A Dual Version of Asset-Liability Risk Modeling

James Bridgeman
University of Connecticut

University of St.Thomas Center for Applied Mathematics Seminar

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INTRODUCTION - RISK MODELING

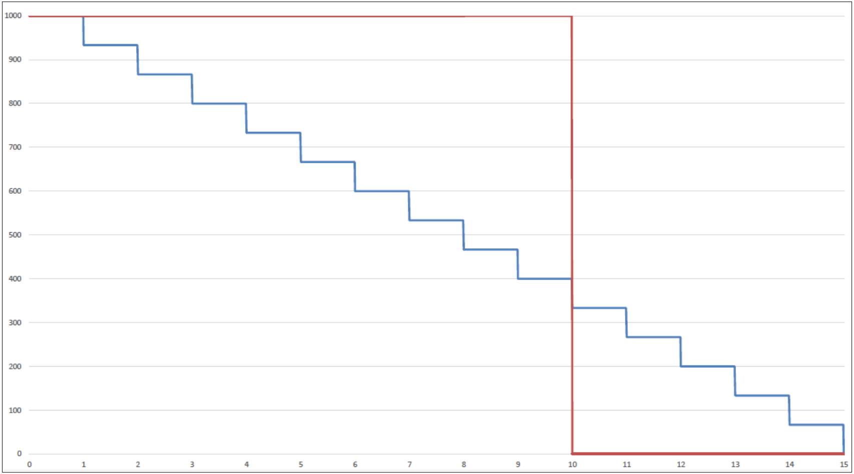
- Often, we model risk with the same models as for pricing, planning, forecasting and reserving.
- In risk modeling we just assume extreme inputs, or look at the tails of randomly generated outcomes, or maybe even use statistical extreme value theory
- But we fine-tune pricing, planning, forecasting and reserving models to work well on typical inputs; can this foreclose good risk modeling in a holistic sense?
- Maybe risk needs radically different models; but somehow related to usual models, grounded in them, perhaps a formal duality?

MODELING ASSET-LIABILITY INTEREST RATE RISK

- Traditionally we model known cash flows and take present values - a balance sheet view
  - Ignore future cash unless implied by balance sheet
  - Test future interest rates' effect on present values:
    - duration/convexity etc.
    - stochastic future interest rates
    - risk-neutral calibrations to market values
- A radically different model could start with going concern assumptions - an income view
  - Embrace future cash flow - on some normalized, on-going basis
  - Test future interest rates' effect on future spreads between asset earnings and liability requirements
- Strictly a work-in-progress: what tools would give a model dual to the balance sheet?

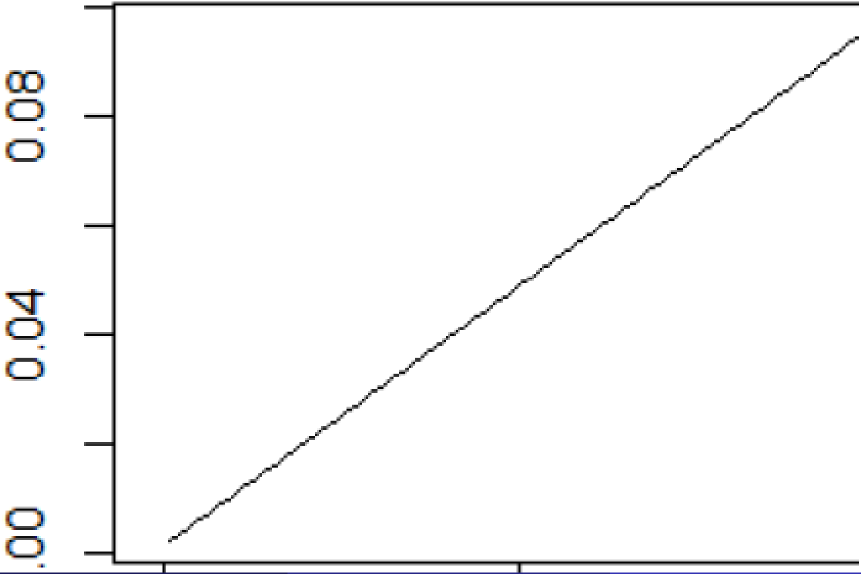
A SIMPLE GOING CONCERN WITH A-L MISMATCH

Take in a steady stream of 10-year bullet liabilities and invest steadily in 15 year ladder asset maturities.



A SIMPLE GOING CONCERN WITH A-L MISMATCH

Let interest rates increase steadily:



A SIMPLE GOING CONCERN WITH A-L MISMATCH

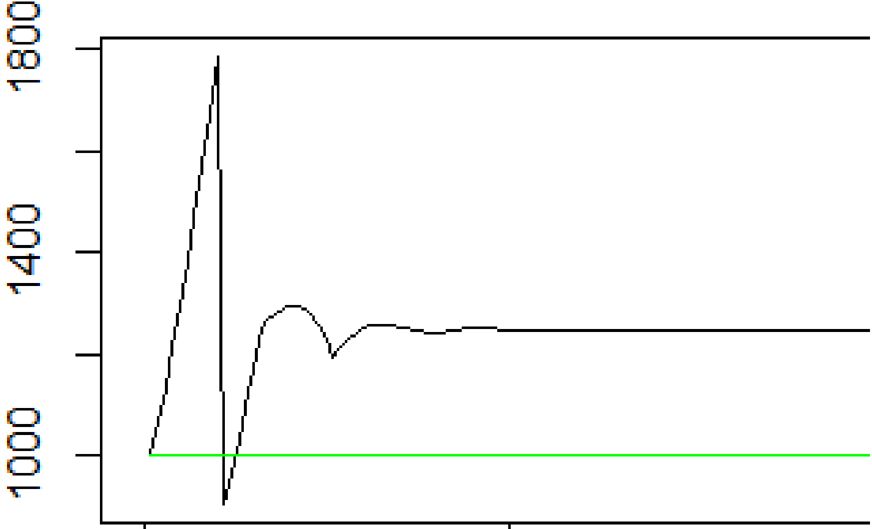
Take in a steady stream of 10-year bullet liabilities and invest steadily in 15 year ladder asset maturities. The following spreads result if interest rates increase steadily:

Linear Inputs Gave Oscillating Output - What's Going On?

- In signal analysis – electronics or optics – this would be the clue to something called an "Edge Effect" or a "Gibbs' Phenomenon"
- Essentially, the linear inputs have encountered or involve some sort of underlying process that combines a jumble of hidden oscillating pieces
- Some beginning (or ending) of the linear inputs splatters the underlying oscillations into the output piecemeal so we can see some of them
- Fourier analysis is the technical tool to explore this
- REALLY?? For ALM Work?
- Maybe there is a simpler explanation?

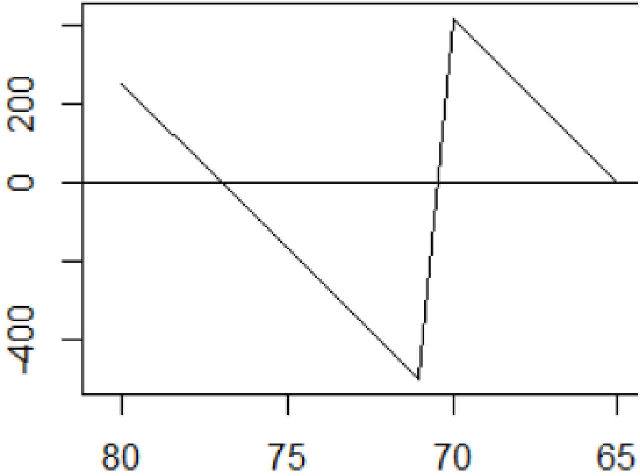
Linear Input Gave Oscillating Output - Why?

Maturity mismatch creates investment mismatch in going concern – early asset maturities need reinvesting



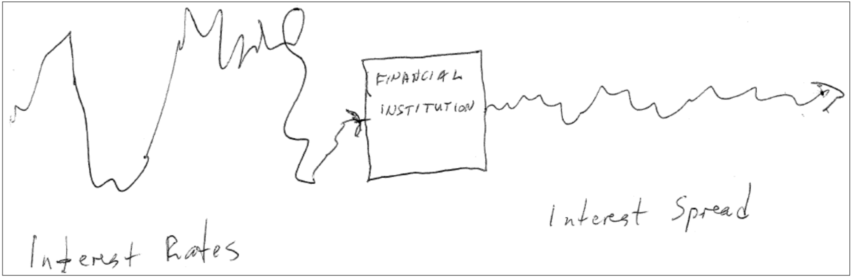
Linear Input Gave Oscillating Output - Why?

And looking backwards even in steady state the net survivors still at each past rate oscillates, too – 12 and 9.5 year cycles observable

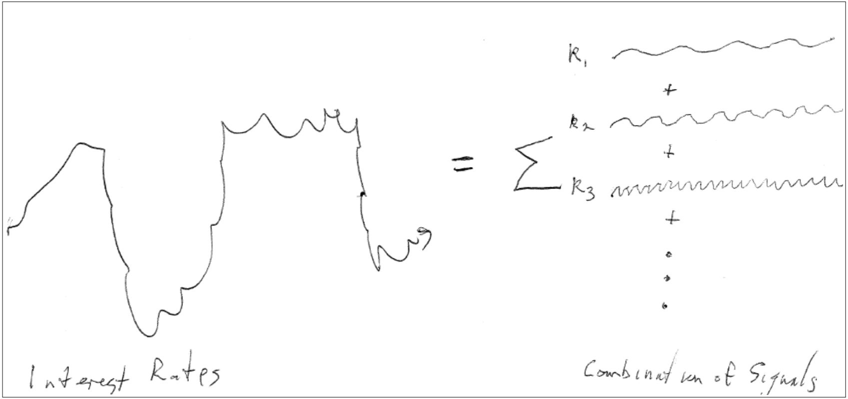


OK, BUT THIS STILL CRIES OUT FOR FOURIER ANALYSIS

A financial institution is a receiver of a stream of interest rates that modulates them into an output stream of interest spreads (gain/loss)

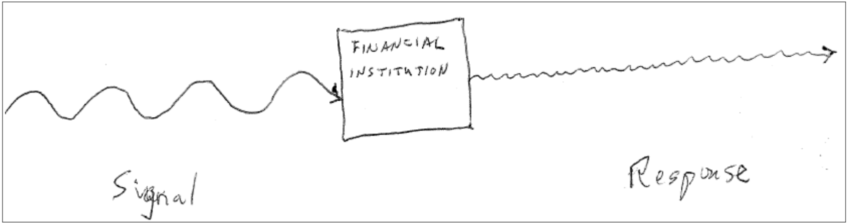


The interest rate stream consists of component signals

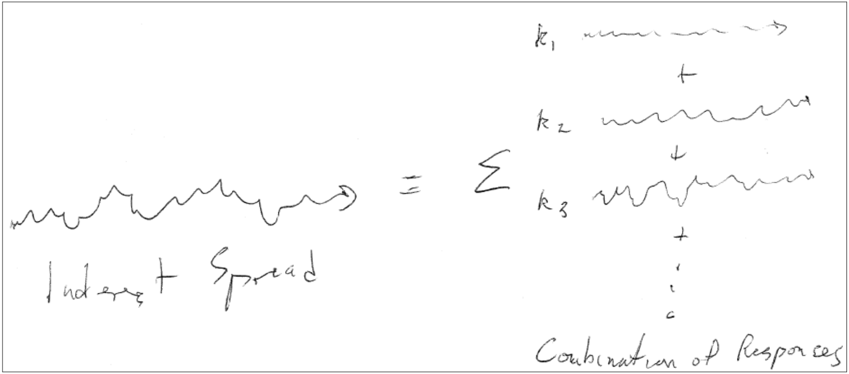


THIS STILL CRIES OUT FOR FOURIER ANALYSIS

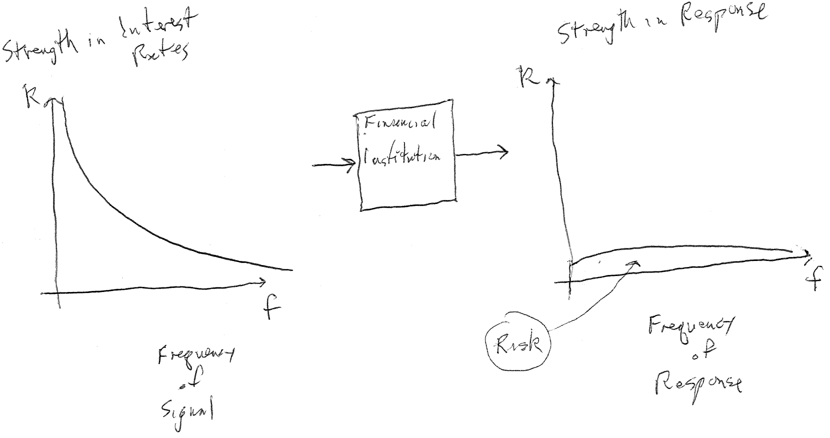
Suppose we know how the financial institution modulates each component of the input signal into an output response



Then we can reconstruct the total response (the spread) to the original interest rate stream



FOURIER ANALYSIS JUST CODIFIES THIS



- Strengths are given by |FT|, the modulus of the Fourier Transform
- Interest Rate Risk is the <u>area</u>

THIS IS THE DUAL VIEW OF INTEREST RATE RISK

- It looks at the institutional response to the entire spectrum of interest rate volatility
  - Dual to duration, etc. which puts most focus on the lowest frequency component(s) of the interest rate spectrum

- It looks at the going-concern interest rate spread (income statement)
  - Dual to the balance-sheet view of traditional immunization
  - Like the duality between position and momentum in physics

- Area under the spectrum is the proper risk measure
  - If random phases align against you the whole area contributes to your woe

CAN'T GET THIS FROM YOUR USUAL MODELS

(Or at least not directly from them)

WHAT WE NEED IS

- A model of the external interest rate spectrum
  - As an abstract random phenomenon, not just past x years or a fitted time series
  - FORECASTING DISTRACTS FROM RISK ANALYSIS!
- A model of the modulation process
  - Unique to each financial institution
  - Applicable to all possible external signals
- And, of course, the Fourier Analysis technique

Fourier Transform

- Given a function $r(t)$, say the interest rate over time, you can write it as a sum of oscillating components

$$r(t) = \int_{-\infty}^{\infty} FT[r](f)\, e^{2\pi i f t} df$$

where $FT[r](f)$, called the Fourier Transform of $r(t)$ at f, determines the component of $r(t)$ that oscillates with frequency f.

- The standard oscillation with frequency f is $e^{2\pi i f t}$.
- Note, $e^{2\pi i f t}$ is a complex number at each time t that corkscrews around the complex unit circle as time passes.
- The frequency f determines how fast and in which direction it spins.

Fourier Transform

$$r(t) = \int_{-\infty}^{\infty} FT[r](f)\, e^{2\pi i f t} df$$

- So $FT[r](f)$ is a complex number
- Whose modulus $|FT[r](f)|$ at each frequency f tells us how large $e^{2\pi i f t}$ looms inside $r(t)$ for that f
- Whose phase at each frequency f tells us how much the version of $e^{2\pi i f t}$ inside $r(t)$ is rotated from its usual starting point (at $t = 0$) for that f.
- With all these complex numbers spinning around, how can we get a real function $r(t)$ back out of the formula?
- It just requires that $FT[r](f)$ and $FT[r](-f)$ be complex conjugates for each frequency f.

Fourier Transform Properties

- There is a formula for $FT[r](f) = \int_{-\infty}^{\infty} r(t) e^{-2\pi i f t} df$. It just "unscrews" the corkscrew at frequency f so we can see it
- If $a(t)$ and $b(t)$ are two functions of t, and k and j are constants, then for each frequency f we have

$$FT[ka + jb](f) = kFT[a](f) + jFT[b](f)$$

- If we define $(a * b)(t) = \int_{-\infty}^{\infty} a(t-s)\, b(s)\, ds$ ("convolution") then

$$FT[a * b](f) = FT[a](f)\, FT[b](f)$$
$$FT[a \cdot b](f) = FT[a](f) * FT[b](f)$$

- If we define $\Delta(t) = 1$ for $t \geq 0$ and $= 0$ for $t < 0$ then

$$FT[\Delta](f) = \frac{1}{2\pi i f} 1_{f \neq 0} + \frac{1}{2}\delta(f), \text{ for } \delta = \text{impulse at } 0$$

- Note: $(\Delta * b)(t) = \int_{-\infty}^{t} b(s)\, ds = B(t)$, etc. for any b, B

START WITH THE MODULATION PROCESS

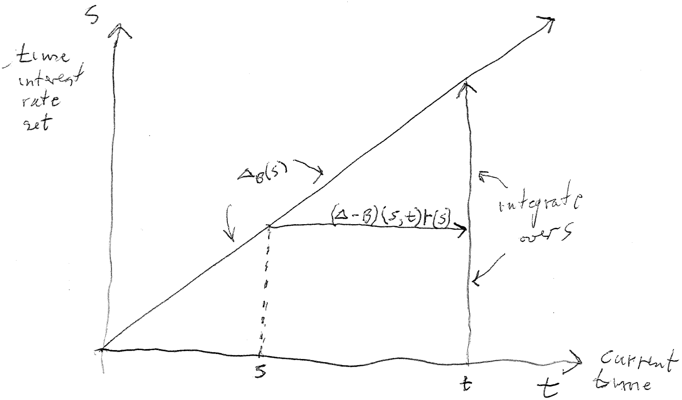
Let $r(s)$ = the interest rate at time s
$\Delta_B(s)$ = new Liabilities taken on at time s
(Assume $\Delta_B(s)$ takes a simple going-concern form)
$B(s,t)$ = Liabilities matured out of $r(s)$ by time t
$b(s,t) = \frac{\partial}{\partial t}B(s,t)$ the rate of Liabilities maturing out of $r(s)$ at time t
$\Delta(s,t) = 1$ for $t \geq s$ and $= 0$ for $t < s$
$(\Delta - B)(s,t)$ = Liabilities still owed $r(s)$ at time t = survival function of $B(s,t)$ viewed as a cdf.

This gives a crude going-concern model of interest requirements on the Liabilities

Rate of interest required (at time t) $= \frac{[(\Delta - B)*(r\Delta_B)](t)}{[(\Delta - B)*\Delta_B](t)}$

START WITH THE MODULATION PROCESS

Interest requirements on the Liabilities (going concern)



$\Delta_B(s)$ is given

$$\text{Rate of Liability Interest} = \frac{\int_0^t r(s)(\Delta-B)(s,t)\Delta_B(s)\,ds}{\int_0^t (\Delta-B)(s,t)\Delta_B(s)\,ds} = \frac{[(\Delta-B)*(r\Delta_B)](t)}{[(\Delta-B)*\Delta_B](t)}$$

That's a generalization of the usual definition of convolution and it won't be commutative

A Few Other Things We Need

$a^{*k} = a * a * ... * a$ k times makes sense and we will use it

When we need it, $\delta =$ Dirac delta function (impulse at 0)

In particular, $a^{*(0)} = \delta$ and

$FT\left[\delta\right]\left(f\right) = 1$ for all f and $FT\left[k\right]\left(f\right) = k\delta\left(f\right)$ for all constants k

START WITH THE MODULATION PROCESS

If $\Delta_A(s) =$ new Assets taken on at time s then $\Delta_A(s)$ will be a function of everything else in the model

In fact, $\Delta_A(s) = \left(\left(\sum_{k=0}^{\infty} a^{*k} \right) * (\delta - b) * \Delta_B \right)(s)$

$A(s,t) =$ Assets matured out of $r(s)$ by time t

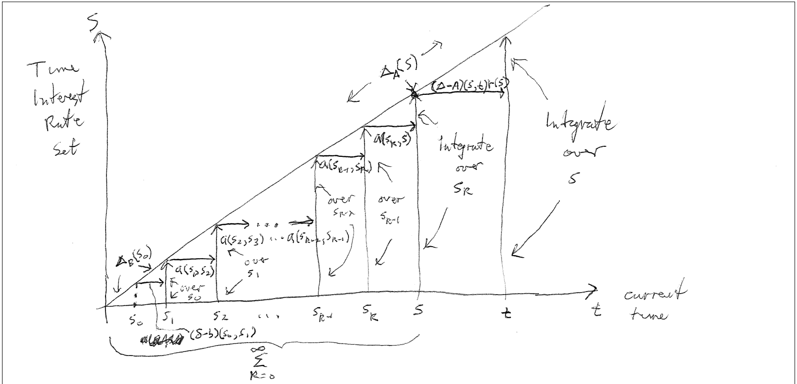
$a(s,t) = \frac{\partial}{\partial t} A(s,t)$ the rate of Assets maturing out of $r(s)$ at time t

$(\Delta - A)(s,t) =$ Assets still earning $r(s)$ at time $t =$ survival function of $A(s,t)$ viewed as a cdf.

Rate of interest available (at time t) $= \dfrac{\left[(\Delta - A) * \left(r \left[\left(\sum_{k=0}^{\infty} a^{*k} \right) * (\delta - b) * \Delta_B \right] \right) \right](t)}{\left[(\Delta - A) * \left(\left(\sum_{k=0}^{\infty} a^{*k} \right) * (\delta - b) * \Delta_B \right) \right](t)}$

START WITH THE MODULATION PROCESS

Interest generated by the Assets (going concern)



$$\sum_{k=0}^{\infty}$$

$$\Delta_A(s) = \sum_{k=0}^{\infty} \int_0^s a(s_k,s) \int_0^{s_k} a(s_{k-1},s_k) \int_0^{s_{k-1}} \cdots a(s_2,s_3) \int_0^{s_2} a(s_1,s_2) \int_0^{s_1} (\delta-b)(s_0,s_1)\, \Delta_B(s_0)\, ds_0\, ds_1 \cdots ds_{k-2}\, ds_{k-1}\, ds_k$$

$$\text{Rate of Asset Interest} = \frac{\int_0^t r(s)(\Delta-A)(s,t)\Delta_A(s)\,ds}{\int_0^t (\Delta-A)(s,t)\Delta_A(s)\,ds} = \frac{[(\Delta-A)*(r\Delta_A)](t)}{[(\Delta-A)*\Delta_A](t)}$$

$$\Delta_A(s) = \left[\left(\sum_{k=0}^{\infty} a^{*k}\right) * (\delta-b) * \Delta_B\right](s)$$

- Going concern interest rate spread s at time t is the difference

$$s\left(t\right) = \frac{\left[(\Delta - A) * \left(r\left[\left(\sum_{k=0}^{\infty} a^{*k}\right) * (\delta - b) * \Delta_B\right]\right)\right](t)}{\left[(\Delta - A) * \left(\left(\sum_{k=0}^{\infty} a^{*k}\right) * (\delta - b) * \Delta_B\right)\right](t)} - \frac{[(\Delta - B) * (r\Delta_B)](t)}{[(\Delta - B) * \Delta_B](t)}$$

where the denominators are equal (a good test of your convolution algebra)

- At this point I don't know how to progress without assuming homogeneous business strategy, ie. $B(s,t) = B(t-s)$, $A(s,t) = A(t-s)$, $\Delta\left(s,t\right) = \Delta\left(t-s\right)$ etc. for all s and t
- Among other things this makes the convolutions the usual commutative definition.

CONTINUING WITH THE MODULATION PROCESS

Some useful facts are

$$\begin{aligned}(\Delta - A) * \left(\sum_{k=0}^{\infty} a^{*k}\right) &= \Delta * (\delta - a) * \left(\sum_{k=0}^{\infty} a^{*k}\right) \\ &= \Delta * \delta \\ &= \Delta\end{aligned}$$

and $\lim_{t\to\infty} \left(\sum_{k=0}^{\infty} a^{*k}\right)(t) = \frac{1}{\mu_A}$ where μ_A is the mean of A considered as a cdf.

Also, those survival functions $(\Delta - A)$ and $(\Delta - B)$ involved in convolutions (= integrals) suggests that some more means are lurking in these formulas, for example $\Delta * (\Delta - A)(t) \to \mu_A$ for $t \to \infty$, called "the surface interpretation of the mean"

CONTINUING WITH THE MODULATION PROCESS

- If we assume a level stream of new Liabilities $\Delta_B = \Delta$ the formula for the spread s is (after a lot of algebra to get a term with r alone)

$$s = \frac{\left[\frac{\mu_B}{\mu_A}(\Delta - A) - (\Delta - B)\right] * r}{[(\Delta - B) * \Delta]} - \frac{(\Delta - A) * \left\{\left[\left(\sum_{k=0}^{\infty} a^{*k}\right) * \left(\frac{\mu_B}{\mu_A}(\Delta - A) - (\Delta - B)\right)\right] r\right\}}{[(\Delta - B) * \Delta]}$$

- Amazingly, the messy term is a transient that goes to 0 as the homogenous going-concern reaches steady-state. It is the exact formula for that oscillation we saw at the beginning.
- In the permanent steady state, the denominator $[(\Delta - B) * \Delta] \to \mu_B$
- For stable growing new Liabilities you just use distortions of A, B, a, b, μ_A and μ_B.
- The permanent steady-state term is made-to-order for a Fourier Transform

CONCLUSION FOR THE MODULATION PROCESS

For each frequency f the Fourier transform of the steady-state going-concern spread with a level stream of new liabilities is

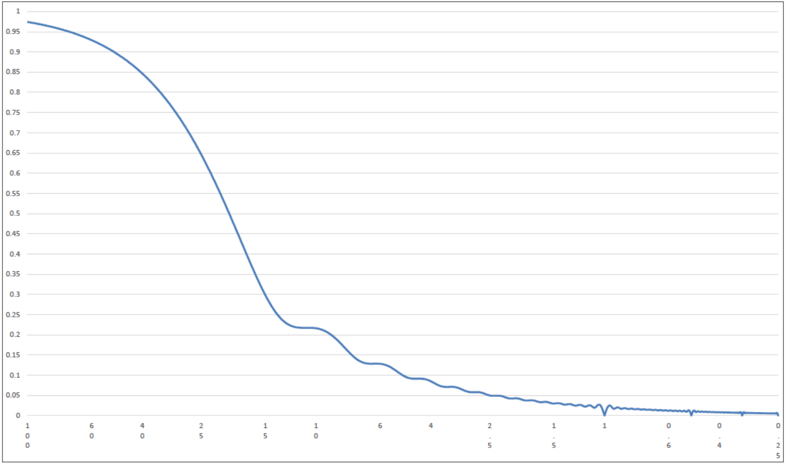
$$
\begin{aligned}
FT[s](f) &= FT\left[\frac{\Delta - A}{\mu_A} - \frac{\Delta - B}{\mu_B}\right](f)\, FT[r](f) \\
&= FT\left[\Delta * \left(\frac{\delta - a}{\mu_A} - \frac{\delta - b}{\mu_B}\right)\right](f)\, FT[r](f) \\
&= FT[\Delta](f)\, FT\left[\frac{\delta - a}{\mu_A} - \frac{\delta - b}{\mu_B}\right](f)\, FT[r](f) \\
&= \left(\frac{1}{2\pi i f} 1_{f \neq 0} + \frac{1}{2}\delta(f)\right)\left(\frac{1 - FT[a](f)}{\mu_A} - \frac{1 - FT[b](f)}{\mu_B}\right) \cdot \\
&\quad \cdot FT[r](f) \\
&= \frac{1}{2\pi i f}\left(\frac{1 - FT[a](f)}{\mu_A} - \frac{1 - FT[b](f)}{\mu_B}\right) FT[r](f) \text{ if } f \neq 0
\end{aligned}
$$

CONCLUSION FOR THE MODULATION PROCESS

- In other words $\frac{1}{2\pi if}\left(\frac{1-FT[a](f)}{\mu_A}-\frac{1-FT[b](f)}{\mu_B}\right)$ represents how the financial institution modulates the external interest rate frequency strengths $FT[r](f)$ into interest spread frequency responses $FT[s](f)$ when there is a level stream of new liabilities.
- We should note that if the new liability stream Δ_B grows at a stable rate g the Fourier Transform of the interest rate spread works out to $FT[s](f)=\frac{1}{\ln(1+g)+2\pi if}\left(\frac{1-FT[a](f)}{\mu_A}-\frac{1-FT[b](f)}{\mu_B}\right)FT[r](f)$ where the distorted versions of the functions and means must be used if the assumed growth g is not 0, and also in that case the equation works for $f=0$ too.
- The factor $\frac{1}{\ln(1+g)+2\pi if}$ in the modulation already teaches an important lesson for risk management: a stable, well-managed level of growth is a very effective risk-control mechanism.
- But let's illustrate the basic modulation for our simple financial institution with 15-year ladder assets and 10-year bullet liabilities.

ASSET MODULATION SPECTRUM (15 Year Ladder)

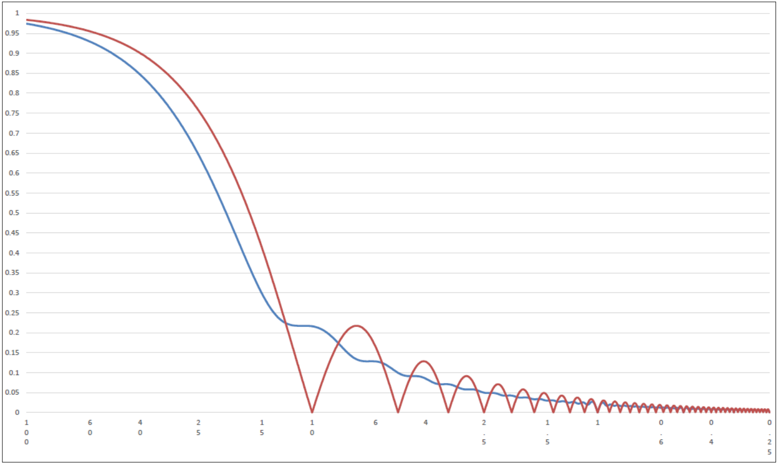
Modulus $\left|\frac{1}{2\pi i f}\left(\frac{1-FT[a](f)}{\mu_A}\right)\right|$; don't forget there's a phase, too



Horizontal axis labeled by wavelength $\left(\frac{1}{f}\right)$ on a logarithmic scale.

NOW THE LIABILITY SPECTRUM (10 Year Bullet)

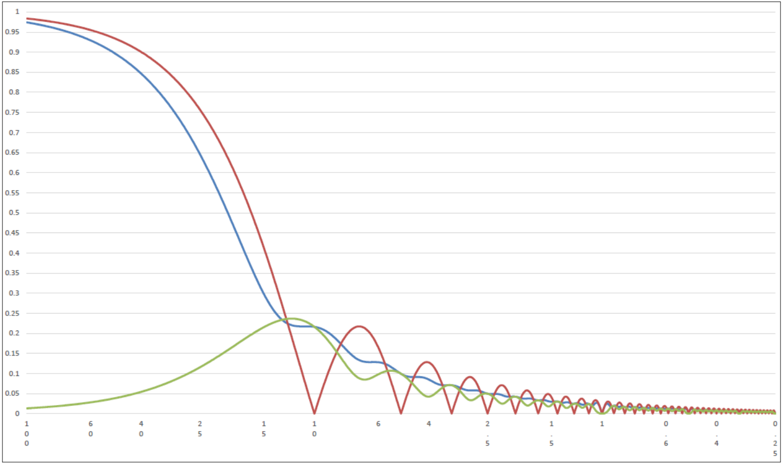
Modulus $\left|\frac{1}{2\pi i f}\left(\frac{1-FT[a](f)}{\mu_A}\right)\right|$ and $\left|\frac{1}{2\pi i f}\left(\frac{1-FT[b](f)}{\mu_B}\right)\right|$; don't forget phases



Horizontal axis labeled by wavelength on a logarithmic scale.

SUBTRACT FOR THE NET MODULATION SPECTRUM

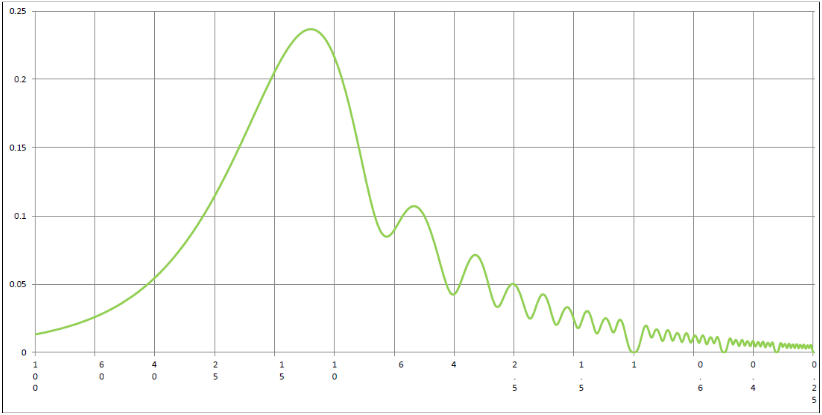
Modulus $\left| \frac{1-FT[a](f)}{2\pi i f \mu_A} - \frac{1-FT[b](f)}{2\pi i f \mu_B} \right|$; phases matter!



Wavelength on a logarithmic scale from 100 years to 0.25 years

NET SPECTRUM = RISK CONTROL PROFILE

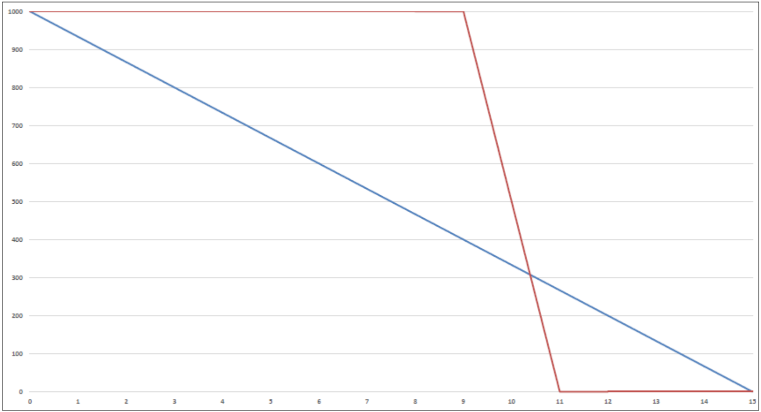
Modulus $\left| \frac{1-FT[a](f)}{2\pi i f \mu_A} - \frac{1-FT[b](f)}{2\pi i f \mu_B} \right|$; phases matter!



- Risk max's at 12 year interest rate cycle (!); next 5.4 $(= \frac{1}{2}\frac{12+9.5}{2})$
- But true risk exposure is the entire area under curve - what if the phases all line up?

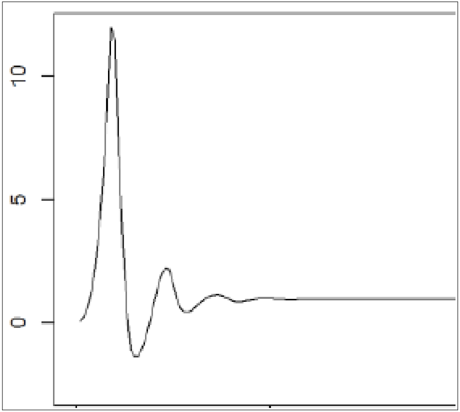
A SLIGHTLY DIFFERENT FINANCIAL INSTITUTION

Steady stream of Liabilities that mature straight-line 9 to 11 years
Assets straight line 0 to 15 years



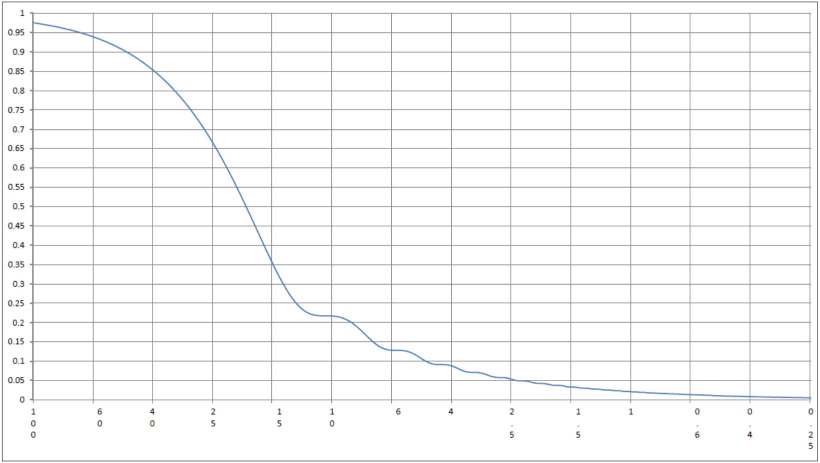
A DIFFERENT GOING CONCERN WITH A-L MISMATCH

Take in a steady stream of 9-11 year st. line liabilities and invest steadily in 0-15 year st.line asset maturities. The following spreads result if interest rates increase steadily:

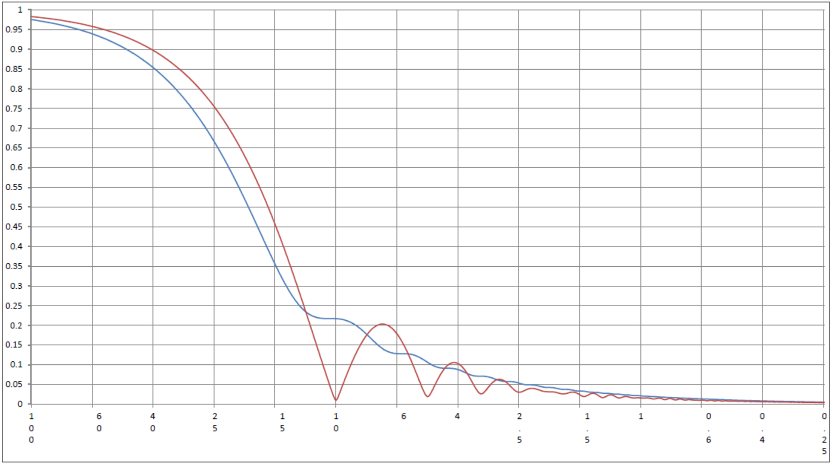


Steady-state profit, not loss

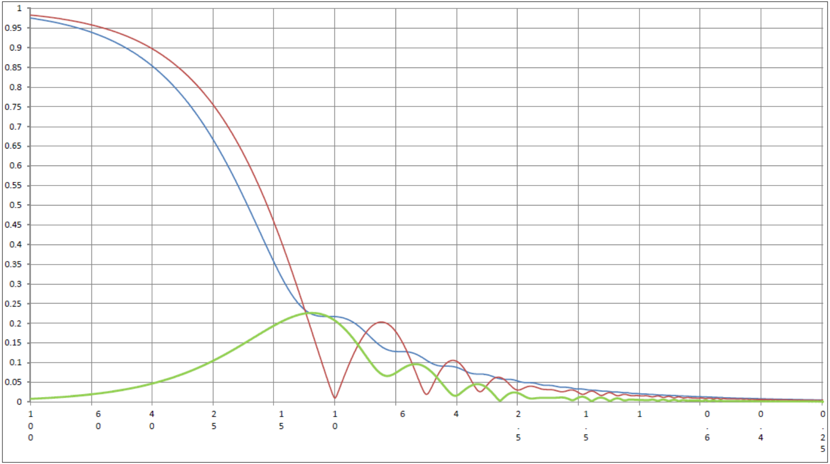
ASSET SPECTRUM (15 Year Straight Line)



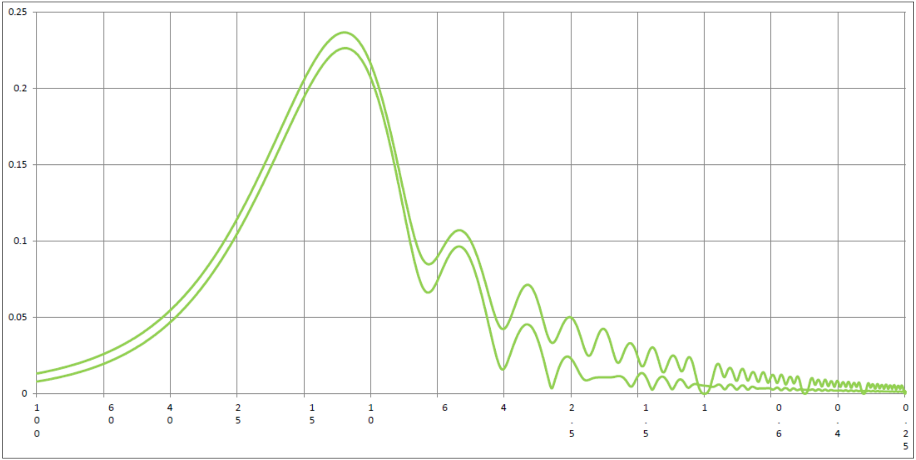
LIABILITY SPECTRUM (9 to 11 Year Straight Line)



NET SPECTRUM



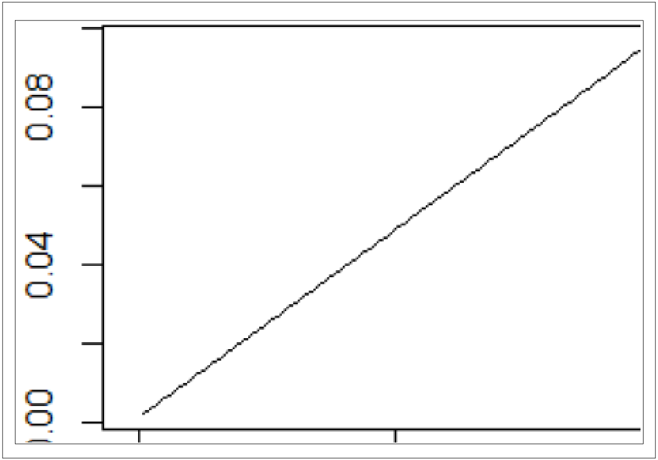
COMPARATIVE NET SPECTRA



PART TWO

WHAT CAN WE SAY ABOUT THE EXTERNAL INTEREST RATE SPECTRUM

FT[r](f) ?



If $r(t)$ is linear with constant slope r' then $\frac{dr}{dt}(t) = r'$, so $r(t) = \left(\Delta * \frac{dr}{dt}\right)(t)$ and $FT[r](f) = FT[\Delta](f)\, FT\left[\frac{dr}{dt}\right](f)$

SPECTRUM OF STRAIGHT LINE INCREASE

$$\begin{aligned} FT\left[r\right](f) &= FT\left[\Delta\right](f)\, FT\left[\frac{dr}{dt}\right](f) \\ &= \left(\frac{1_{f\neq 0}}{2\pi i f} + \frac{1}{2}\delta(f)\right) FT\left[r'\right](f) \\ &= \left(\frac{1_{f\neq 0}}{2\pi i f} + \frac{1}{2}\delta(f)\right) r'\delta(f) \end{aligned}$$

because $FT\left[\text{constant}\right] = \text{constant}\cdot\delta$.
Thus $FT\left[r\right](f) = 0$ unless $f = 0$, in which case it's an impulse function.
For interest rate spread $s(t)$ we know that
$FT\left[s\right](f) = \left[\frac{1-FT[a](f)}{2\pi i f\mu_A} - \frac{1-FT[b](f)}{2\pi i f\mu_B}\right] FT\left[r\right](f)$ but
$\left[\frac{1-FT[a](f)}{2\pi i f\mu_A} - \frac{1-FT[b](f)}{2\pi i f\mu_B}\right] = 0$ when $f = 0$, so we'll need l'Hôpital's help
to unravel $FT\left[s\right](0) = 0\cdot\infty$.

SPECTRUM OF SPREAD FROM STRAIGHT LINE

$FT\left[s\right]\left(0\right) =$

$$= \lim_{f\to 0}\left[\frac{1-FT\left[a\right]\left(f\right)}{2\pi i f\mu_A}-\frac{1-FT\left[b\right]\left(f\right)}{2\pi i f\mu_B}\right]\left(\frac{1_{f\neq 0}}{2\pi i f}+\frac{1}{2}\delta\left(f\right)\right)r'\delta\left(f\right)$$

$$\begin{aligned}
&= \lim_{f\to 0}\left[\frac{1-FT\left[a\right]\left(f\right)}{\left(2\pi i f\right)^2\mu_A}-\frac{1-FT\left[b\right]\left(f\right)}{\left(2\pi i f\right)^2\mu_B}\right]2\pi i f\left(\frac{1_{f\neq 0}}{2\pi i f}+\frac{1}{2}\delta\left(f\right)\right)r'\delta\left(f\right)\\
&= \lim_{f\to 0}\left[\frac{1-FT\left[a\right]\left(f\right)}{\left(2\pi i f\right)^2\mu_A}-\frac{1-FT\left[b\right]\left(f\right)}{\left(2\pi i f\right)^2\mu_B}\right]\left(1_{f\neq 0}+\pi i f\delta\left(f\right)\right)r'\delta\left(f\right)\\
&= \lim_{f\to 0}\left[\frac{1-FT\left[a\right]\left(f\right)}{\left(2\pi i f\right)^2\mu_A}-\frac{1-FT\left[b\right]\left(f\right)}{\left(2\pi i f\right)^2\mu_B}\right]1_{f\neq 0}r'\delta\left(0\right)\\
&= \lim_{f\to 0}\left[\frac{1-FT\left[a\right]\left(f\right)}{\left(2\pi i f\right)^2\mu_A}-\frac{1-FT\left[b\right]\left(f\right)}{\left(2\pi i f\right)^2\mu_B}\right]r'\delta\left(0\right)
\end{aligned}$$

because $\lim_{f\to 0}1_{f\neq 0}=1$

Now use l'Hôpital twice

$$
\begin{aligned}
FT[s](0) &= \lim_{f\to 0}\left[\frac{1-FT[a](f)}{(2\pi i f)^2\mu_A}-\frac{1-FT[b](f)}{(2\pi i f)^2\mu_B}\right]r'\delta(0) \\
&= \lim_{f\to 0}\left[-\frac{\frac{d^2}{df^2}FT[a](f)}{\frac{d^2}{df^2}(2\pi i f)^2\mu_A}+\frac{\frac{d^2}{df^2}FT[b](f)}{\frac{d^2}{df^2}(2\pi i f)^2\mu_B}\right]r'\delta(0) \\
&= \left[-\frac{FT\left[(2\pi i)^2 t^2 a(t)\right](0)}{2(2\pi i)^2\mu_A}+\frac{FT\left[(2\pi i)^2 t^2 b(t)\right](0)}{2(2\pi i)^2\mu_B}\right]r'\delta
\end{aligned}
$$

because for any density $h(t)$ (for example, $h = a$ or b) it's true that $\frac{d}{df}FT[h(t)](f) = -FT[2\pi i t h(t)](f)$. Now 2nd moments appear

$$
\begin{aligned}
FT\left[(2\pi i)^2 t^2 h(t)\right](0) &= \int_{-\infty}^{\infty}(2\pi i)^2 t^2 h(t)\, e^{-2\pi i t\cdot 0}dt \\
&= (2\pi i)^2\int_{-\infty}^{\infty} t^2 h(t)\, dt = (2\pi i)^2\mu'_{H2}
\end{aligned}
$$

SPECTRUM OF SPREAD FROM STRAIGHT LINE

- So
$$FT\left[s\right](0) = -\frac{1}{2}\left[\frac{\mu'_{A2}}{\mu_A} - \frac{\mu'_{B2}}{\mu_B}\right] r'\delta(0)$$
is the impulse function at $f = 0$ that constitutes the entire spectrum of the stable state interest rate spead $s(t)$ caused by a straight line movement of the external interest rate $r(t)$.
- This spectrum implies that $s(t)$ is a constant
$$s(t) = -\frac{1}{2}\left[\frac{\mu'_{A2}}{\mu_A} - \frac{\mu'_{B2}}{\mu_B}\right] r'$$
- If we calculate ratios of second to first moments for the simple asset and liability maturity schedules that we showed earlier, this formula gives exactly the small constant steady-state loss (or gain) that we saw earlier by brute force calculation..
- $\frac{1}{2}\frac{\mu'_{H2}}{\mu_H}$ is the mean of what risk theory calls "the equilibrium distribution" of the distribution H, with density $\frac{S_H(t)}{\mu_H}$, $H = A$ or B

1ST CONCLUSION FOR THE MODULATED PROCESS

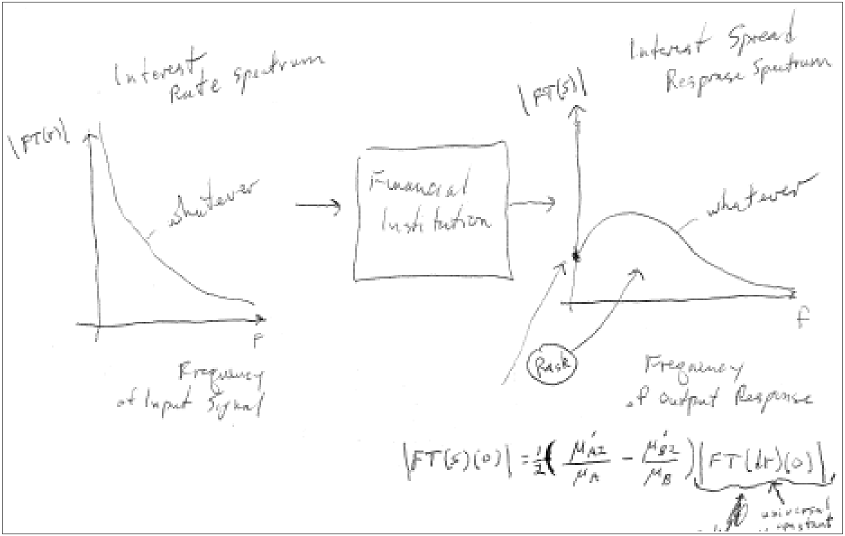
- This rather specialized result at $f = 0$ generalizes to any external interest rate process $r(t)$:

$$|FT[s](0)| = \frac{1}{2}\left|\frac{\mu'_{A2}}{\mu_A} - \frac{\mu'_{B2}}{\mu_B}\right| |FT[dr](0)|$$

where $|FT[dr](0)|$ is the frequency 0 (i.e. drift) component of the Fourier Transform of the process $dr(t)$ that generates $r(t)$.

- If the model has growing new liabilities $\Delta_B(t)$ then distortions of both the means and the second moments must be used
- It should be no surprise that these equilibrium distribution means $\frac{1}{2}\frac{\mu'_{A2}}{\mu_A}$ and $\frac{1}{2}\frac{\mu'_{B2}}{\mu_B}$ can be formally related (a duality) with the traditional duration concept.
- All of the risk area beyond $f = 0$ still remains, however, untouched by this dual version of "duration".

1ST CONCLUSION FOR THE MODULATED PROCESS



NOW THE EXTERNAL INTEREST RATE SPECTRUM

- Assume that Brownian motion drives whatever process we end up with
- $dW(t)$ is the random Brownian increment at each time t; $\{dW(t)\}_{-\frac{1}{2dt}<t\leq\frac{1}{2dt}}$ are $\frac{1}{dt}$ independent $N(0, dt)$ RVs
  - A fixed world runs from $-\frac{1}{2dt}$ to $\frac{1}{2dt}$; Brownian sampling happens to be the mechanism that gives a $dW(t)$ at each of the $\frac{1}{dt}$ values of t which then is fixed once and for all. This amounts to random choice of a single Brownian sample path.
- We are using actual infinitesimals and infinites, following Robinson's Non-Standard Analysis
- Usual Fourier analysis gets around the infinities by weaving together locally finite measures based on the Fourier transform (called the "spectral density") of the autocovariance function of the Brownian process and taking a limit. Brémaud's Fourier Analysis and Stochastic Processes gives a good account of this way of coming at it.
- It seems less distracting to just go ahead and use Robinson's actual infinites and infinitesimals.

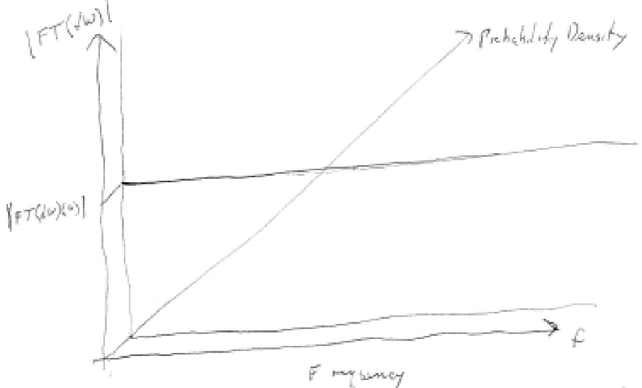
THE SPECTRUM OF dW(t)

- $FT\left[dW\right](f) = \int_{-\infty}^{\infty} e^{-2\pi i f t} dW(t) = e^{2\pi i \phi(f)} N\left(0, \frac{1}{dt}\right)$ with uniformly random phase $\phi(f)$.
- Why is that?
    - $\mathbb{E}\left[FT\left[dW\right](f)\right] = \int_{-\frac{1}{2dt}}^{\frac{1}{2dt}} e^{-2\pi i f t} \mathbb{E}\left[dW(t)\right] = 0$
    - $\mathbb{E}\left[\left|FT\left[dW\right](f)\right|^2\right] = \mathbb{E}\left[FT\left[dW\right](f)\, FT\left[dW\right](-f)\right]$ (conjugates)
    - $= \int_{-\frac{1}{2dt}}^{\frac{1}{2dt}} \mathbb{E}\left[dW^2(t)\right]$ ($dW(t_1)$ independent of $dW(t_2)$ for $t_1 \neq t_2$)
    - $= \int_{-\frac{1}{2dt}}^{\frac{1}{2dt}} dt = \frac{1}{2dt} - \left(-\frac{1}{2dt}\right) = \frac{1}{dt}$

THE SPECTRUM OF dW(t)

- We even can identify the specific $N\left(0, \frac{1}{dt}\right)$ RV in the expression for $FT\left[dW\right](f)$
- $\left|FT\left[dW\right](f)\right|^2 = FT\left[dW\right](f)\, FT\left[dW\right](-f) = \int_{-\frac{1}{2dt}}^{\frac{1}{2dt}} dW^2(t) = \left(FT\left[dW\right](0)\right)^2$, where the final two $=$'s are because $dW(t_1)$ independent of $dW(t_2)$ for $t_1 \neq t_2$
- So $\left|FT\left[dW\right](f)\right| = \left|FT\left[dW\right](0)\right|$, same value for all f, and the phase of $FT\left[dW\right](f)$ is totally random in f (and unknowable)
- $FT\left[dW\right](f) = e^{2\pi i \phi(f)} FT\left[dW\right](0)$ with $\phi(f)$ a uniformly random phase and $FT\left[dW\right](0)$ a random real number, drawn from $N\left(0, \frac{1}{dt}\right)$
- $\phi(f)$ and $FT\left[dW\right](0)$ are unknowable, random, and fixed (randomly) for all time when we drew our $\left\{dW(t)\right\}_{-\frac{1}{2dt}<t\leq\frac{1}{2dt}}$ world, i.e. by the random Brownian sample path we chose (see picture)

SPECTRUM OF THE BROWNIAN INCREMENT



- dW(t) has all Fourier frequencies equally represented at one random value
- Its phases are completely randomized

THE SPECTRUM OF dW(t)

- This unknowable $FT\left[dW\right](0)$ is a little like renormalization in physics. It sounds strange but it works since everything we actually observe will just be relative to this unknowable thing $FT\left[dW\right](0)$
- Remember, we promised that true risk models will be very different from our usual pricing, planning, forecasting, and reserving models!
- All of the Fourier frequencies are equally represented in $FT\left[dW\right](f)$
- Random walk comes from randomized phase relationships.

THE SPECTRUM OF BROWNIAN MOTION W(t)

Now we know everything we need to get from $FT[dW]$ to $FT[W]$

$$W(t) = (\Delta * dW)(t) \text{ so}$$

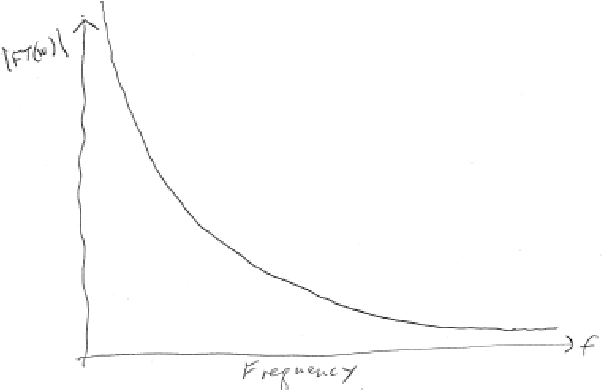
$$\begin{aligned}
FT[W](f) &= FT[\Delta * dW](f) \\
&= FT[\Delta](f)\, FT[dW](f) \\
&= \left(\frac{1_{f\neq 0}}{2\pi i f} + \frac{1}{2}\delta(f)\right) e^{2\pi i \phi(f)} FT[dW](0)
\end{aligned}$$

where $\phi(f)$ is a totally random phase. This gives

$$\begin{aligned}
|FT[W](f)| &= \left|\frac{1_{f\neq 0}}{2\pi i f} + \frac{1}{2}\delta(f)\right| |FT[dW](0)| \\
&= \frac{|FT[dW](0)|}{2\pi |f|} 1_{f\neq 0} + \frac{|FT[dW](0)|}{2}\delta(f)
\end{aligned}$$

with phases totally randomized (note: last step required $1_{f\neq 0} \cdot \delta(f) = 0$). The random phases are what makes a random walk random.

THE SPECTRUM OF BROWNIAN MOTION



- $|FT(w)(f)| = \frac{|FT(dw)(0)|}{2\pi f} \cdot 1_{f \neq 0} + \frac{1}{2}|FT(dw)(0)|\,\delta(f)$
- Remember: $|FT(dW)(0)|$ is a fixed (random) value
- Phases of $FT(w)(f)$ are totally randomized

MEAN-REVERTING GEOMETRIC BROWNIAN MOTION

- The external interest rate $r\left(t\right)$ is likely to have a mean-reversion of some kind and a zero-avoidance of some kind.
- An example is mean-reverting Geometric Brownian Motion (Black-Karisinski). It looks like:

$$dr(t) = (R(\ln T - \ln r(t)) + \tfrac{1}{2}\sigma^2 - D)r(t)dt + \sigma r(t)dW(t)$$

- Decoding:
  - σ is a volatility factor that brings the random $dW(t)$ into play
  - R is a mean-reversion factor that tends to pull $r(t)$ back toward a target level T
  - D is a drift compensation factor that will make $\mathbb{E}\left[r(t)\right] = T$
  - This assumes we start process at $r(-\frac{1}{2dt}) = T$
  - Drift compensation is very important when $R = 0$
  - $r(t)dW(t)$ and $r(t)dt$ keep the process from hitting $r(t) = 0$

So, just like we did to go from $FT\left[dW\right]$ to $FT\left[W\right]$

- $r(t) = (\Delta * dr)(t)$
- $FT[r](f) = FT[\Delta](f)FT[dr](f)$, and off we go
- Alas, $dr(t) = (R(\ln T - \ln r(t)) + \frac{1}{2}\sigma^2 - D)r(t)dt + \sigma r(t)dW(t)$
- So $FT[dr](f)$ has $FT[r](f)$ in it and also $FT[\ln r]$ in it
- So the expression for $FT[r](f)$ is circular
- We have to do some real work to get a solution
- Strategy is to set up equations from which we can eliminate $FT\left[dr\right]$ and $FT\left[d\ln r\right]$

- Set up an integration by parts:

$$d[e^{-2\pi ift}r(t)] = e^{-2\pi ift}dr(t) - 2\pi ife^{-2\pi ift}r(t)dt$$
$$e^{-2\pi ift}dr(t) = 2\pi ife^{-2\pi ift}r(t)dt + d[e^{-2\pi ift}r(t)]$$

- Integrate from $-\frac{1}{2dt}$ to $\frac{1}{2dt}$

$$FT[dr](f) = 2\pi ifFT[r](f) + e^{-2\pi if\frac{1}{2dt}}r(\frac{1}{2dt}) + e^{2\pi if\frac{1}{2dt}}r(-\frac{1}{2dt})$$

- We can rigorously ignore the two infinite spinning variables if we stay away from $f = 0$ which we will do from now on
  - It requires an argument, but isn't worth belaboring the details
- This leaves us with Equation I

$$FT[dr](f) = 2\pi ifFT[r](f)$$

- An exactly parallel integration by parts gives Equation II

$$FT[d\ln r](f) = 2\pi ifFT[\ln r](f)$$

MEAN-REVERTING GEOMETRIC BROWNIAN MOTION

Now we look for two other expressions for $FT[dr](f)$ and $FT[d\ln r](f)$

- Itô Lemma gives $d\ln r(t) = \frac{1}{r(t)}dr(t) - \frac{1}{2}\frac{1}{r^2(t)}dr(t)dr(t)$
- Make the definition (MRGBM)

$$d\ln r(t) = (R(\ln T - \ln r(t)) - D)dt + \sigma dW(t)$$

- Equate, rearrange, and multiply through by $r(t)$

$$dr(t) = (R(\ln T - \ln r(t)) - D)r(t)dt + \sigma r(t)dW(t) + \frac{1}{2}\frac{1}{r(t)}dr(t)dr(t)$$

- $dr(t)dr(t) = \sigma^2 r^2(t)dt$ by the usual Itô calculus rules
- So $dr(t) = (R(\ln T - \ln r(t)) + \frac{1}{2}\sigma^2 - D)r(t)dt + \sigma r(t)dW(t)$
- Integrate against $e^{-2\pi i f t}$ to get Equation III

$$FT\left[dr\right](f) =$$
$$(R\ln T + \frac{1}{2}\sigma^2 - D)FT\left[r\right](f) - R\cdot FT\left[r\ln r\right] + \sigma FT\left[rdW\right](f)$$

MEAN-REVERTING GEOMETRIC BROWNIAN MOTION

- Equation III

$$FT\left[dr\right](f) =$$
$$(R\ln T + \tfrac{1}{2}\sigma^2 - D)FT\left[r\right](f) - R\cdot FT\left[r\ln r\right] + \sigma FT\left[rdW\right](f)$$

- Go back to the MRGBM definition

$$d\ln r(t) = (R(\ln T - \ln r(t)) - D)dt + \sigma dW(t)$$

- Integrate against $e^{-2\pi ift}$ to get Equation IV

$$FT\left[d\ln r\right](f) = \ (R\ln T - D)\delta(f) - R\cdot FT\left[\ln r\right](f) + \sigma FT\left[dW\right](f)$$

- An important step in there was

$FT\left[(R\ln T - D)\right](f) = (R\ln T - D)\delta(f)$ which is true for any constant.

- Now we can equate RHS Equation I and Equation III and RHS Equation II and Equation IV to eliminate $FT\left[dr\right]$ and $FT\left[d\ln r\right]$

- Equating I and III
  $2\pi i f FT[r](f) = (R\ln T + \frac{1}{2}\sigma^2 - D)FT[r](f) - R\cdot FT[r\ln r] + \sigma FT[rdW](f)$

$FT[r](f) = \frac{\sigma FT[rdW](f) - R\cdot FT[r\ln r]}{2\pi i f - (R\ln T + \frac{1}{2}\sigma^2 - D)}$ & use $FT[ab] = FT[a] * FT[b]$

$FT[r](f) = \frac{\sigma(FT[r]*FT[dW])(f) - R(FT[r]*FT[\ln r])(f)}{2\pi i f - (R\ln T + \frac{1}{2}\sigma^2 - D)}$

- Now equate II and IV

$2\pi i f FT[\ln r](f) = (R\ln T - D)\delta(f) - R\cdot FT[\ln r](f) + \sigma FT[dW](f)$

$FT[\ln r](f) = \frac{\sigma FT[dW](f) + (R\ln T - D)\delta(f)}{2\pi i f + R}$

- Substitute into the expression for $FT[r](f)$

$FT[r](f) = \frac{\sigma(FT[r]*FT[dW])(f) - R\left[\left(FT[r]*\frac{\sigma FT[dW]}{2\pi i f + R}\right)(f) + \frac{FT[r](f)(R\ln T - D)}{R}\right]}{2\pi i f - (R\ln T + \frac{1}{2}\sigma^2 - D)}$, & solve

$FT[r](f) = \frac{\sigma\left[(FT[r]*FT[dW])(f) - (FT[r]*(\frac{R}{2\pi i f + R}FT[dW]))(f)\right]}{2\pi i f - \frac{1}{2}\sigma^2}$

- This still appears to be circular

$$FT[r](f) = \frac{\sigma\left[(FT[r]*FT[dW])(f) - (FT[r]*(\frac{R}{2\pi if+R}FT[dW]))(f)\right]}{2\pi if - \frac{1}{2}\sigma^2}$$

- But there is a big difference: $FT[r]$ appears on the RHS only within convolutions that involve all values of $FT[r](g)$, not the specific value $FT[r](f)$ on the LHS (although 2nd term concentrates there.)
- Moreover, the convolutions are against the randomized phases in $FT\left[dW\right](g) = e^{2\pi i\phi(g)}FT\left[dW\right](0)$
- This takes some delicate analysis but allows an approximation:

$$FT[r](f) \approx \frac{\sigma\left[\tilde{r}(f)FT[dW](0) - \frac{1}{2}R\cdot FT[r](f)\right]}{2\pi if - \frac{1}{2}\sigma^2}$$

- with $\tilde{r}(f)$ a random phase on $|\tilde{r}(f)|$ a random draw (for each f) from the set of "$r(0)$" over all the random $\{dW(t)\}_{-\frac{1}{2dt}<t\leq\frac{1}{2dt}}$ worlds (sample paths) from which we first sampled
- think of $FT\left[r\right] * e^{2\pi i\phi(g)}$ as integrating an alternate version of $FT\left[r\right]$ with randomly scrambled phases $\int FT\left[\bar{r}\right](g)\,dg = \bar{r}(0) \triangleq \tilde{r}(f)$

- The approximation comes in the second term

$$\left(FT\left[r\right] * \left(\frac{R}{2\pi if+R}FT\left[dW\right]\right)\right)(f) \approx \frac{1}{2}R \cdot FT\left[r\right](f)$$

- The idea is that in the convolution integral the term $FT\left[r\right](g)\frac{R}{2\pi i(f-g)+R}$ concentrates on values near $FT\left[r\right](f)$, so $FT\left[r\right](f)$ approximately factors out; $\frac{1}{2}R$ is the value of the remaining integral
- Now $FT[r](f) \approx \frac{\sigma\left[\tilde{r}(f)FT[dW](0) - \frac{1}{2}R \cdot FT[r](f)\right]}{2\pi if - \frac{1}{2}\sigma^2}$ solves to
- $FT[r](f) \approx \frac{\sigma\tilde{r}(f)FT[dW](0)}{2\pi if - \frac{1}{2}\sigma(\sigma - R)}$
- This is the Fourier Spectrum for the external interest rate $r(t)$
- The randomness in the modulus and phase of $\tilde{r}(f)$ is implied in our original random choice of a $\{dW(t)\}_{-\frac{1}{2dt}<t\leq\frac{1}{2dt}}$ world, a sample path
- Let's see what it looks like and how our institution's risk profile fares

SPECTRUM FOR EXTERNAL INTEREST RATE

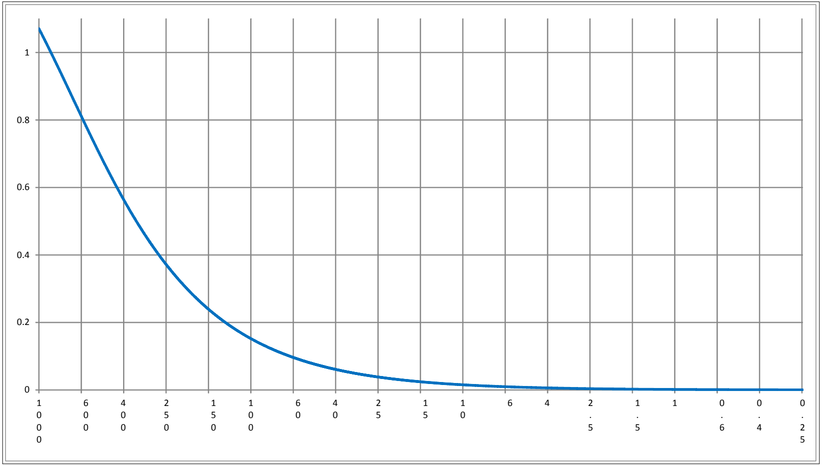
$$FT[r](f) \approx \frac{\sigma \tilde{r}(f) FT[dW](0)}{2\pi i f - \frac{1}{2}\sigma(\sigma - R)}$$

- We need parameters.
- 60 years' history of the 10 year US Treasure rate suggest $\sigma = .16$ and $R = .08$ and $\mathbb{E}\left[|\tilde{r}(f)|\right] = .06$ with standard deviation .028

$$|FT[r](f)| \approx \frac{.0096 FT[dW](0)}{2\pi i f - .0064} \pm \frac{.00448 FT[dW](0)}{2\pi i f - .0064}$$

- $\pm \frac{.00448 FT[dW](0)}{2\pi i f - .0064}$ is a random density (in f) of actual values for $|FT[r](f)|$, not some approximation error.
- This expression for $FT[r](f)$ is completely useless for forecasting because of the randomized phases and, to some extent, the random $\pm$ density. You can't even estimate the current phases because you need $t \to \infty$ to calculate a phase.
- But $|FT[r](f)|$ carries the full potential risk exposure to random interest rate fluctuation. Here is a graph of the central values of the density:

SPECTRUM FOR EXTERNAL INTEREST RATE



Wider scale (1000 year wave length) to see inflection point

MODULATED RISK SPECTRUM

- Now we need to multiply by the risk-control spectrum in our asset-liability structure

$$\left(\frac{1-FT[a](f)}{2\pi i f \mu_A} - \frac{1-FT[b](f)}{2\pi i f \mu_B}\right)$$

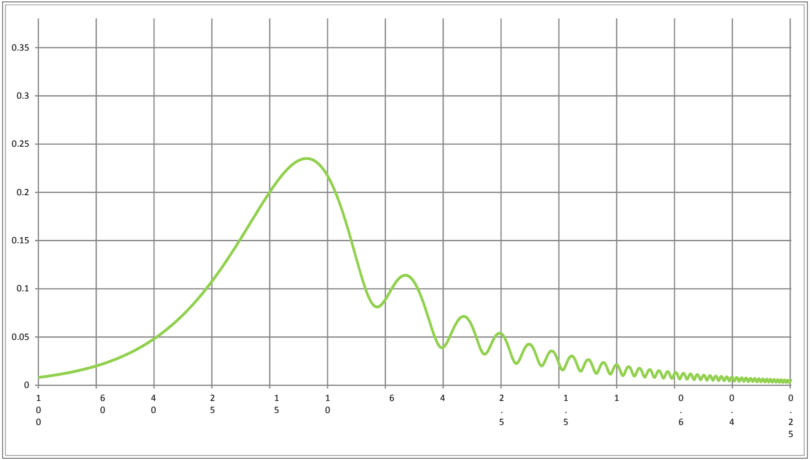
- $FT[s](f) = \left(\frac{1-FT[a](f)}{2\pi i f \mu_A} - \frac{1-FT[b](f)}{2\pi i f \mu_B}\right) FT[r](f)$ shows the modulated Fourier Spectrum for the interest rate spread $s(t)$ that results when the external interest rate $r(t)$ hits our asset and liability structure
- Again it is useless for forecasting but its modulus expresses the full potential interest rate fluctuation exposure of the asset and liability structure

$$|FT[s](f)| \approx \left|\left(\frac{1-FT[a](f)}{2\pi i f \mu_A} - \frac{1-FT[b](f)}{2\pi i f \mu_B}\right)\left(\frac{.0096 FT[dW](0)}{2\pi i f - .0064} \pm \frac{.00448 FT[dW](0)}{2\pi i f - .0064}\right)\right|$$

- Let's illustrate with the first asset-liability structure we looked at

RISK CONTROL PROFILE

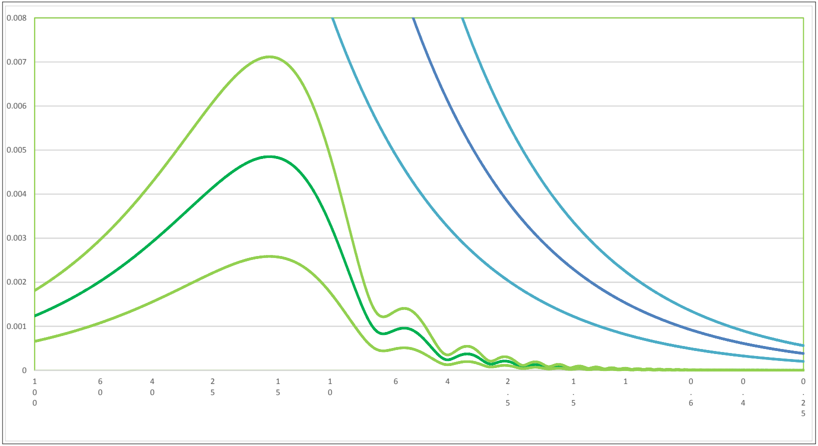
Modulus $\left| \frac{1 - FT[a](f)}{2\pi i f \mu_A} - \frac{1 - FT[b](f)}{2\pi i f \mu_B} \right|$



Back to 100 year wave length scale

RESULTING MODULATED RISK SPECTRUM

$$\left|\left(\frac{1-FT[a](f)}{2\pi i f\mu_A}-\frac{1-FT[b](f)}{2\pi i f\mu_B}\right)\left(\frac{.0096FT[dW](0)}{2\pi i f-.0064}\pm\frac{.00448FT[dW](0)}{2\pi i f-.0064}\right)\right|$$



Insensitve to our alternative model; relatively insensitive to R

CONCLUDING REMARKS

- Low frequencies dominate even in Geometric and/or Mean-Reverting
- Banks avoid those frequencies; bond market hedges them (but fragile hedges can rupture and $FT\left[r\right]\left(f\right)$ gives huge stress at small f!)
- Justifies traditional insurance attention to duration as first-order priority
- Our first order $\frac{1}{2}\left|\frac{\mu'_{A2}}{\mu_A} - \frac{\mu'_{B2}}{\mu_B}\right|$ at $f = 0$ may be an improvement
- Still just a toy:
- Interest rate model should be at least 4 dimensional (time, maturity, currency, credit); 4 dimensional Fourier analysis can be done.
- Even a time-only model probably should be multi-scaled (more than one σ? beyond FT to wavelets?)
- Modulation structure should reflect time-varying asset-liability strategies (treat this model as a local tangent and integrate?); we already have exact formulas for a constant growth strategy

TECHNICAL NOTE

- In $FT[r](f) \approx \frac{\sigma \tilde{r}(f) FT[dW](0)}{2\pi i f - \frac{1}{2}\sigma(\sigma - R)}$, what happened to the T and D parameters in the Mean Reverting Geometric Brownian Motion $dr(t) = (R(\ln T - \ln r(t)) + \frac{1}{2}\sigma^2 - D) r(t) dt + \sigma r(t) dW(t)$?
- It turned out that, from the Fourier Analysis perspective, the only roles T and D played were to prevent (almost always) random excursions from escaping to infinity and to ensure that $\mathbb{E}[r(t)] = T$, for some value T (rather than vanishing or oscillating.)
- Given σ and R (even the case $R = 0$), the suppression of unbounded random excursions together with any particular $\mathbb{E}[r(t)] = T$ turned out to have the same effect on the oscillation structure encoded in $FT[r](f)$ as for any other value of T and the only role for D was to make the suppression precise, given σ, R, and T.
- Having performed their jobs, T and D disappear from the oscillation scene, their presence encoded in the σ and R that determined them together with the precise absence of unbounded excursions, precise in the sense that $\mathbb{E}[r(t)]$ neither vanishes nor oscillates nor explodes.

References

- Stationary Immunization Theory, 1998 International Congress of Actuaries

link at bottom of
www.math.uconn.edu/~bridgeman/Papers_and_Presentations/index.htm
Gives details of the modulation process; but anyone who has programmed an ALM model has done this, whether they know it or not

- For the Fourier Analysis - any good text:
- for theory I like
  - Rudin's Real and Complex Analysis
  - Brémaud's Fourier Analysis and Stochastic Processes
- for visualization I like
  - Brigham's Fast Fourier Transform
  - Meikle's A New Twist To Fourier Transforms
- For actual infinites and infinitesimals
  - Robinson's Non-Standard Analysis

THANKS