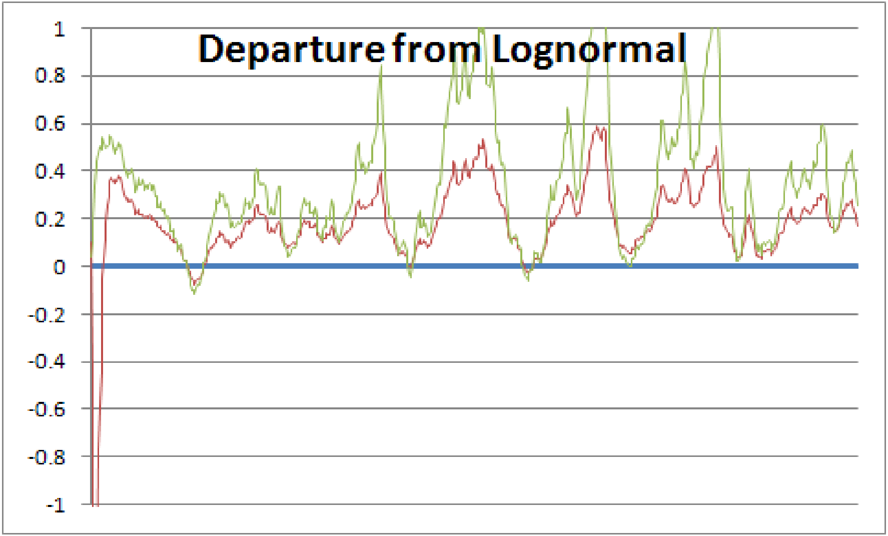

Kurtosis 74% of Lognormal (vs historical 58%)

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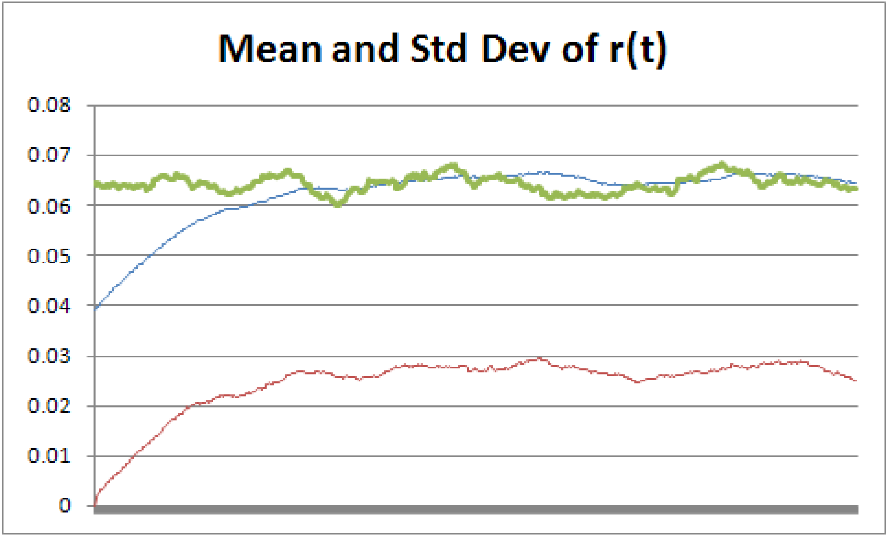

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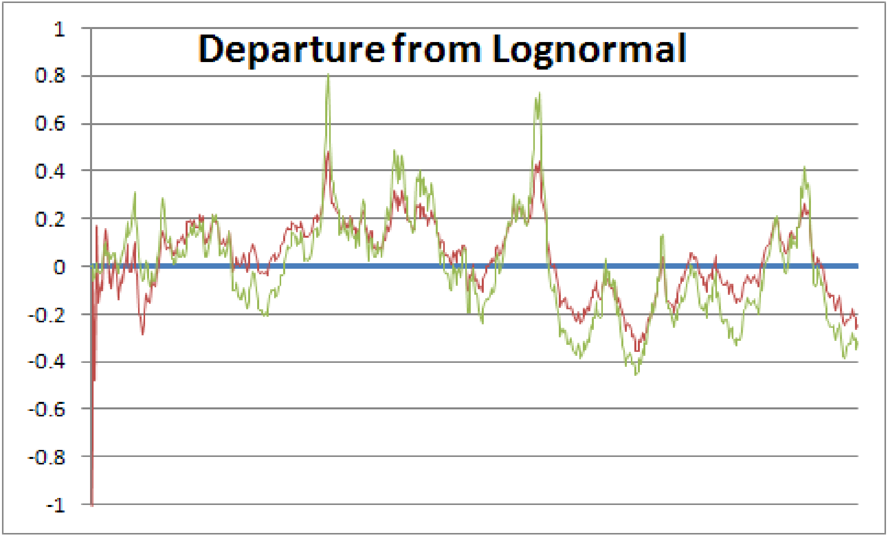

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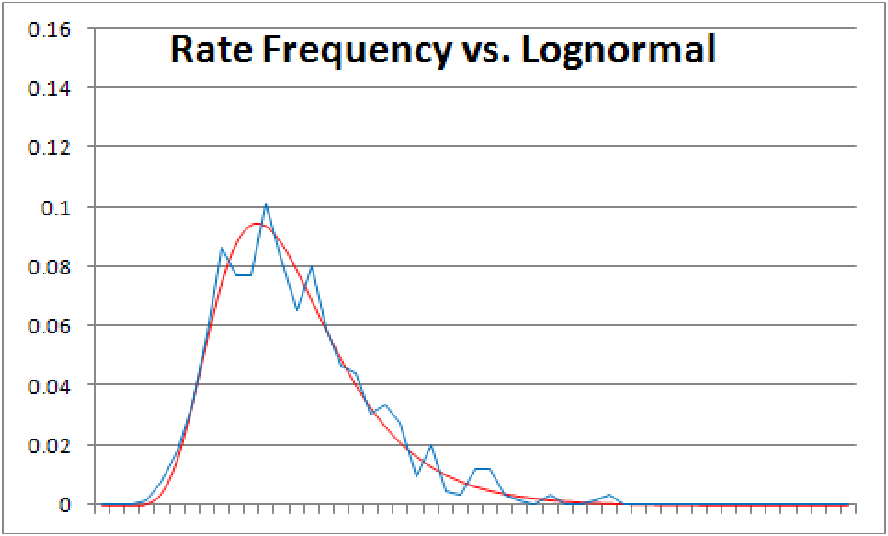

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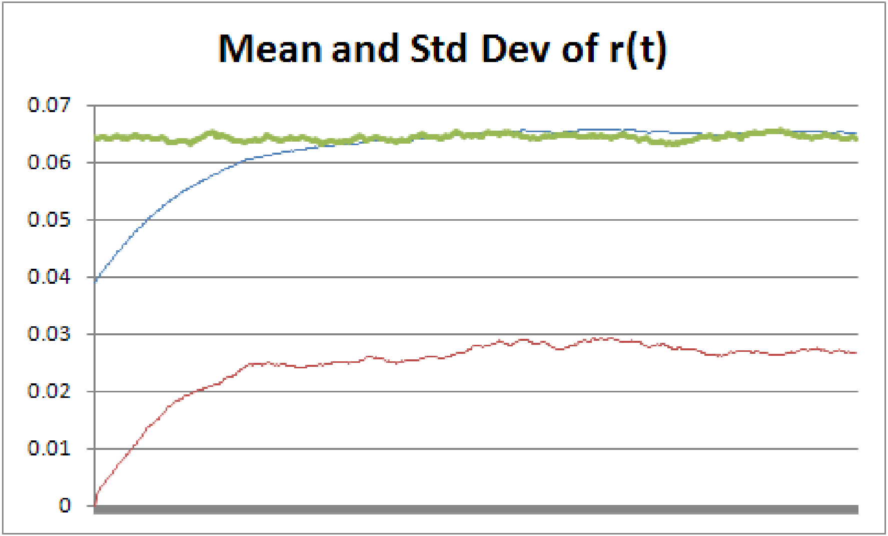

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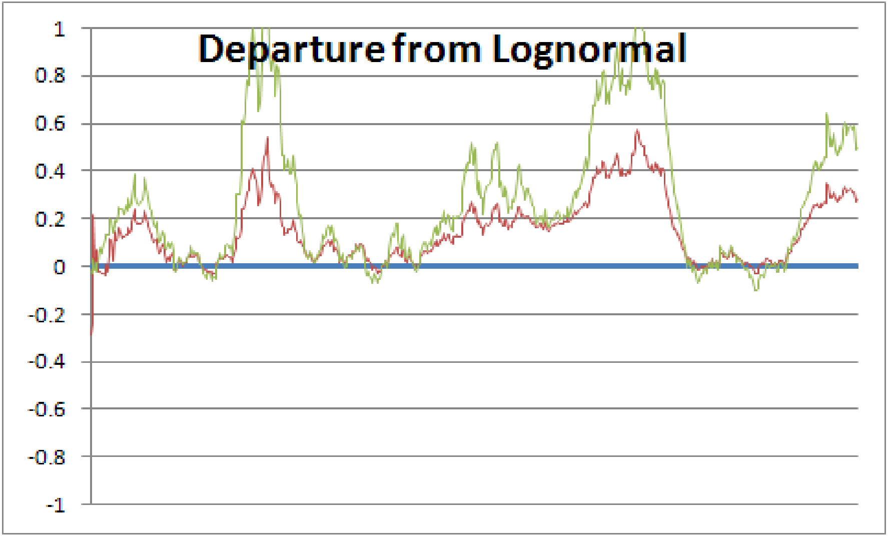

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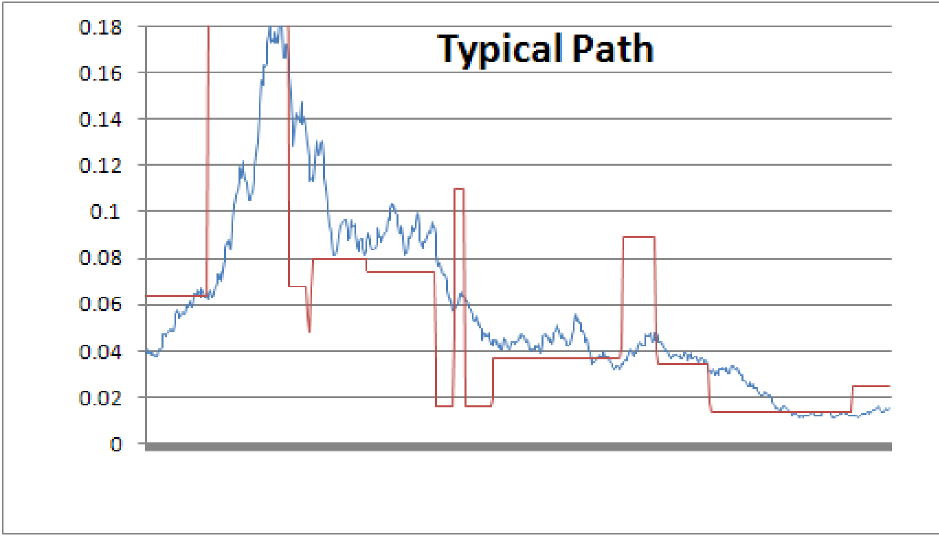

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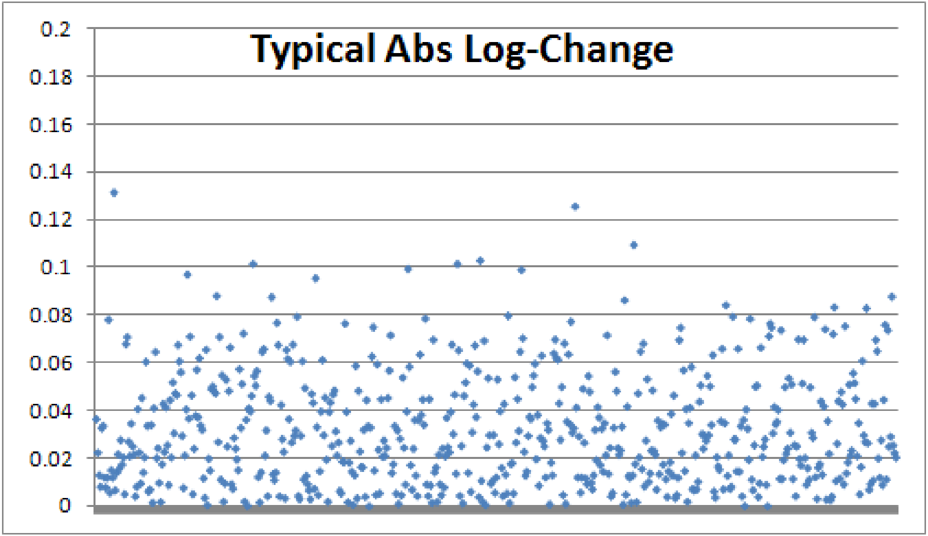

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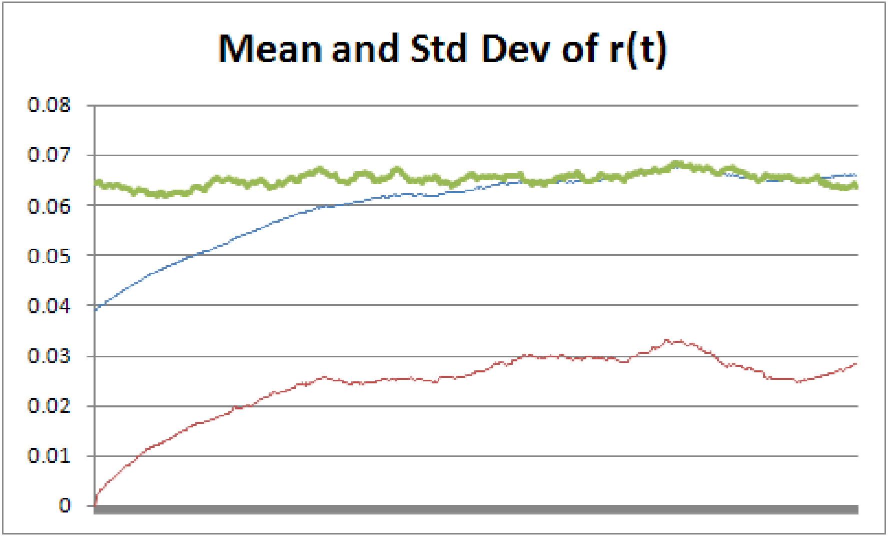

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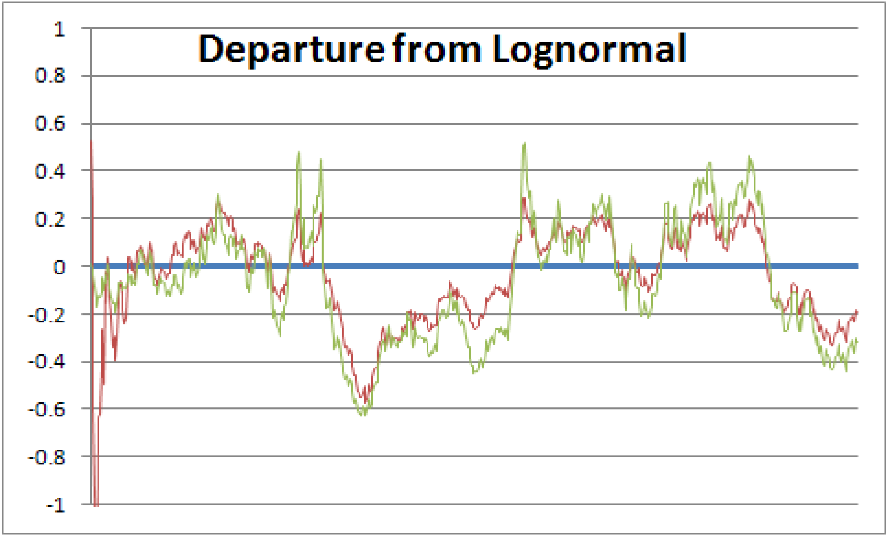

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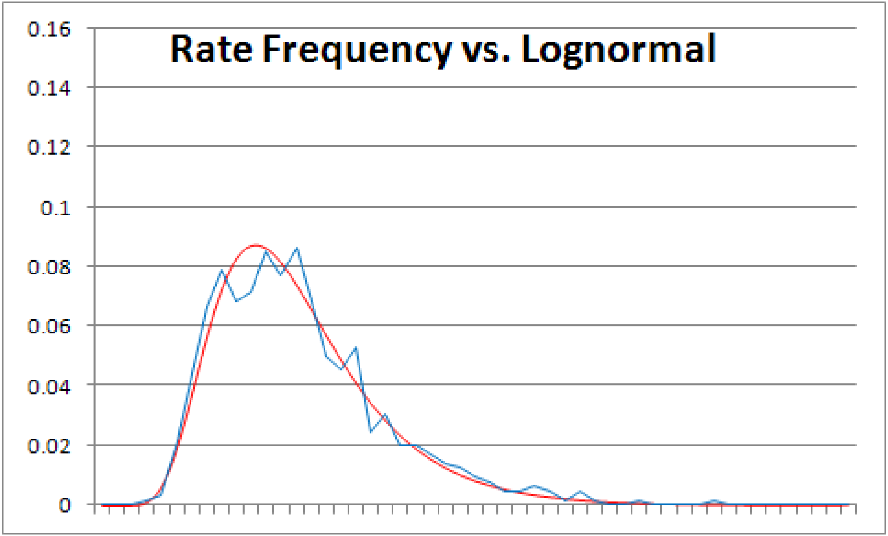

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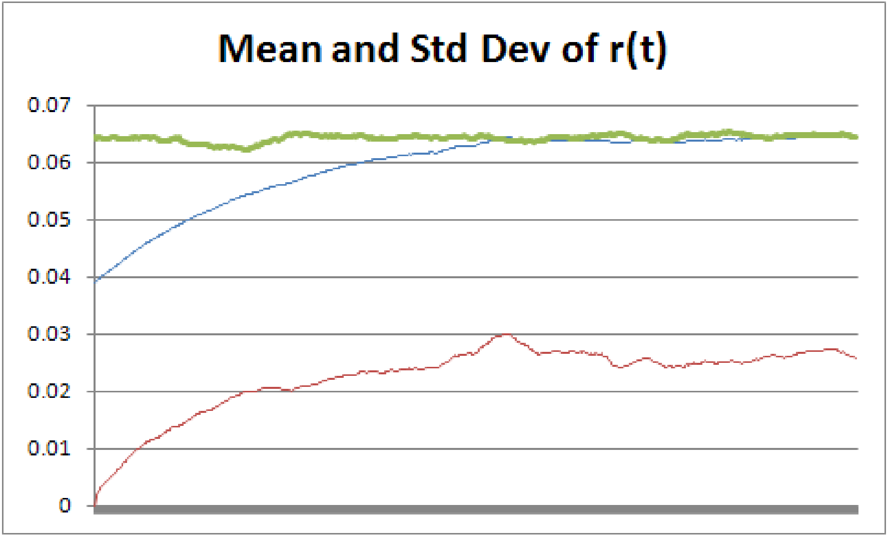

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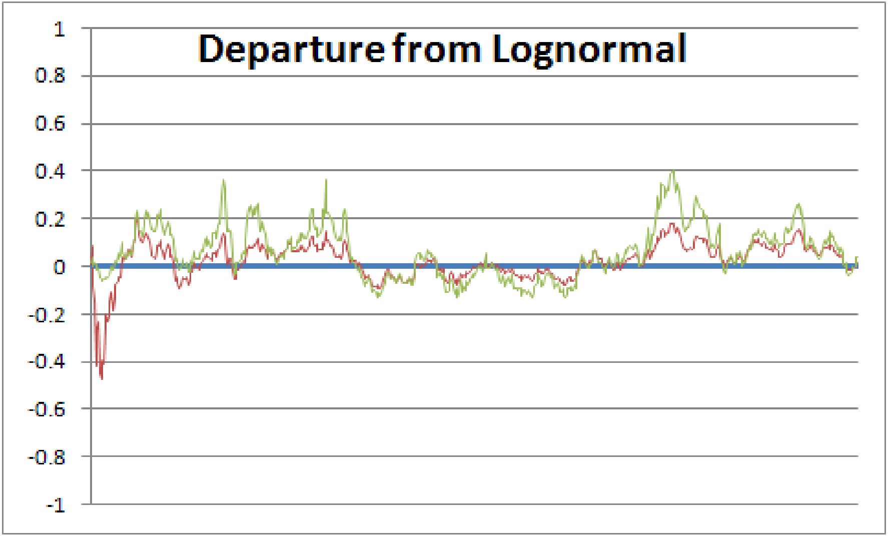

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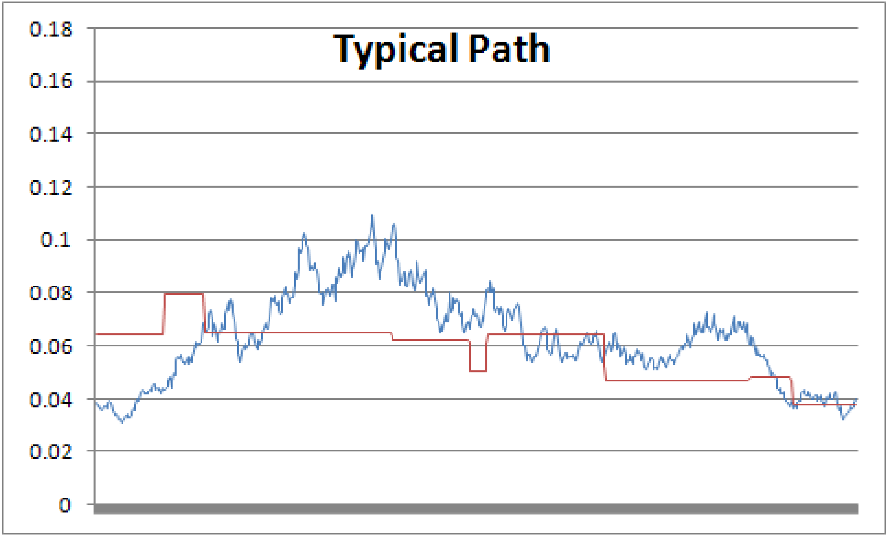

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