POWER SERIES — A BRIEF SUMMARY

1. The basic definitions

Weierstrass approached complex variable using power series. It is the way that Cartan starts his book.

Definition 1.1. A *formal power series* is an expression

$$\sum_{k=0}^{\infty} a_n Z^k.$$

$\mathbb{C}[[Z]]$ denotes the vector space of formal power series with coefficients a_n in $\mathbb{C}$.

Our goal is to be able to give complex values to certain series when we substitute a complex number for the indeterminate Z.

Definition 1.2. A *sequence* in a set X is a map

$$\begin{array}{ccccc} (\ ) & : & \mathbb{N} & \to & X \\ & & n & \mapsto & a_n \end{array}$$

The sequence is usually denoted (a_n). Here $\mathbb{N}$ is called the *index set.* Often it is replaced by $\mathbb{N} \cup \{0\}$. (We wont be very fussy about this.)

A sequence (a_n) in $\mathbb{C}$ (respectively, in any metric space (X, d)) converges to L if, for any $\epsilon > 0$, there is an $N_0 \in \mathbb{N}$ so that $|a_n - L|$ (respectively, $d(a_n, L)$) $< \epsilon$ whenever $n \geq N_0$. We write $a_n \to L$.

Next, we gather together a whole lot of definitions.

Definition 1.3. Let (c_k) be a sequence in $\mathbb{C}$.

(1) A *series* of complex numbers is a formal expression

$$\sum_{k=0}^{\infty} c_k.$$

(2) The *n-th partial sum* of $\sum_{k=0}^{\infty} c_k$ is the complex number

$$s_n = \sum_{k=0}^{n} c_k.$$

(3) $\sum_0^{\infty} c_k$ *converges to* $C \in \mathbb{C}$ if the sequence (s_n) converges to C. We then say that $\sum c_k$ has *sum* or *limit* C; we even write that $\sum c_k = C$.

(4) If $\sum c_k$ converges to some $C \in \mathbb{C}$ then we say that $\sum c_k$ is *convergent*; otherwise, it is *divergent*.
(5) $\sum c_k$ is *Cauchy convergent* if, for each $\epsilon > 0$, there is a number $N_0 \in \mathbb{N}$ so that
$$|s_n - s_m| < \epsilon \qquad \text{when } n, m \geq N_0.$$
(6) $\sum c_k$ is *absolutely convergent* if the series $\sum |c_k|$ is convergent.
(7) If $\sum c_k$ is convergent but not absolutely convergent, it is called *conditionally convergent*.

2. Properties of series

This is just a list of properties of series.

(1) Suppose $c_k = r_k + it_k$ with $r_k, t_k \in \mathbb{R}$, then $\sum c_k$ is convergent if and only if both $\sum r_k$ and $\sum t_k$ are convergent and, in that case, $\sum c_k = \sum r_k + i \sum t_k$.
(2) If $\sum c_k$ is convergent, then $c_k \to 0$. The converse is false.
(3) If $\sum c_k$ is absolutely convergent, then $\sum c_k$ is convergent. Again the converse is false.
(4) If $\sum c_k$ is absolutely convergent, you can rearrange the order of the terms and still get the same sum.
(5) A conditionally convergent series with real coefficients can be rearranged so that it converges to any real number or to $\pm\infty$. (Conditionally convergent series are hard to deal with.)
(6) If $\sum a_k$ and $\sum b_k$ are series, then the *Cauchy product* of $\sum a_k$ and $\sum b_k$ is the series
$$\sum a_k * \sum b_k := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right).$$
(7) If $\sum a_k = A$ and $\sum b_k = B$ and both series are absolutely convergent, then the Cauchy product of $\sum a_k$ and $\sum b_k$ is absolutely convergent to AB.

Most important to us is the following

Theorem 2.1. *Absolutely convergent series with complex coefficients form an algebra over $\mathbb{C}$. You can add them, multiply by constants and multiply them by each other just as you would do with polynomials.*

2.1. Two basic tests for convergence of series.

Suppose we are given a series $\sum c_k$.

Theorem 2.2 (The Root Test)**.** *Let $\gamma = \limsup \sqrt[k]{|c_k|}$. If $\gamma < 1$, then the series converges. If $\gamma > 1$, then the series diverges.*

Theorem 2.3 (The Ratio Test)**.** *Let*
$$\gamma = \limsup \frac{|c_{k+1}|}{|c_k|}.$$
If $\gamma < 1$, then the series converges. If $\gamma > 1$, then the series diverges.

Blue Rudin has proofs of both theorems.

3. FUNCTIONS DEFINED BY SERIES

Everything above can be thought of as the value at a point, call it z_0 of a sum of functions u_k. Even if we have some form of convergence at z_0, we might need to know what happens when we wiggle z_0 to a nearby point z_1 or just call it z.

Here's Weierstrass' idea:

Assume

$$\sum_{k=0} M_k = M$$

is a convergent series of non-negative real numbers.

Theorem 3.1. *(The Weierstrass M-test) Suppose $u_n : S \to \mathbb{C}$ is a collection of functions with $|u_n| \le M_n$ for all $z \in S$ and $\sum M_n$ is convergent. Then $\sum u_k(z)$ is absolutely convergent.*

Proof. $\sum M_n$ is convergent hence it is Cauchy convergent. It follows that, for all $\epsilon > 0$, there is a number N_0 so that, for $n.m \ge N_0$ with, say $n > m$,

$$\sum_{m+1}^{n} M_k < \epsilon.$$

But

$$\sum_{m+1}^{n} |u_k(z)| \le \sum_{m+1}^{n} M_k < \epsilon.$$

It follows that, everywhere on S,

$$\sigma = \sum_{k=0}^{\infty} |u_k(z)|$$

is absolutely Cauchy convergent hence is absolutely convergent. □

An Aside — When we write $|u_k(z)| \le M_k$, we are not only making a statement about the behavior of u_k at the one point z. It is a statement about every point in the set S. i.e. it is a uniform statement about the sums on S. In particular, the Weierstrass criterion tells us that the partial sums converge uniformly and absolutely on S.

4. Back to Power Series

Suppose $\sum a_k Z^k$ is a formal power series. Then, for any fixed z and z_0,

$$\sum a_k (z - z_0)^k$$

is a formal series and we should discuss its convergence behavior. Suppose it converges for some $z \ne z_0$. That means the series $\sum a_k u^k$ has that convergence behavior when we make the substitution $u = z - z_0$. So, most often, we can assume that $z_0 = 0$ — it makes most computations easier.

We've discussed the convergence, even absolute convergence and Cauchy convergence, of series. These are meaningful when we plug in some value for z in $\sum a_n Z^n$. Those values of z which give us a convergent series form the *convergence set* X of the power series $\sum a_n z^n$.

5. The convergence properties of complex power series

We need estimates to develop the theory of complex power series. The starting point always seems to be the geometric series. Let's recall its properties.

5.1. **The geometric series.**

When we divide complex polynomials we find that, for $z \neq 1$,

$$\frac{1}{1-z} = \left(\sum_{k=0}^{n} z^k\right) + \frac{z^{n+1}}{1-z}.$$

The last term is a remainder and goes to zero precisely when $|z| < 1$.

I would put a proof here but you should have seen it many times.

5.2. **Abel's Theorem and its consequences.**

Suppose $\sum c_n Z^n$ is a formal power series. If we substitute $Z = 0$, we can evaluate it as a function and the only term that doesn't get killed is the term $c_0 0^0$. In this context, 0^0 is defined to be 1 and the value of the series is c_0. The value of the series at zero is overwhelmingly uninteresting. So we'll assume there is a point $z_1 \neq 0$ where our series converges. Here's what Abel taught us:

Theorem 5.1. *Suppose $\sum c_n Z^n$ converges for some $z_1 \neq 0$. Then, for all $z \in \mathbb{C}$ with $|z| < |z_1|$, $\sum c_n z^n$ converges absolutely.*

Proof. there should be a picture here.

Since $\sum c_n z_1^n$ converges, there is a number $M \in \mathbb{R}^+$ so that

$$|c_n z_1^n| \leq M$$

for $n \geq M$ — that's what we mean by $c_n z_1^n \to 0$. It follows that

$$|c_n z^n| \leq M \left|\frac{z}{z_1}\right|^n.$$

It follows that

$$\sum |c_n z^n| \leq \sum M \left(\left|\frac{z}{\bar{z}}\right|\right)^n = M \sum r^n$$

where $r < 1$. ☐

Note: — Each term in $\sum M_n$ is non-negative. It follows that the partial sums are increasing. If they don't diverge to infinity, they converge to some finite positive number.

Definition 5.2. Let $\sum c_n z^n$ be a power series which converges for some $z_1 \neq 0$. Let ρ be defined by

$$\rho = \limsup\{R : \sum c_n z^n \text{converges in } \ B_R(0)\}.$$

Then ρ is called the *radius of convergence* of the series $\sum c_n z^n$. If no $z_1 \neq 0$ lies in the convergence set of the series, we say $\rho = 0$. If the series converges for all $z_1 \in \mathbb{C}$, we write $\rho = \infty$.

The root test implies

Theorem 5.3. *Given a formal power series* $\sum c_n Z^n$. *Then its radius of convergence* ρ *satisifes*

$$\frac{1}{\rho} = \limsup \sqrt[n]{|c_n|}.$$

In the set $B = B_\rho(0)$, *the series converges absolutely and, on compact subsets of* B, *the series converges uniformly. If* $|z_1| > \rho$, *then the series* $\sum c_n z_1^n$ *diverges.*

5.3. **Analytic/Holomorphic functions and their basic properties.** The word analytic is used in so many different contexts that it almost becomes meaningless. Unfortunately, it is commonly used in complex analysis. Let me offer a few definitions that, hopefully, will make sense of the concept. X will be a set, often in a vector space, where these definitions can be made meaningful — ask if they're not clear.

Definition 5.4. Suppose f is a real or complex-valued function — call the target F; we can write

$$\begin{array}{cccc} f & : & X & \to & F \\ & & x & \mapsto & f(z) \end{array} \tag{1}$$

Then we say that $f \in \mathcal{C}^k = \mathcal{C}^k(X) = \mathcal{C}^k(X, K)$ if f has a continuous k-th derivative at every point in X. f is $\mathcal{C}^\infty$ if f is in $\mathcal{C}^k$ for all k and we write that f is *infinitely differentiable*.

f is *real analytic* in X if f can be written as a power series, with real coefficients, which is convergent near any point of X. f is *complex analytic* or, more frequently called, *analytic* if near each point of x, f can be written as a convergent power series with complex coefficients. We then write $f \in \mathcal{C}^\omega(X, \mathbb{R})$ or $f \in \mathcal{C}^\omega(X, \mathbb{C})$. The later case is the classical definition of an analytic function. If $X \subset \mathbb{C}$ is open and f is complex analytic in x, we will call f holomorphic in X. The set of holomorphic functions in an open set $D \in \mathbb{C}$ is denoted $\mathcal{O}(D)$. Usually D is taken to be a domain.

Here's a summary of what we've proved.

Theorem 5.5. *Suppose* $\sum c_n z^n$ *has radius of convergence* $\rho > 0$, *then there is a function*

$$\begin{array}{cccc} f & : & B_\rho(0) & \to & \mathbb{C} \\ & & z & \mapsto & \sum c_n z^n \end{array}$$

which, on any disk where $|z| < r < \rho$ *is the uniform limit of* $\sum c_n z^n$, *hence is continuous on* $B_\rho(0)$.

An essential property of of $f(z) = \sum c_n z^n$ on $B_\rho(0)$ is

Theorem 5.6. *Suppose* $\sum c_n z^n$ *is convergent on* $B := B_\rho)(0)$ *and*

$$f(z) = \sum c_n z^n.$$

Then f *is differentiable on* B *and*

$$f'(z) = \sum n c_n z^{n-1}.$$

We can repeat this process and obtain

Corollary 5.7. *If $f(z) = \sum c_n z^n$ is convergent on $B = B_\rho(0)$ then the k-th derivative of f is also absolutely convergent on B and*

$$f^{(k)}(z) = \sum n(n-1)(n-2)\cdots(n-(k+1))c_n z^{n-k}.$$