

Math 2142: PROJECT #1

Newton's Method

Due date: April 25, 2013.

Instructions: Write up your solution accurately and carefully. Your project will be graded both for what you say and how you say it. Presentation matters!! Your pride in your work is reflected by the presentation — and that will be considered in the grading. I also have to be able to read your work. You may work with others to discover the solutions, but your write up must be your own work. You are also encouraged to discuss your work in progress with me.

A project isn't a homework assignment. In your writeup of the project, I expect a lucid presentation of the problem you are studying, the method(s) you are using, the results you have obtained and the conclusions you draw. If the methods don't work, please explain what you think might be the problem and how you might address it.

The due date is firm. Papers will be collected at the beginning of the class hour. Hand in whatever you have done by the due date as no late papers will be accepted.

Newton's method is an algorithmic technique for solving equations. While linear equations in any (finite) number of variables can be solved by known techniques (you'll see that in later Advanced Calc courses or take more advanced courses), nonlinear equations are much more difficult to solve. Furthermore, the techniques don't always work and can yield bizarre results under some circumstances.

In this project we will examine Newton's method. We will look both at its utility and its shortcomings.

Here's a quick review of Newton's method. Let $f(x)$ be some function. We wish to find roots of $f(x) = 0$. We form the *iterator*

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Newton's method tells us to start with some point x_0 and let $x_1 = g(x_0)$. Then define $x_2 = g(x_1)$ and in general, $x_n = g(x_{n-1})$. In other words,

$$x_n = g_n(x_0) := \underbrace{g \circ g \circ \cdots \circ g}_n(x_0).$$

The process by which Newton's method tries to approximate a root of the equation $f(x) = 0$ is shown by using the above iterator function in Mathematica (the necessary description is attached) or some other computer software.

A) Use an iterator function to find the first approximant to a root x_1 of $f(x) = x^2 - 4$ using the following points as initial points, $x_0 = -1, 1, 0, 2$. Describe the differences in the behavior depending on the initial point. (Remember that you are trying to convince the world that you have a valid approach to a problem. Computations

and descriptions should be interwoven. Data can be in an appendix or in the text.

In Part A, the function $f(x)$ actually has a root although Newton's method may not always work when there is a root. When the function has no (real) root, other things can happen.

B) Experiment using different starting points with the iterator function for the function $f(x) = x^2 + 4$. Explain what you observe.

We now need to use the notion of limit of the sequence (x_n) . (You can think of the points x_n that we just defined.) Informally, we say that (x_n) has *limit* ξ if, when n is large enough, x_n is a good approximant to ξ — to any prescribed degree of accuracy. The formal definition is: Given a sequence (x_n) in \mathbb{R} , then (x_n) converges to ξ , written

$$\lim_{n \rightarrow \infty} x_n = \xi \quad \text{or} \quad x_n \rightarrow \xi$$

if, for any $\epsilon > 0$, there is a number N so that

$$|x_n - \xi| < \epsilon \quad \text{whenever} \quad n \geq N.$$

Suppose $f(x)$ has a root ξ . A point x is said to lie in the basin of attraction $B(\xi)$ for ξ if $g_n(x) \rightarrow \xi$ as $n \rightarrow \infty$.

Informally the basin of attraction of a root ξ is the set of good initial points for the use of Newton's method to approximate the root ξ .

C) Suppose $x \in B(\xi)$ then prove that $g_k(x) \in B(\xi)$ for any $k \geq 0$. (You have to show that $g_n(g_k(x)) \rightarrow \xi$ exactly when $g_n(x) \rightarrow \xi$.)

C says that the basin of attraction of a root is *invariant* under g , i.e. $g(B(\xi)) \subset B(\xi)$.

D) Now we'll do a more intensive study of the roots of $f(x) = x^3 - 4x$. First we'll start with the basin of attraction of $\xi = 0$. Start with $x_0 = 0$ and its associated iterator function to show that $x_0 \in B(0)$.

E) Suppose $-\sqrt{\frac{1}{12}} < x_0 < g_2(x_0) < 0$. Then show that $x_0 \in B(0)$. You needn't give a formal proof — explain what is happening in words or pictures (which could be graphs).

F) If $g_2(x_0) < x_0 < 0$ then g_2 is not moving x_0 closer to zero and we might expect that Newton's method, with initial point x_0 , would converge to a different zero than $\xi = 0$. Try using the Mathematica function FindRoot and see what happens when $-\sqrt{\frac{4}{3}} < x_0 < 0$.

Other kinds of behavior can occur — besides those mentioned in parts E and F. For example, we could find that x_0 lies in a periodic orbit for g (this only means that $g_n(x_0) = x_0$ for some non-zero value of n). In this case, Newton's method never converges to a zero (because it keeps returning to x_0).

G) The condition $g_2(x) = x$ or $G(x) = g_2(x) - x = 0$ is solved at a root of the function $G(x)$. Determine $G(x)$ and use FindRoot to find a root of $G(x)$ which lies between -1 and 0. (Newton's method can be used to solved problems arising from Newton's method!)

The behavior of Newton's method is relatively well controlled even near the periodic orbit that we just found. As we move our initial point from one side of the periodic orbit to the other, we move from one basin of attraction to another. More complicated things occur near critical points.

H) If $x_0 < -\sqrt{4/3}$, does x_0 have to lie in the basin of attraction of a single root of $f(x)$? Experiment with it and try to prove your guess.

I) If $x_0 > -\sqrt{4/3}$ but is close to that critical point, what happens?

For our last question,

J) The curve $f(x) = x^3 - x$ is very symmetric. How does that affect the basins of attraction for the various roots? Formulate a guess by experimentation or use prose and then try to prove that your guess is correct.

Constructing the iterator function in Mathematica Mathematica allows you to construct user-defined functions quite easily — you will need to define the iterator function. A simple example is $x^2 + 3$. Key in

$$g[x_] = x^2 + 3$$

and then hit enter on the numeric keypad. You have saved the function $g(x)$ in Mathematica. If you key in

$$y1 = g[4]$$

and hit the same return key, you'll get

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However, if you again key in $y1$ and hit that return, you also get the number 19. That means that the storage space for the variable $y1$ now contains the number 19. If you now key in

$$y2 = g[y1]$$

the process will repeat itself. This gives you the building block of the iteration process. Using this method, you can iterate by hand. Loop constructions are the way to have the computer do the repetition automatically — learn them if you like. If we are trying to find the roots of the function $f(x)$, the iterator is

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

You enter this into Mathematica as follows:

You will get a prompt of the form

In[some number] :=

When you key something in, it will appear to the right of the prompt. For example,

$$f[x_] := x^4 - 3x + 2.$$

On Macs, the return key doesn't tell Mathematica to compute. Use the enter key instead.¹ Also, when you define a function $f(x)$ on the left hand side, write it as $f[x_]$.

At this point, you have entered f.

To enter the iterator, key in

$$g[x_] := x - f[x]/f'[x].$$

Then hit enter

Now pick a starting point x_0 and key in

$$g[x_0]$$

and hit enter.

Note that you want x_0 to some number not a symbol (you can do that too but we're trying to avoid complications.)

Instead of using a number as the argument of a function, you can say " use the result of the last computation. The syntax for that is to use the % sign instead of entering the number.

Next enter

$$N[%]$$

and hit enter.

From then on, Mathematica will give the answers in decimals rather than as terribly long fractions.

If you next enter

$$g[%],$$

you will be iterating.

¹I don't know the current enter keys for Windows or Linux. Look at the first example of a computation in the Help system for Mathematica or ask for help in using your operating system.