A Dual Version of Asset-Liability Risk Modeling

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INTRODUCTION - RISK MODELING

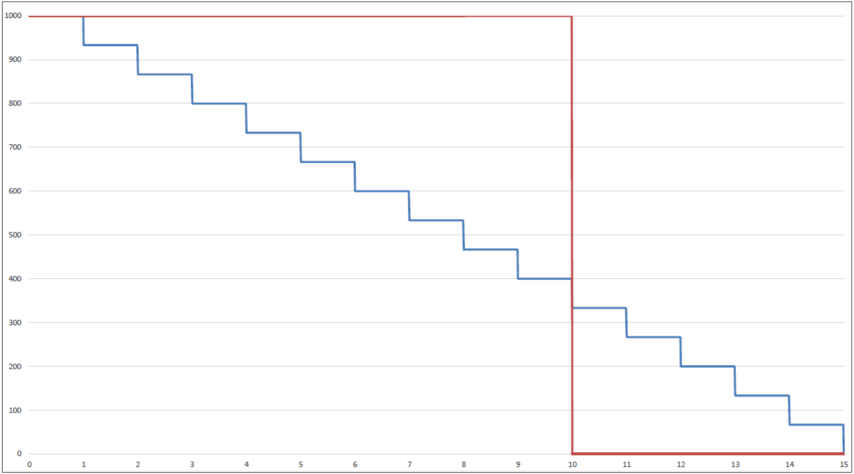
- Often, we model risk with the same models as for pricing, planning and forecasting
- In risk modeling we just assume extreme inputs, or look at the tails of random outcomes, or maybe even use statistical extreme value theory
- But we fine-tune pricing, planning and forecasting models to work well on normal inputs; can this foreclose good risk modeling in a holistic sense?
- Maybe risk needs radically different models; but somehow related to normal models, grounded in them, perhaps a formal duality?

MODELING ASSET-LIABILITY INTEREST RATE RISK

- Traditionally we model known cash flows and take present values - a balance sheet view
  - Ignore future cash unless implied by balance sheet
  - Test future interest rates' effect on present values:
    - duration/convexity etc.
    - stochastic future interest rates
    - risk-neutral calibrations to market values
- A radically different model could start with going concern assumptions - an income view
  - Embrace future cash flow - on some normalized, on-going basis
  - Test future interest rates' effect on future spreads between asset earnings and liability requirements
- Strictly a work-in-progress: what tools would give a dual model to the balance sheet?

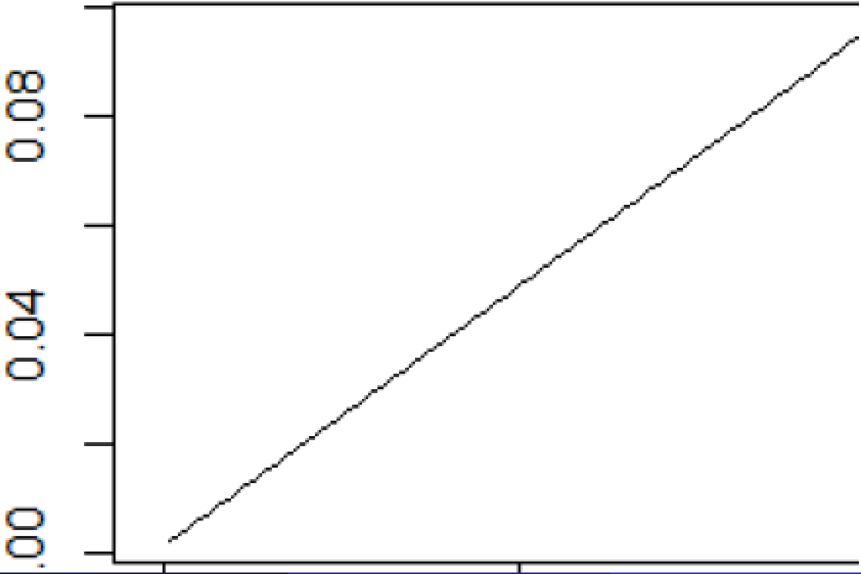
A SIMPLE GOING CONCERN WITH A-L MISMATCH

Take in a steady stream of 10-year bullet liabilities and invest steadily in 15 year ladder asset maturities.



A SIMPLE GOING CONCERN WITH A-L MISMATCH

Let interest rates increase steadily:



A SIMPLE GOING CONCERN WITH A-L MISMATCH

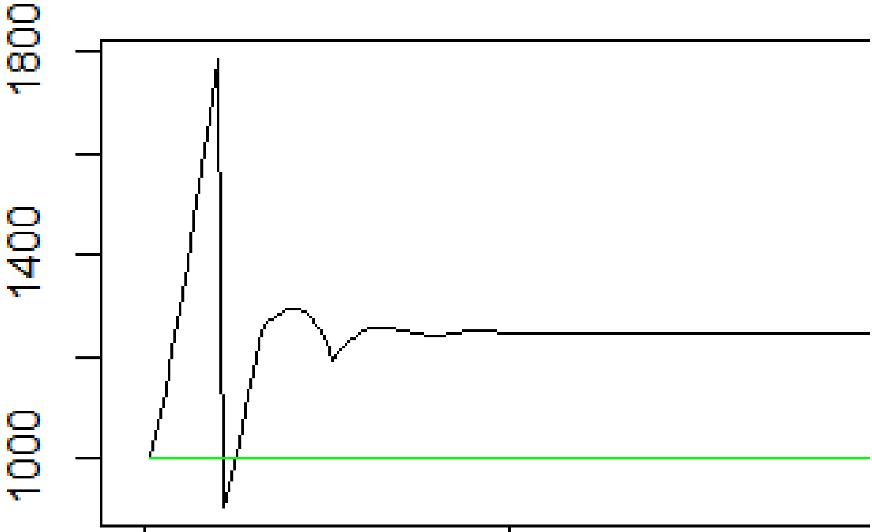
Take in a steady stream of 10-year bullet liabilities and invest steadily in 15 year ladder asset maturities. The following spreads result if interest rates increase steadily:

Linear Inputs Gave Oscillating Output - What's Going On?

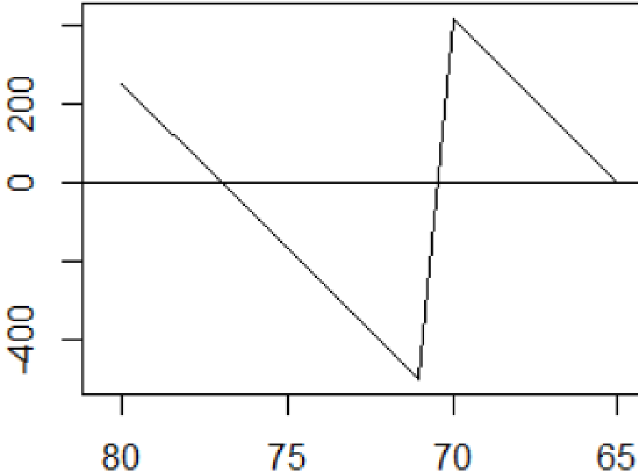
- In signal analysis – electronics or optics – this would be the clue to something called an "Edge Effect" or a "Gibbs' Phenomenon"
- Essentially, the linear inputs have encountered or involve some sort of underlying process that combines a jumble of hidden oscillating pieces
- The beginning (or ending) of the linear inputs splatters the underlying oscillations into the output piecemeal so we can see some of them
- Fourier analysis is the technical tool to explore this
- REALLY?? For ALM Work?
- Maybe there is a simpler explanation?

Linear Input Gave Oscillating Output - Why?

Maturity mismatch creates investment mismatch in going concern – early asset maturities need reinvesting

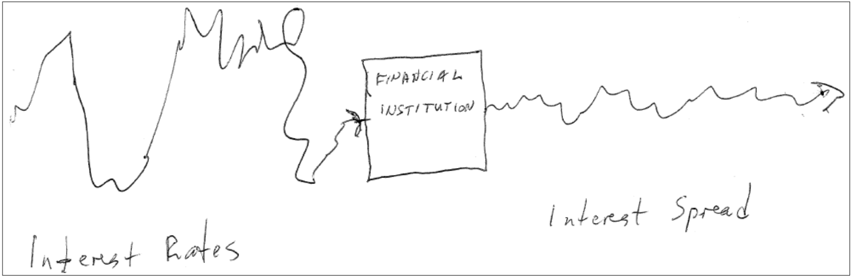


And looking backwards even in steady state the net survivors still at each past rate oscillates, too – 12 and 9.5 year cycles observable

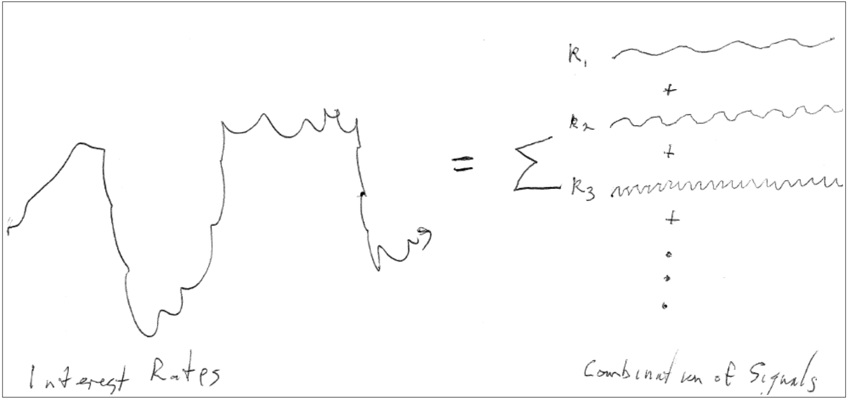


OK, BUT THIS STILL CRIES OUT FOR FOURIER ANALYSIS

A financial institution is a receiver of a stream of interest rates that modulates them into an output stream of interest spreads (gain/loss)

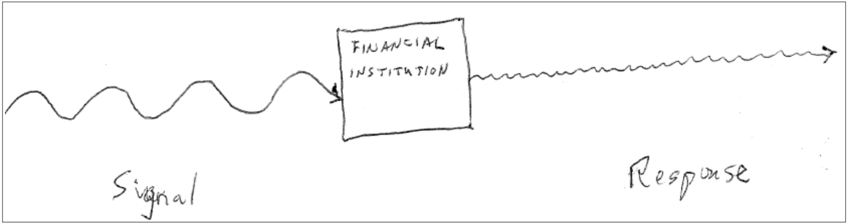


The interest rate stream consists of component signals



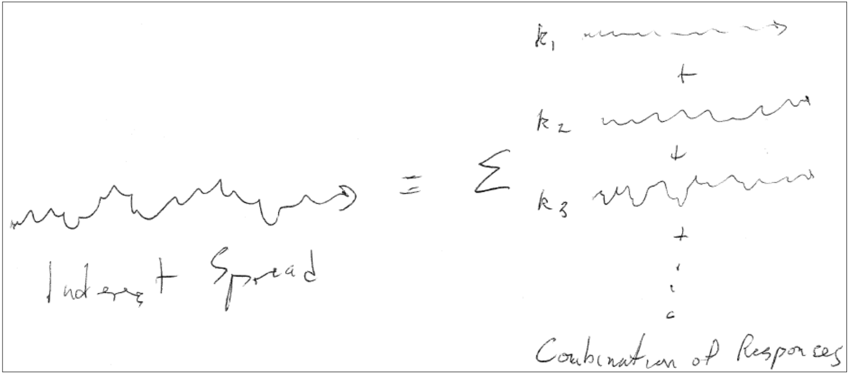
THIS STILL CRIES OUT FOR FOURIER ANALYSIS

Suppose we know how the financial institution modulates each component of the input signal into an output response

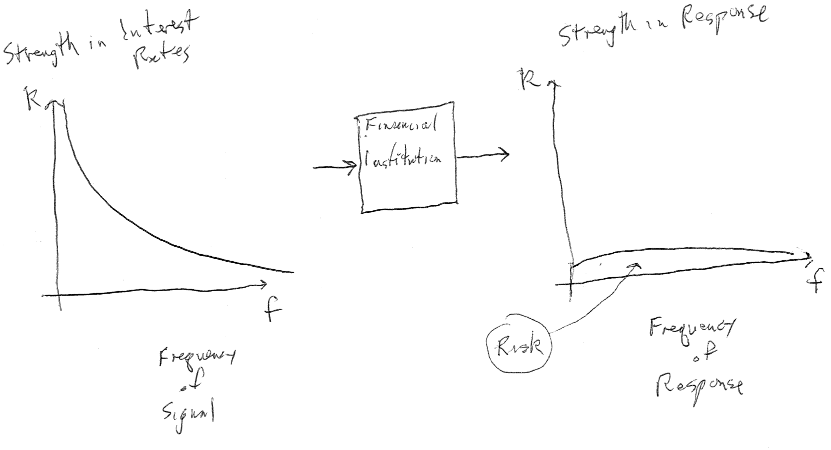


THIS STILL CRIES OUT FOR FOURIER ANALYSIS

Then we can reconstruct the total response (the spread) to the original interest rate stream



FOURIER ANALYSIS JUST CODIFIES THIS



- Strengths are given by $|FT|$, the modulus of the Fourier Transform
- Interest Rate Risk is the <u>area</u>

THIS IS THE DUAL VIEW OF INTEREST RATE RISK

- It looks at the institutional response to the entire spectrum of interest rate volatility
  - Dual to duration, etc. which puts most focus on the lowest frequency component(s) of the interest rate spectrum
- It looks at the going-concern interest rate spread (income statement)
  - Dual to the balance-sheet view of traditional immunization
  - Like the duality between position and momentum in physics
- Area under the spectrum is the proper risk measure
  - If random phases align against you the whole area contributes to your woe

CAN'T GET THIS FROM YOUR NORMAL MODELS

(Or at least not directly from them)

WHAT WE NEED IS

- A model of the external interest rate spectrum
  - As an abstract random phenomenon, not just past x years or a closed time series
  - FORECASTING DISTRACTS FROM RISK ANALYSIS!
- A model of the modulation process
  - Unique to each financial institution
  - Applicable to all possible external signals
- And, of course, the Fourier Analysis technique

Fourier Transform

- Given a function $r(t)$, say the interest rate over time, you can write it as a sum of oscillating components

$$r(t) = \int_{-\infty}^{\infty} FT[r](f)\, e^{2\pi i f t} df$$

where $FT[r](f)$, called the Fourier Transform of $r(t)$ at f, determines the component of $r(t)$ that oscillates with frequency f.

- The standard oscillation with frequency f is $e^{2\pi i f t}$.
- Note, $e^{2\pi i f t}$ is a complex number at each time t that corkscrews around the complex unit circle as time passes.
- The frequency f determines how fast and in which direction it spins.

Fourier Transform

$$r(t) = \int_{-\infty}^{\infty} FT[r](f)\, e^{2\pi i f t} df$$

- So $FT[r](f)$ is a complex number
- Whose modulus $|FT[r](f)|$ at each frequency f tells us how large $e^{2\pi i f t}$ looms inside $r(t)$ for that f
- Whose phase at each frequency f tells us how much the version of $e^{2\pi i f t}$ inside $r(t)$ is rotated from its usual starting point (at $t = 0$) for that f.
- With all these complex numbers spinning around, how can we get a real function $r(t)$ back out of the formula?
- It just requires that $FT[r](f)$ and $FT[r](-f)$ be complex conjugates for each frequency f.

Fourier Transform Properties

- There is a formula for $FT[r](f) = \int_{-\infty}^{\infty} r(t)\, e^{-2\pi i f t} df$
- If $a(t)$ and $b(t)$ are two functions of t, and k and j are constants, then for each frequency f we have

$$FT[ka + jb](f) = kFT[a](f) + jFT[b](f)$$

- If we define $(a * b)(t) = \int_{-\infty}^{\infty} a(t-s)\, b(s)\, ds$ (the "convolution") then

$$FT[a * b](f) = FT[a](f)\, FT[b](f)$$

- If we define $\Delta(t) = 1$ for $t \geq 0$ and $= 0$ for $t < 0$ then

$$FT[\Delta](f) = \tfrac{1}{2\pi i f} 1_{f \neq 0} + \tfrac{1}{2}\delta(f),\ \text{for } \delta = \text{impulse at } 0$$

- Note: $(\Delta * b)(t) = \int_{-\infty}^{t} b(s)\, ds = B(t)$, etc. for any b, B

START WITH THE MODULATION PROCESS

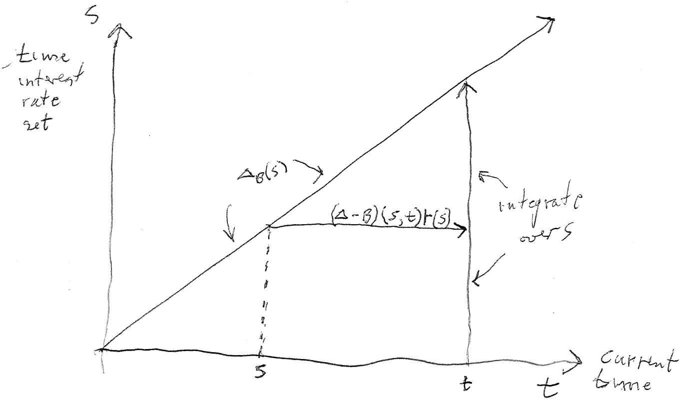
Let $r(s)$ = the interest rate at time s
$\Delta_B(s)$ = new Liabilities taken on at time s
(Assume $\Delta_B(s)$ takes a simple going-concern form)
$B(s,t)$ = Liabilities matured out of $r(s)$ by time t
$b(s,t) = \frac{\partial}{\partial t}B(s,t)$ the rate of Liabilities maturing out of $r(s)$ at time t
$\Delta(s) = 1$ for $s \geq 0$ and $= 0$ for $s < 0$
$(\Delta - B)(s,t)$ = Liabilities still owed $r(s)$ at time t = survival function of $B(s,t)$ viewed as a cdf.

This gives a crude going-concern model of interest requirements on the Liabilities

Rate of interest required (at time t) $= \frac{[(\Delta - B) * (r\Delta_B)](t)}{[(\Delta - B) * \Delta_B](t)}$

START WITH THE MODULATION PROCESS

Interest requirements on the Liabilities (going concern)



$\Delta_B(s)$ is given

$$\text{Rate of Liability Interest} = \frac{\int_0^t r(s)(\Delta-B)(s,t)\Delta_B(s)\,ds}{\int_0^t (\Delta-B)(s,t)\Delta_B(s)\,ds} = \frac{[(\Delta-B)*(r\Delta_B)](t)}{[(\Delta-B)*\Delta_B](t)}$$

That's a generalization of the usual definition of convolution and it won't be commutative

A Few Other Things We Need

$a^{*k} = a * a * ... * a$ k times makes sense and we will use it

When we need it, $\delta =$ Dirac delta function (impulse at 0)

In particular, $a^{*(0)} = \delta$

Also, $FT[\delta](f) = 1$ for all f

START WITH THE MODULATION PROCESS

If $\Delta_A(s)$ = new Assets taken on at time s then $\Delta_A(s)$ will be a function of everything else in the model

In fact, $\Delta_A(s) = \left(\left(\sum_{k=0}^{\infty} a^{*k}\right) * (\delta - b) * \Delta_B\right)(s)$

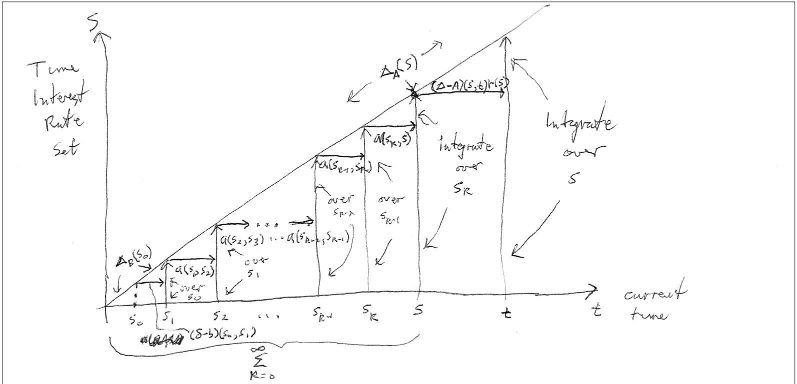
$A(s,t)$ = Assets matured out of $r(s)$ by time t

$a(s,t) = \frac{\partial}{\partial t}A(s,t)$ the rate of Assets maturing out of $r(s)$ at time t

$(\Delta - A)(s,t)$ = Assets still earning $r(s)$ at time t = survival function of $A(s,t)$ viewed as a cdf.

$$\text{Rate of interest available (at time t)} = \frac{\left[(\Delta - A) * \left(r\left[\left(\sum_{k=0}^{\infty} a^{*k}\right) * (\delta - b) * \Delta_B\right]\right)\right](t)}{\left[(\Delta - A) * \left(\left(\sum_{k=0}^{\infty} a^{*k}\right) * (\delta - b) * \Delta_B\right)\right](t)}$$

Interest generated by the Assets (going concern)



$$\Delta_A(s) = \sum_{k=0}^{\infty} \int_0^{s} a(s_k,s)\int_0^{s_k} a(s_{k-1},s_k)\int_0^{s_{k-1}} \cdots a(s_2,s_3)\int_0^{s_2} a(s_1,s_2)\int_0^{s_1} (\delta-b)(s_0,s_1)\,\Delta_B(s_0)\,ds_0\,ds_1\cdots ds_{k-2}\,ds_{k-1}\,ds_k$$

$$\text{Rate of Asset Interest} = \frac{\int_0^t r(s)(\Delta-A)(s,t)\,\Delta_A(s)\,ds}{\int_0^t (\Delta-A)(s,t)\,\Delta_A(s)\,ds} = \frac{[(\Delta-A)*(r\Delta_A)](t)}{[(\Delta-A)*\Delta_A](t)}$$

$$\Delta_A(s) = \left[\left(\sum_{k=0}^{\infty} a^{*k}\right)*(\delta-b)*\Delta_B\right](s)$$

START WITH THE MODULATION PROCESS

- Going concern interest rate spread s at time t is the difference

$$s\left(t\right)=\frac{\left[\left(\Delta-A\right)*\left(r\left[\left(\sum_{k=0}^{\infty}a^{*k}\right)*\left(\delta-b\right)*\Delta_B\right]\right)\right]\left(t\right)}{\left[\left(\Delta-A\right)*\left(\left(\sum_{k=0}^{\infty}a^{*k}\right)*\left(\delta-b\right)*\Delta_B\right)\right]\left(t\right)}-\frac{\left[\left(\Delta-B\right)*\left(r\Delta_B\right)\right]\left(t\right)}{\left[\left(\Delta-B\right)*\Delta_B\right]\left(t\right)}$$

where the denominators are equal (a good test of your convolution algebra)

- At this point I don't know how to progress without assuming homogeneous business strategy, ie. $B(s,t)=B(t-s)$, $A(s,t)=A(t-s)$, etc. for all s and t
- Among other things this makes the convolutions the usual commutative definition.

CONTINUING WITH THE MODULATION PROCESS

Some useful facts are

$$\begin{aligned}(\Delta - A) * \left(\sum_{k=0}^{\infty} a^{*k}\right) &= \Delta * (\delta - a) * \left(\sum_{k=0}^{\infty} a^{*k}\right)\\ &= \Delta * \delta \\ &= \Delta\end{aligned}$$

and $\lim_{t\to\infty} \left(\sum_{k=0}^{\infty} a^{*k}\right)(t) = \frac{1}{\mu_A}$ where μ_A is the mean of A considered as a cdf.

Also, those survival functions $(\Delta - A)$ and $(\Delta - B)$ involved in convolutions ($=$ integrals) suggests that some more means are lurking in these formulas, for example $\Delta * (\Delta - A)(t) \to \mu_A$ for $t \to \infty$, called "the surface interpretation of the mean"

CONTINUING WITH THE MODULATION PROCESS

- If we assume a level stream of new Liabilities $\Delta_B = \Delta$ the formula for the spread s is (after a lot of algebra to get r alone)

$$s = \frac{\left[\frac{\mu_B}{\mu_A}(\Delta - A) - (\Delta - B)\right] * r}{[(\Delta - B) * \Delta]} - \frac{(\Delta - A) * \left\{\left[\left(\sum_{k=0}^{\infty} a^{*k}\right) * \left(\frac{\mu_B}{\mu_A}(\Delta - A) - (\Delta - B)\right)\right] r\right\}}{[(\Delta - B) * \Delta]}$$

- Amazingly, the messy term is a transient that goes to 0 as the homogenous going-concern reaches steady-state.
- In the permanent steady state, the denominator $[(\Delta - B) * \Delta] \rightarrow \mu_B$
- For stable growing new Liabilities you just use distortions of A, B, a, b, μ_A and μ_B.
- The permanent steady-state term is made-to-order for a Fourier Transform

CONCLUSION FOR THE MODULATION PROCESS

For each frequency f the Fourier transform of the steady-state going-concern spread with a level stream of new liabilities is

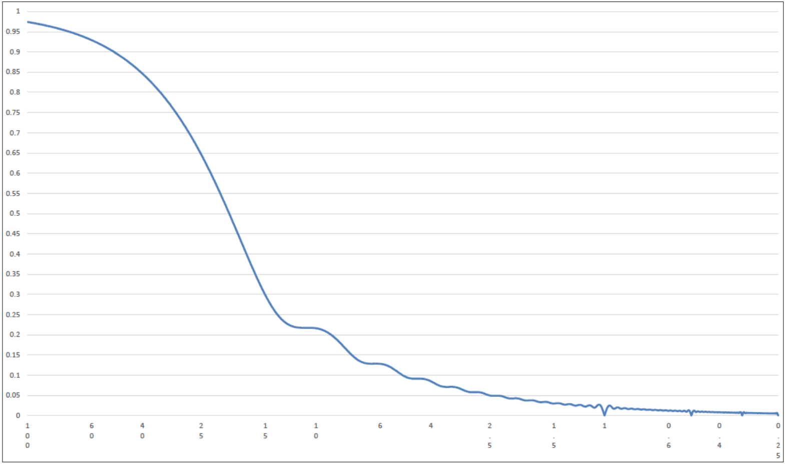
$$
\begin{aligned}
FT[s](f) &= FT\left[\frac{\Delta - A}{\mu_A} - \frac{\Delta - B}{\mu_B}\right](f)\, FT[r](f) \\
&= FT\left[\Delta * \left(\frac{\delta - a}{\mu_A} - \frac{\delta - b}{\mu_B}\right)\right](f)\, FT[r](f) \\
&= FT[\Delta](f)\, FT\left[\frac{\delta - a}{\mu_A} - \frac{\delta - b}{\mu_B}\right](f)\, FT[r](f) \\
&= \left(\frac{1}{2\pi i f} 1_{f \neq 0} + \frac{1}{2}\delta(f)\right)\left(\frac{1 - FT[a](f)}{\mu_A} - \frac{1 - FT[b](f)}{\mu_B}\right) \cdot \\
&\quad \cdot FT[r](f) \\
&= \frac{1}{2\pi i f}\left(\frac{1 - FT[a](f)}{\mu_A} - \frac{1 - FT[b](f)}{\mu_B}\right) FT[r](f) \text{ if } f \neq 0
\end{aligned}
$$

CONCLUSION FOR THE MODULATION PROCESS

- In other words $\frac{1}{2\pi if}\left(\frac{1-FT[a](f)}{\mu_A}-\frac{1-FT[b](f)}{\mu_B}\right)$ represents how the financial institution modulates the external interest rate frequency strengths $FT[r](f)$ into interest spread frequency responses $FT[s](f)$ when there is a level stream of new liabilities.
- We should note that if the new liability stream Δ_B grows at a stable rate g the Fourier Transform of the interest rate spread works out to $FT[s](f)=\frac{1}{\ln(1+g)+2\pi if}\left(\frac{1-FT[a](f)}{\mu_A}-\frac{1-FT[b](f)}{\mu_B}\right)FT[r](f)$ where the distorted versions of the functions and means must be used if the assumed growth g is not 0, and also in that case the equation works for $f=0$ too.
- The factor $\frac{1}{\ln(1+g)+2\pi if}$ in the modulation already teaches an important lesson for risk management: a stable, well-managed level of growth is a very effective risk-control mechanism.
- But let's illustrate the basic modulation for our simple financial institution with 15-year ladder assets and 10-year bullet liabilities.

ASSET MODULATION SPECTRUM (15 Year Ladder)

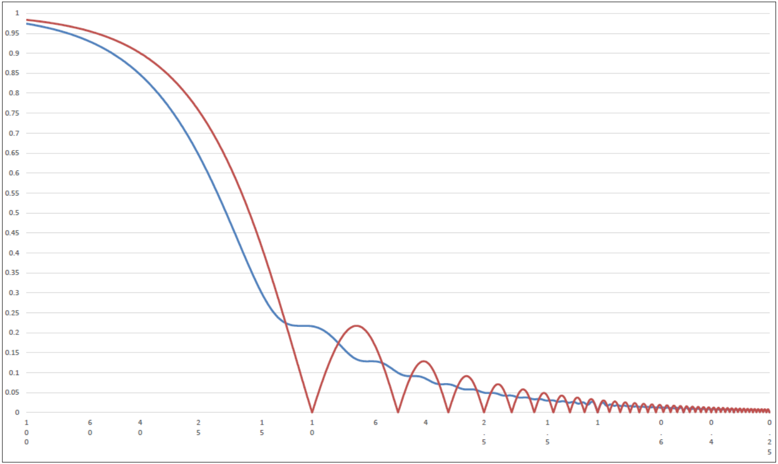
Modulus $\left|\frac{1}{2\pi i f}\left(\frac{1-FT[a](f)}{\mu_A}\right)\right|$; don't forget there's a phase, too



Horizontal axis labeled by wavelength $\left(\frac{1}{f}\right)$ on a logarithmic scale.

NOW THE LIABILITY SPECTRUM (10 Year Bullet)

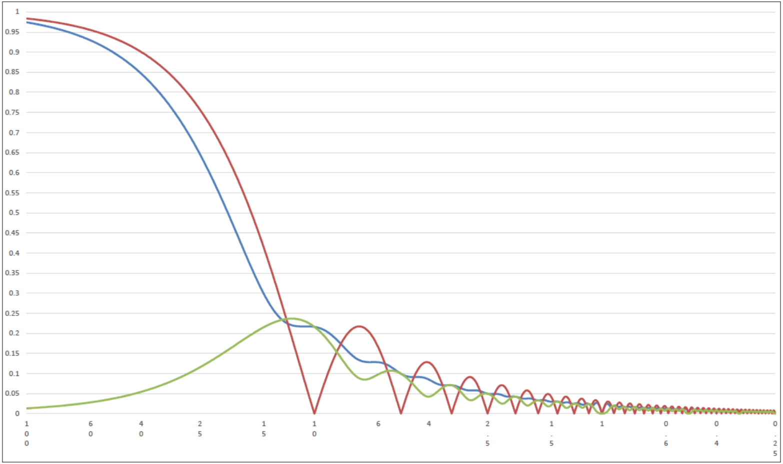
Modulus $\left|\frac{1}{2\pi if}\left(\frac{1-FT[a](f)}{\mu_A}\right)\right|$ and $\left|\frac{1}{2\pi if}\left(\frac{1-FT[b](f)}{\mu_B}\right)\right|$; don't forget phases



Horizontal axis labeled by wavelength on a logarithmic scale.

SUBTRACT FOR THE NET MODULATION SPECTRUM

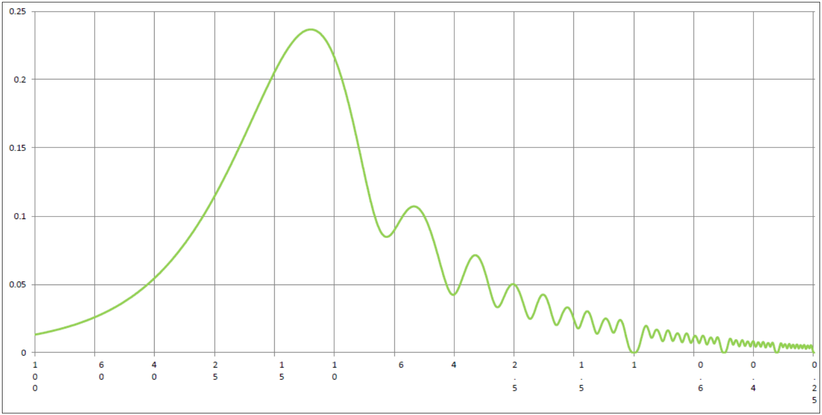
Modulus $\left| \frac{1 - FT[a](f)}{2\pi i f \mu_A} - \frac{1 - FT[b](f)}{2\pi i f \mu_B} \right|$; phases matter!



Wavelength on a logarithmic scale from 100 years to 0.25 years

NET SPECTRUM = RISK CONTROL PROFILE

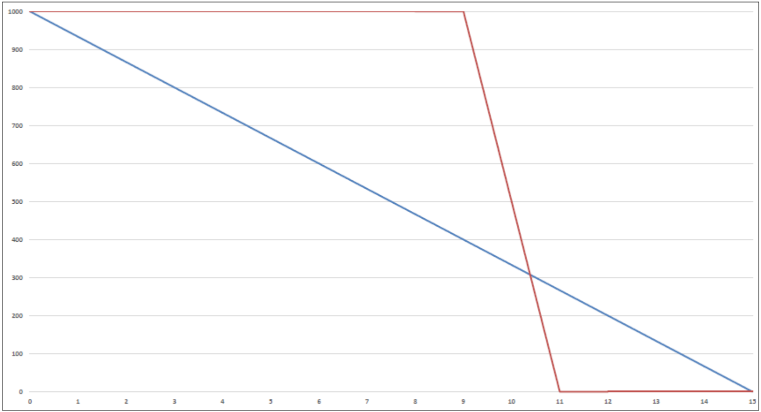
Modulus $\left| \frac{1 - FT[a](f)}{2\pi i f \mu_A} - \frac{1 - FT[b](f)}{2\pi i f \mu_B} \right|$; phases matter!



- Risk max's at 12 year interest rate cycle (!); next 5.4 $\left(= \frac{1}{2}\frac{12+9.5}{2}\right)$
- But true risk exposure is the entire area under curve - what if the phases all line up?

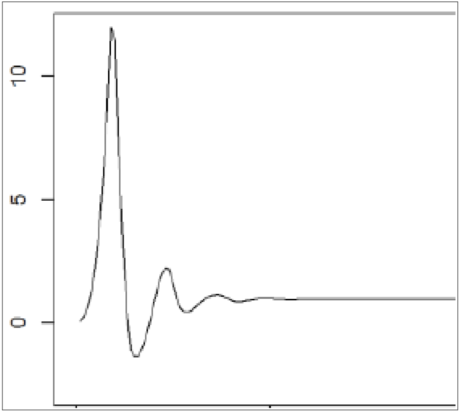
A SLIGHTLY DIFFERENT FINANCIAL INSTITUTION

Steady stream of Liabilities that mature straight-line 9 to 11 years
Assets straight line 0 to 15 years



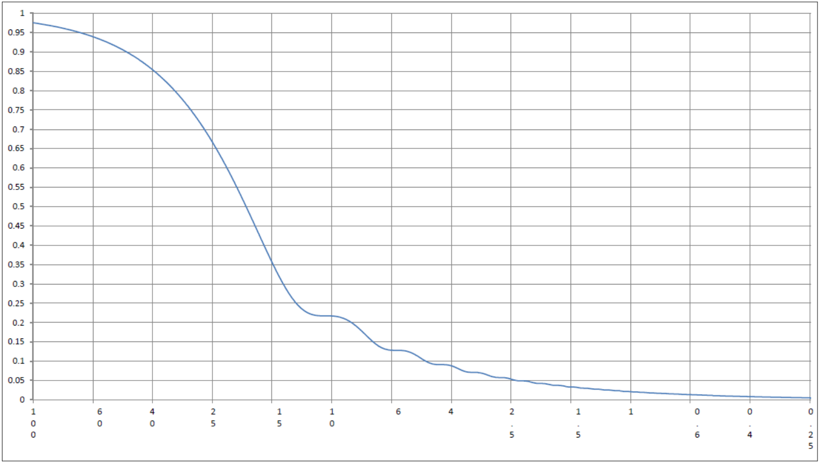
A DIFFERENT GOING CONCERN WITH A-L MISMATCH

Take in a steady stream of 9-11 year st. line liabilities and invest steadily in 0-15 year st.line asset maturities. The following spreads result if interest rates increase steadily:

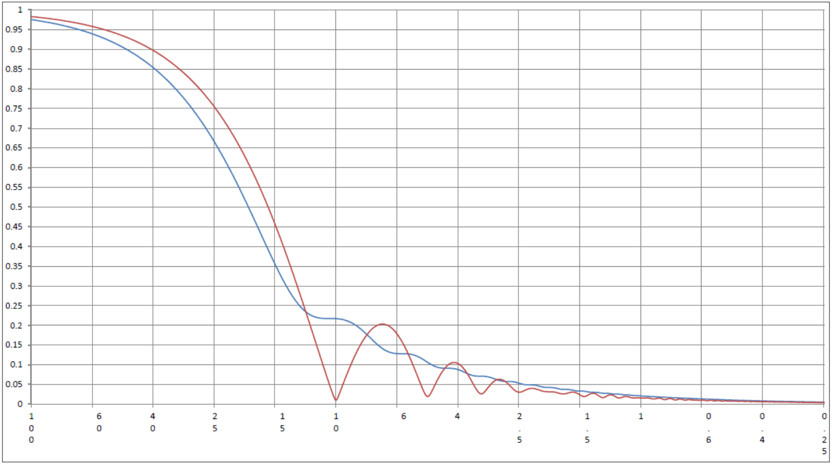


Steady-state profit, not loss

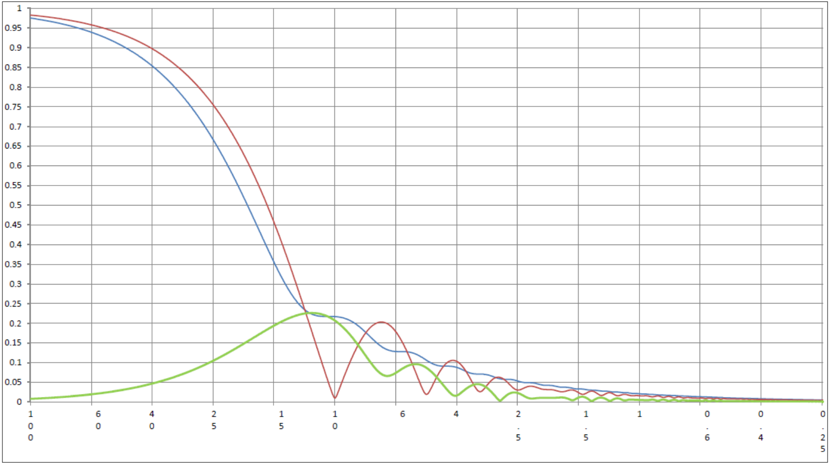
ASSET SPECTRUM (15 Year Straight Line)



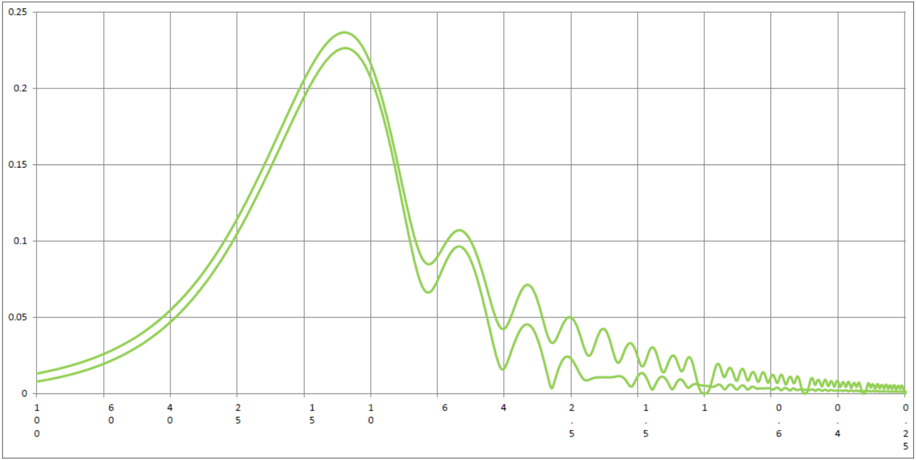
LIABILITY SPECTRUM (9 to 11 Year Straight Line)



NET SPECTRUM



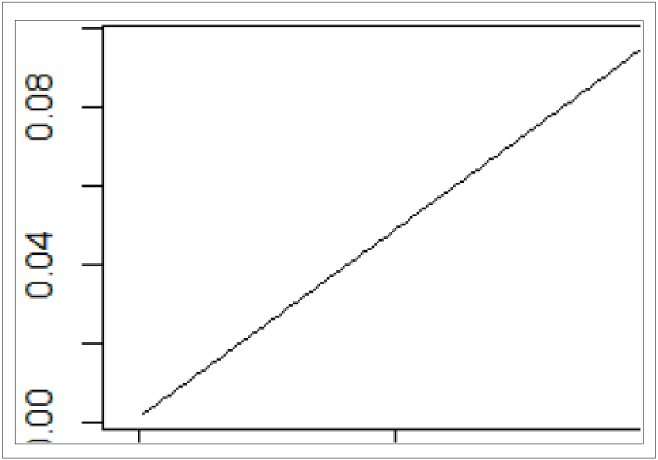
COMPARATIVE NET SPECTRA



WHAT CAN WE SAY ABOUT THE EXTERNAL INTEREST RATE SPECTRUM

FT[r](f) ?

FINGER EXERCISE - STRAIGHT LINE INCREASE



If $r(t)$ is linear with constant slope r' then $\frac{dr}{dt}(t) = r'$, so
$r(t) = \left(\Delta * \frac{dr}{dt}\right)(t)$ and $FT[r](f) = FT[\Delta](f)\, FT\left[\frac{dr}{dt}\right](f)$

SPECTRUM OF STRAIGHT LINE INCREASE

$$
\begin{aligned}
FT[r](f) &= FT[\Delta](f)\, FT\left[\frac{dr}{dt}\right](f) \\
&= \left(\frac{1_{f\neq 0}}{2\pi i f} + \frac{1}{2}\delta(f)\right) FT\left[r'\right](f) \\
&= \left(\frac{1_{f\neq 0}}{2\pi i f} + \frac{1}{2}\delta(f)\right) r'\delta(f)
\end{aligned}
$$

because $FT[\text{constant}] = \text{constant}\cdot\delta$.
Thus $FT[r](f) = 0$ unless $f = 0$, in which case it's an impulse function.
For interest rate spread $s(t)$ we know that
$FT[s](f) = \left[\frac{1-FT[a](f)}{2\pi i f \mu_A} - \frac{1-FT[b](f)}{2\pi i f \mu_B}\right] FT[r](f)$ but
$\left[\frac{1-FT[a](f)}{2\pi i f \mu_A} - \frac{1-FT[b](f)}{2\pi i f \mu_B}\right] = 0$ when $f = 0$, so we'll need l'Hôpital's help to unravel $FT[s](0) = 0 \cdot \infty$.

SPECTRUM OF SPREAD FROM STRAIGHT LINE

$FT\left[s\right](0) =$

$$= \lim_{f\to 0}\left[\frac{1-FT\left[a\right](f)}{2\pi i f\mu_A} - \frac{1-FT\left[b\right](f)}{2\pi i f\mu_B}\right]\left(\frac{1_{f\neq 0}}{2\pi i f} + \frac{1}{2}\delta(f)\right) r'\delta(f)$$

$$= \lim_{f\to 0}\left[\frac{1-FT\left[a\right](f)}{(2\pi i f)^2\mu_A} - \frac{1-FT\left[b\right](f)}{(2\pi i f)^2\mu_B}\right] 2\pi i f\left(\frac{1_{f\neq 0}}{2\pi i f} + \frac{1}{2}\delta(f)\right) r'\delta(f)$$

$$= \lim_{f\to 0}\left[\frac{1-FT\left[a\right](f)}{(2\pi i f)^2\mu_A} - \frac{1-FT\left[b\right](f)}{(2\pi i f)^2\mu_B}\right]\left(1_{f\neq 0} + \pi i f\delta(f)\right) r'\delta(f)$$

$$= \lim_{f\to 0}\left[\frac{1-FT\left[a\right](f)}{(2\pi i f)^2\mu_A} - \frac{1-FT\left[b\right](f)}{(2\pi i f)^2\mu_B}\right] 1_{f\neq 0} r'\delta(0)$$

$$= \lim_{f\to 0}\left[\frac{1-FT\left[a\right](f)}{(2\pi i f)^2\mu_A} - \frac{1-FT\left[b\right](f)}{(2\pi i f)^2\mu_B}\right] r'\delta(0)$$

because $\lim_{f\to 0} 1_{f\neq 0} = 1$

SPECTRUM OF SPREAD FROM STRAIGHT LINE

Now use l'Hôpital twice

$$
\begin{aligned}
FT[s](0) &= \lim_{f\to 0}\left[\frac{1-FT[a](f)}{(2\pi i f)^2\mu_A} - \frac{1-FT[b](f)}{(2\pi i f)^2\mu_B}\right] r'\delta(0) \\
&= \lim_{f\to 0}\left[-\frac{\frac{d^2}{df^2}FT[a](f)}{\frac{d^2}{df^2}(2\pi i f)^2\mu_A} + \frac{\frac{d^2}{df^2}FT[b](f)}{\frac{d^2}{df^2}(2\pi i f)^2\mu_B}\right] r'\delta(0) \\
&= \left[-\frac{FT\left[(2\pi i)^2 t^2 a(t)\right](0)}{2(2\pi i)^2\mu_A} + \frac{FT\left[(2\pi i)^2 t^2 b(t)\right](0)}{2(2\pi i)^2\mu_B}\right] r'\delta
\end{aligned}
$$

because for any density $h(t)$ it's true that $\frac{d}{df}FT[h(t)](f) = -FT[2\pi i t h(t)](f)$. Now 2nd moments appear

$$
\begin{aligned}
FT\left[(2\pi i)^2 t^2 h(t)\right](0) &= \int_{-\infty}^{\infty}(2\pi i)^2 t^2 h(t)\, e^{-2\pi i t\cdot 0}\,dt \\
&= (2\pi i)^2\int_{-\infty}^{\infty} t^2 h(t)\,dt = (2\pi i)^2\mu'_{H2}
\end{aligned}
$$

SPECTRUM OF SPREAD FROM STRAIGHT LINE

- So

$$FT\left[s\right](0) = -\frac{1}{2}\left[\frac{\mu'_{A2}}{\mu_A} - \frac{\mu'_{B2}}{\mu_B}\right] r'\delta(0)$$

  is the impulse function at $f = 0$ that constitutes the entire spectrum of the stable state interest rate spead $s(t)$ caused by a straight line movement of the external interest rate $r(t)$.
- This spectrum implies that $s(t)$ is a constant

$$s(t) = -\frac{1}{2}\left[\frac{\mu'_{A2}}{\mu_A} - \frac{\mu'_{B2}}{\mu_B}\right] r'$$

- If we calculate ratios of second to first moments for the simple asset and liability maturity schedules that we showed earlier, this formula gives exactly the small constant steady-state loss (or gain) that we saw earlier by brute force calculation.
- $\frac{1}{2}\frac{\mu'_{H2}}{\mu_H}$ is the mean of what risk theory calls "the equilibrium distribution" of the distribution H, with density $\frac{S_H(t)}{\mu_H}$, $H = A$ or B

1ST CONCLUSION FOR THE MODULATED PROCESS

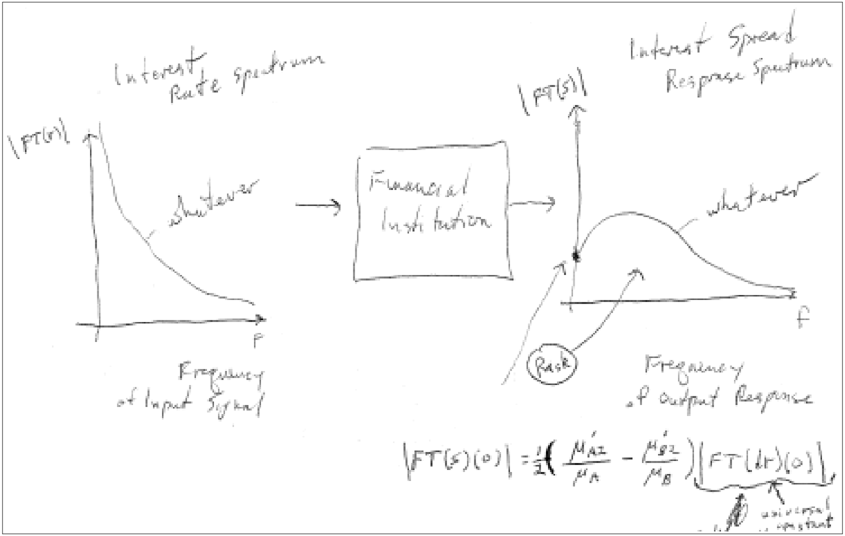
- This rather specialized result at $f = 0$ generalizes to any external interest rate process $r(t)$:

$$|FT[s](0)| = \frac{1}{2}\left|\frac{\mu'_{A2}}{\mu_A} - \frac{\mu'_{B2}}{\mu_B}\right| |FT[dr](0)|$$

where $|FT[dr](0)|$ is the frequency 0 (i.e. drift) component of the Fourier Transform of the process $dr(t)$ that generates $r(t)$.

- If the model has growing new liabilities $\Delta_B(t)$ then distortions of both the means and the second moments must be used
- It should be no surprise that these equilibrium distribution means $\frac{1}{2}\frac{\mu'_{A2}}{\mu_A}$ and $\frac{1}{2}\frac{\mu'_{B2}}{\mu_B}$ can be formally related (a duality) with the traditional duration concept.
- All of the risk area beyond $f = 0$ still remains, however, untouched by this dual version of "duration".

1ST CONCLUSION FOR THE MODULATED PROCESS



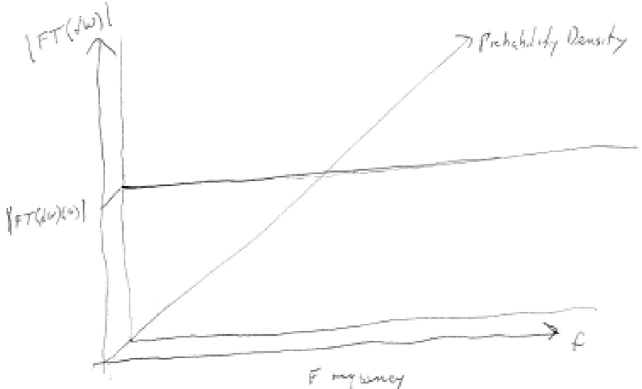
NOW THE EXTERNAL INTEREST RATE SPECTRUM

- Assume that Brownian motion drives whatever process we end up with
- $dW(t)$ is the random Brownian increment at each time t; $\{dW(t)\}_{-\frac{1}{2dt}\leq t\leq \frac{1}{2dt}}$ are $\frac{1}{dt}$ independent $N(0, dt)$ RVs
  - A fixed world runs from $-\frac{1}{2dt}$ to $\frac{1}{2dt}$; Brownian sampling gives the $dW(t)$ at each t which then is fixed once and for all
- $FT[dW](f) = \int_{-\infty}^{\infty} e^{-2\pi ift} dW(t) = e^{2\pi i\phi(f)} N\left(0, \frac{1}{dt}\right)$ with random phase $\phi(f)$. Why?
  - $\{dW(t)\}$ is a set of $\frac{1}{dt}$ independent samples from $N(0, dt)$ so $\int_{-\frac{1}{2dt}}^{\frac{1}{2dt}} e^{-2\pi ift} dW(t)$ is normal with $\mathbb{E} = 0, \mathbb{V} = \left(\frac{1}{dt}\right)^2 dt = \frac{1}{dt}$, i.e.
  - $\mathbb{E}\left[|FT[dW](f)|^2\right] = \mathbb{E}\left[FT[dW](f)\, FT[dW](-f)\right]$ (conjugates)
  - $= \int_{-\frac{1}{2dt}}^{\frac{1}{2dt}} \mathbb{E}\left[dW^2(t)\right]$ (independent)
  - $= \int_{-\frac{1}{2dt}}^{\frac{1}{2dt}} dt = \frac{1}{2dt} - \left(-\frac{1}{2dt}\right) = \frac{1}{dt}$

THE SPECTRUM OF dW(t)

- We even can identify the specific $N\left(0, \frac{1}{dt}\right)$ RV: $|FT[dW](f)|^2 = FT[dW](f)\,FT[dW](-f) = \int_{-\frac{1}{2dt}}^{\frac{1}{2dt}} dW^2(t) = (FT[dW](0))^2$
- So $|FT[dW](f)| = |FT[dW](0)|$, same value for all f, and the phase of $FT[dW](f)$ is totally random in f (and unknowable)
- $FT[dW](0)$ is a random real number, drawn from $N\left(0, \frac{1}{dt}\right)$, fixed for all time, and unknowable (see picture)
- This is a little like renormalization in physics. It sounds strange but it works since everything we actually observe will just be relative to this unknowable thing.
- Remember, we promised that true risk models will be very different from our usual models!
- All of the Fourier frequencies are equally represented in $FT[dW](f)$
- Random walk comes from randomized phase relationships.

SPECTRUM OF THE BROWNIAN INCREMENT



- dW(t) has all Fourier frequencies <u>equally</u> represented at <u>one random</u> value
- Its phases are completely randomized

THE SPECTRUM OF BROWNIAN MOTION W(t)

Now we know everything we need to get from $FT[dW]$ to $FT[W]$

$$W(t) = (\Delta * dW)(t) \text{ so}$$

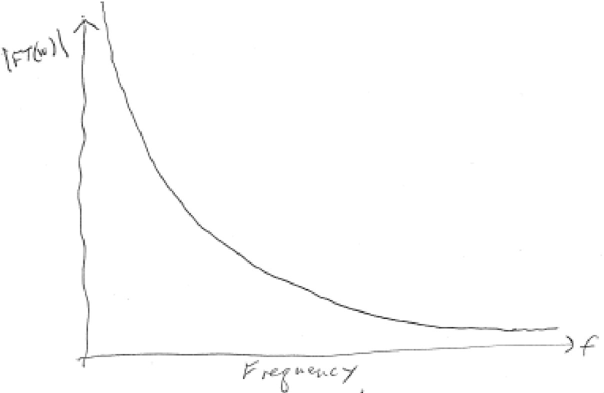
$$\begin{aligned}
FT[W](f) &= FT[\Delta * dW](f) \\
&= FT[\Delta](f)\, FT[dW](f) \\
&= \left(\frac{1_{f\neq 0}}{2\pi i f} + \frac{1}{2}\delta(f)\right) e^{2\pi i \phi(f)} FT[dW](0)
\end{aligned}$$

where $\phi(f)$ is a totally random phase. This gives

$$\begin{aligned}
|FT[W](f)| &= \left|\frac{1_{f\neq 0}}{2\pi i f} + \frac{1}{2}\delta(f)\right| |FT[dW](0)| \\
&= \frac{|FT[dW](0)|}{2\pi|f|} 1_{f\neq 0} + \frac{|FT[dW](0)|}{2}\delta(f)
\end{aligned}$$

with phases totally randomized (note: last step required $1_{f\neq 0} \cdot \delta(f) = 0$). The random phases are what makes the walk random.

THE SPECTRUM OF BROWNIAN MOTION



- $|FT(w)(f)| = \dfrac{|FT(dw)(0)|}{2\pi f} \cdot 1_{f \neq 0} + \frac{1}{2} |FT(dw)(0)| \, \delta(f)$
- Remember: $|FT(dW)(0)|$ is a fixed (random) value
- Phases of $FT(w)(f)$ are totally randomized

MEAN-REVERTING GEOMETRIC BROWNIAN MOTION

- The external interest rate $r(t)$ is likely to have a mean-reversion of some kind and a zero-avoidance of some kind.
- Mean-reverting Geometric Brownian (Black-Karisinski) looks like:

$$dr(t) = \{\tfrac{1}{2}\sigma^2 - \ln(1-F)[\ln T - \ln r(t)] - \tfrac{1}{4}\sigma^2[1 + 1_{F=0}]\}r(t)dt + \sigma r(t)dW$$

i.e. $dr(t) = (L + \ln(1-F)\ln(r)(t))r(t)dt + \sigma r(t)dW(t)$
with $L = \frac{1}{2}\sigma^2 - \ln(1-F)\ln T - \frac{1}{4}\sigma^2[1 + 1_{F=0}]$

- Decoding:
  - $0 \leq F \leq 1$ is an annualized mean-reversion strength
  - $1_{F=0}$ picks up the (little-known) need to change drift compensation from $-\frac{1}{2}\sigma^2$ toward $-\frac{1}{4}\sigma^2$ when there's mean-reversion
  - With this drift compensation $\mathbb{E}[r(t)] = T$ in the steady-state
  - Start process at $r(-\frac{1}{2dt}) = T$; drift comp. important when $F = 0$
- $r(t) = (\Delta * dr)(t)$ so $FT[r](f) = FT[\Delta](f)FT[dr](f)$ and off we go
- Alas, $FT[dr](f)$ has $FT[r](f)$ in it and also $FT[\ln r]$ in it.
- We have to work for a solution

- Start with an integration by parts:

$$d[e^{-2\pi ift}r(t)] = e^{-2\pi ift}dr(t) - 2\pi ife^{-2\pi ift}r(t)dt$$

$2\pi ife^{-2\pi ift}r(t)dt = e^{-2\pi ift}dr(t) - d[e^{-2\pi ift}r(t)]$ and now $\int_{-\frac{1}{2dt}}^{\frac{1}{2dt}} dt$

$$\begin{aligned} 2\pi ifFT[r](f) &= FT[dr](f) - e^{-2\pi if\frac{1}{2dt}}r(\tfrac{1}{2dt}) + e^{2\pi if\frac{1}{2dt}}T \\ &= FT[dr](f) - e^{-2\pi if\frac{1}{2dt}}[T + FT[dr](0)] + e^{2\pi if\frac{1}{2dt}}T \\ &= FT[dr](f) - e^{-2\pi if\frac{1}{2dt}}FT[dr](0) + 2\pi ifT\delta(f) \end{aligned}$$

by a careful application of l'Hôpital's rule

- Now substitute

$$FT[dr](f) = L\cdot FT[r](f) + \ln(1-F)FT[\ln(r)\cdot r](f) + \sigma FT[r\cdot dW](f)$$
$$= L\cdot FT[r](f) + \ln(1-F)[FT[\ln r] * FT[r]](f) + \sigma[FT[r] * FT[dW]](f)$$

and solve for $FT[r](f)$:

$$FT[r](f) = \tfrac{1}{2\pi if - L}\{\ln(1-F)[FT[\ln r] * FT[r]](f)$$
$$+\sigma[FT[r] * FT[dW]](f) - e^{-2\pi if\frac{1}{2dt}}FT[dr](0) + 2\pi ifT\delta(f)\}$$

MEAN-REVERTING GEOMETRIC BROWNIAN MOTION

- A similar calculation gives an expression for $FT[\ln r]$ using:

$d[e^{-2\pi ift}\ln r(t)] = e^{-2\pi ift} d\ln r(t) - 2\pi ife^{-2\pi ift}\ln r(t)dt$ to integrate by parts

- $d\ln r(t) = -Kdt + \sigma dW(t)$ for

  $K = \{\ln(1-F)[\ln T - \ln r(t)] + -\frac{1}{4}\sigma^2[1+1_{F=0}]\}$ to substitute in the resulting integral, giving

- $FT[\ln r](f) = \frac{1}{2\pi if - \ln(1-F)}\{-K\delta(f) + \sigma FT[dW](f)$

  $-e^{-2\pi if}\frac{1}{2dt}FT[d\ln r](0) + 2\pi if\delta(f)\}$

- That can be substituted into the expression for $FT[r](f)$
- The resulting expression takes a lot of work but leads to $FT[r](f) = \frac{\sigma[FT[r]*FT[dW]](f) - FT[dr](0)1_{f=0} + 2\pi ifT\delta(f)}{2\pi if - \frac{1}{2}\sigma^2 1_{F\neq 0} - Q[r](f)\sigma[FT[r]*FT[dW]](f)}$

MEAN-REVERTING GEOMETRIC BROWNIAN MOTION

- In $FT[r](f) = \frac{\sigma[FT[r]*FT[dW]](f) - FT[dr](0)1_{f=0} + 2\pi i f T\delta(f)}{2\pi i f - \frac{1}{2}\sigma^2 1_{F\neq 0} - Q[r](f)\sigma[FT[r]*FT[dW]](f)}$ it looks like we have we $FT[r](f)$ on both sides of the equation (it is part of those convolutions.)
- However, we can interpret the expression $[FT[r]*FT[dW]](f)$ to be a randomly weighted average of $FT[r](h)$ over all values of h with the specific value f contributing primarily a random phase and perhaps (needs more analysis) a small random fluctuation in the modulus as f varies.
- Specifically, $[FT[r]*FT[dW]](f) = FT[dW](0)\cdot(\text{factor})(f)$ where the factor depends upon f only for a phase and at most a small random fluctuation in modulus
- The expression $Q[r](f)\sigma[FT[r]*FT[dW]](f)$ is a real number.
- $Q[r](f)$ depends upon f only for a phase and (perhaps) a randomly fluctuating modulus < 1
- $Q[r](f) = 0$ when F=0

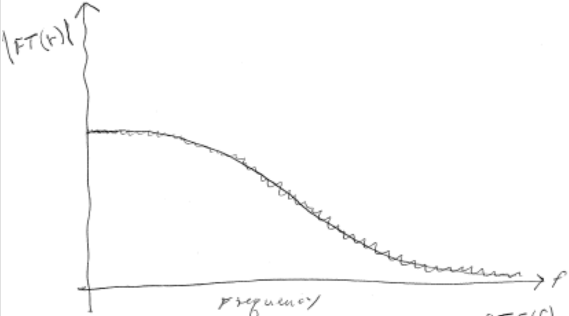
For its modulus,
$|FT\,[r]\,(f)|\,1_{f\neq 0}=\frac{\sigma|[FT(r)*FT(dW)](f)|1_{f\neq 0}+2\pi fT\delta(f)}{\left|2\pi if-\left\{\frac{1}{2}\sigma^2+Q(f)\sigma[FT(r)*FT(dW)](f)\right\}1_{F\neq 0}\right|}$ with totally randomized phase.

The expression for $FT\,[r]\,(f)$ is completely useless for forecasting because of the phase randomization. You can't even estimate the current phases because you need $t\to\infty$ to calculate a phase.

But the modulus carries the full risk exposure.

Note that when $F=0$ this looks a lot like Brownian Motion (goes to ∞ when $f=0$, but mean reversion sets a maximum on value at $f=0$)

THE EXTERNAL RATE SPECTRUM (F not 0)



- $$|FT(r)(f)| = \frac{|\sigma[FT(r) * FT(dw)](f)| \cdot 1_{f \neq 0} + 2\pi f T S(f)}{|2\pi i f - \frac{1}{2}\sigma^2 - Q(f)\sigma[FT(r) * FT(dw)](f)|}$$
- $[FT(r) * FT(dw)](f)$ is a "nearly" fixed random variable (dependence on f is a totally random phase and tiny fluctuations)
- $Q(f)$ is a "nearly" fixed random variable (dependence on f is a phase making $Q(f)[FT(r) * FT(dw)](f)$ real, and tiny fluctuations)

THAT'S OUR DUAL MODEL FOR INTEREST RISK

So the interest rate spread in our going-concern has a risk spectrum of:
$|FT[s](f)| = \left| \frac{1}{\ln(1+g)+2\pi if} \left(\frac{1-FT[a](f)}{\mu_A} - \frac{1-FT[b](f)}{\mu_B} \right) FT[r](f) \right|$ where the phase is totally randomized, where the distorted versions of the functions and means must be used if the assumed growth g is not 0, and where

$|FT[r](f)| = \frac{\sigma|[FT(r)*FT(dW)](f)|\mathbf{1}_{f\neq 0}+2\pi fT\delta(f)}{\left|2\pi if-\left\{\frac{1}{2}\sigma^2+Q(f)\sigma[FT(r)*FT(dW)](f)\right\}\mathbf{1}_{F\neq 0}\right|}$ is the external rate spectrum.

I'm still working on numerical illustration of the external rate spectrum, which is needed to get a feel for where on the going-concern risk spectrum we most need to compress the response.

References

- For the modulation process
  - My paper at the 1998 International Congress of Actuaries

link to it at bottom of
http://www.math.uconn.edu/~bridgeman/Papers_and_Presentations/index.

-
  - But really, anyone who has programmed an ALM model has done this, whether they know it or not
- For the Fourier Analysis - any good text; I like
  - Rudin's Real and Complex Analysis
  - Brigham's Fast Fourier Transform
  - Meikle's A New Twist To Fourier Transforms
- For the application to random walk - you need to be careful; I used an actual-infinitesimals approach following the ideas in
  - Robinson's Non-Standard Analysis
  - But he didn't apply it to random walk ... I did and am confident I got it right

SPECIAL THANKS TO

Chengyun Li