COMPUTATIONAL TOPOLOGY FOR RECONSTRUCTION OF SURFACES WITH BOUNDARY, PART I: APPLICATIONS

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Abstract. This paper presents computational topology techniques for reconstruction of surfaces with boundary, where all manifolds considered are assumed to be embedded in $\mathbf{R}^3$. The focus here is upon examples and applications, with the theoretical basis being presented in a companion paper. As a step towards these results, we consider any C^2 compact 2-manifold M with boundary and then we define and construct its envelope $E(M)$, such that $E(M)$ has no boundary. Then $E(M)$ can be used to approximate M, even though $E(M)$ need not be C^2. This construction supports extensions of many previous results on surface reconstruction, where the assumption of an empty boundary of M had been crucial. Note, also that the original surface M need not be orientable, again extending previously known techniques. Our prototype code is discussed and examples are shown to demonstrate the effectivenss of this approach, with specific demonstration of reconstruction improvements along a boundary where refined normal approximations have been crucial.

Keywords: Computational topology; surface approximation; surface reconstruction, computer graphics; topology methods for shape understanding, ambient isotopy. and visualization.

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1. Introduction and Motivation

The state of the art in topology-preserving surface approximation has been restricted to 2-manifolds without boundary which are also C^2 [4, 6, 8, 13, 24]. However, practical surface reconstruction also requires consideration of manifolds with boundary, where a heuristic method has been successfully used in some cases [5]. The primary new contribution of this article is to show how the theory of its companion paper [1] can be adapted to develop practical new algorithms which produce better boudaries during surface reconstruction.

While the examples presented rely upon software integration with a specific implementation, we expect these new techniques to also apply to other surface reconstruction algorithms, which presently assume the absence of boundary and output a piecewise-linear approximation. For the examples presented here, the implementation of a pre-processing interface was the only new code required. The examples presented rely upon some expediencies for computational efficiency, as is discussed.

In a related article [13] on surface reconstruction for computer-aided geometric design, general questions were posed about the possibility of creating algorithms which could handle surfaces with boundary. The approach offered here is responsive, postulating new assumptions for input to be sufficient for approximation of surfaces with boundary.

The paper is organized as follows: In Section 2, we summarize related work. Section 3 provides an overview of the theory needed for the extensions presented. Section 4 contains the discussion of implementation and presents graphical examples of these new techniques. Closing remarks are given in Section 5.

2. Related Work

An emphasis upon topological guarantees for surface approximants has recently appeared in the literature on surface reconstruction [4, 6, 8, 13]. For surface reconstruction, it is typical that only point cloud data is assumed to be available. The methods presented here rely formally on maximal curvature and minimal separation distances. For our foot example, which is based upon point cloud data, we used additional known characteristics of the object to approximate these geometric values in order to perform an improved surface reconstruction. This practice remains the subject of further study to minimize the additional user-specific knowledge required for execution. Furthermore, these curvature and separation values are often available. This will be true for many graphics applications, where the surface definitions will already be given and the problem is to produce a topologically correct approximation, rather than a reconstruction. It will also often be true for reverse engineering of manufactured objects [20]. Some previous methods relied upon an approximation of the medial axis [4, 6, 11, 14], which implicitly captures this curvature and separation information. The overview is that it should be clear that some estimation of a bound on surface curvature is crucial to any well-defined surface approximation method.

The theoretical concerns in providing topological guarantees for surface approximations near boundaries have been presented in the literature [5, 13, 17] within the context of approximants created during surface reconstruction. In particular, the paper [5] presents a heuristic argument to reconstruct a surface with boundary. This is presented under the terminology of 'holes', with a relevant example being the reconstruction of a surface for a foot, where data was only sampled below the ankle. In a different approach [13], a similar example of a foot is reconstructed as a manifold without boundary to avoid undersampling problems often experienced near the boundary. Both of these approaches for boundaries [5, 13] were pragmatic responses to the known difficulties of reconstruction of boundaries from unorganized sample points. Many surface reconstruction methods [2, 3, 4, 5, 13] rely upon creation of an approximation of the medial axis of M, which is often assumed to be topologically correct. Hence, it is relevant that it has since been shown [17] that the typical sampling input for surface reconstruction is not sufficient, in general, to permit a topologically faithful reconstruction of the medial axis of the surface, M.

For curves, a theorem has been published that provides for ambient isotopic piecewise linear (PL) approximations of a specifically described class of curves [21]. The proof utilizes the notion of 'pipe surfaces' from classical differential geometry [23]. The curves to which this method is applicable includes both those with and without boundary points, motivating the present extension to surfaces with boundary. This previous work for curves [21] relied upon geometric properties of the curve, as opposed to any consideration of the medial axis. Similarly, the approach taken here relies only upon the medial axis to the extent that it is used within the Power Crust algorithm, which is utilized here for convenience. However, the theory presented in the companion paper [1] does *not* specifically require the creation of an approximation of the medial axis, but, instead relies upon geometric characteristics that can be computed from the surface, namely curvature and global separation. These geometric parameters are sufficient for constructing a very specifically defined surface without boundary, which we call the envelope. The envelope can then be used to obtain an ambient approximation of the original surface.

The value in preferring ambient isotopy for topological equivalence [8, 24] versus the more traditional equivalence by homeomorphism [26] has previously been presented [8, 24] and the interested reader is referred to those papers for formal definitions. Intuitively, two closed curves will not be ambient isotopic if they form different knots, which can only be converted into each other by untying one knot and retying it to conform to the other. It has been shown that approximants can change knot type [8, 24] or, correspondingly, the way a surface is embedded in $\mathbf{R}^3$, but the work presented here can prevent these difficulties by appropriately constraining the approximations produced. A formal definition of isotopy is given in recent computational application papers [8, 24] or in a standard mathematical text [19].

The work presented here will be of interest to the computer graphics community, for generating approximations with appropriate topology. For instance, related earlier work by some of the present authors [9] has been used to prevent undesirable topological changes during object deformations [16] for animations. The methods presented here will provide even more general criteria for an animator to preserve the critical topological characteristics of an object as it changes across successive frames.

The present work emphasizes the integration of concepts from low-dimensional topology and differential geometry into the emerging sub-discipline of computational topology, as a complementary contribution to the incorporation of combinatorial topology and computational geometry formalisms that have already appeared [7, 12, 22] within the computational topology literature.

3. Approximating Manifolds

A careful examination of the proofs previously presented [8, 24, 25] for reconstructing C^2 manifolds without boundary reveals that a critical property was a positive minimum distance between the surface and its medial axis. While the details are contained in the companion theory paper [1], this aspect underlies the following practical algorithm that is presented in this section. A subtle distinction about the new theory is that it does *not* depend upon an *explicit* calculation of the medial axis. In particular, it is not necessary to determine the distance from the medial axis to the surface. As a temporary practical coding expedient, the current examples were produced by implemented code that does rely upon a construction of the medial axis, but the theory and the case studies presented here show promise for elimination of this medial axis dependancy.

It is essential to the supporting theory that there be careful definitions of the differentiable properties of the boundary, as given in the texts [18, 10] and summarizeded in our companion paper [1]. Once this is established, the methods presented here are possible, which generalize techniques that had previously appeared in the literature [2, 3, 4, 5, 13]. The theorems that are supportive of the methods presented here are explicated in the companion paper [1]. The following defintion is central to our approach.

Definition 3.1. *For suitably chosen values of ρ, the ρ-*envelope *of M, denoted $E_\rho(M)$ is defined*[1] *as*

$$E_\rho(M) = \{p \in \mathbf{R}^3 : d(p, M) = \rho\} \ .$$

[1]The definition given here is not exactly the same as presented in the companion theory paper [1], but it follows from the results in the companion paper that the different wordings yield equivalent definitions.

It is *not* necessary to assume that M is orientable for our definition of the ρ-envelope, as given here. The following lemma is key to the usefulness of the envelope and its proof is deferred to the companion theory paper [1], which also provides the bounds on ρ.

Lemma 3.1. *If M is C^2, then, for sufficiently small values of ρ, its ρ-envelope admits an ambient isotopic PL approximation via the nearest point mapping.*

4. Extending Known Reconstruction Algorithms

We remark that the theory presented in our companion paper shows how to create approximants that are ambient isotopic to $E_\rho(M)$, as well as approximants that are ambient isotopic to M. The examples presented here were motivated by that theory. They were created with new code as a pre-processing interface to the Power Crust algorithm in order to produce ambient isotopic approximations to $E_\rho(M)$. Complete adherence to the theory of our companion paper would have also required implementation of post-processing code to extract a subset of the Power Crust output to be ambient isotopic to M. This additional code is subtle and has not yet been fully implemented. The examples presented here demonstrate a viable alternative to that full implementation. Namely, the component of the medial axis of $E_\rho(M)$ that lies interior to $E_\rho(M)$ is equal to M. Since the Power Crust algorithm also produces an approximation of this interior component of the medial axis of $E_\rho(M)$, this approximation is taken as an approximation of M. The implications regarding ambient isotopy are discussed further in the next section. In the examples presented that contrast our results with direct reconstructions from the Power Crust, we note, in fairness, that the Power Crust algorithm was not designed to accept surfaces with boundary. However, in practice, the Power Crust has been found to be useful for reconstruction of some surfaces with boundary, with possible reliance upon *ad hoc* modifications [5]. Hence, our interest here in this comparison is to begin to formalize an admissible class of input surfaces for a provable implementation of ambient isotopic approximation of a rich class of surfaces. The algorithm used is presented, below, in pseudocode. We then offer several examples as representative of the scope of our new techniques.

4.1. **Reconstruction Algorithm Pseudocode.**

We first present the algorithm without any reliance upon any calculation of the medial axis and then discuss why the medial axis is used here as a means to interface with existing code that has already gained wide acceptance. The time lag for code to gain acceptance in practice is non-trivial and the pragmatic approach taken here leverages existing user acceptance while proceding to develop further innovations. One of the delicate open issues is the efficient, accurate approximation of normals along a boundary when only point cloud data is available. This is discussed further in Subsection 4.5.

For a compact, C^2 manifold, denote as follows:

- $S =$ a set of sample points from M, with density $\epsilon > 0$, appropriately chosen, (The density requirement is that for every point $x \in M$, there exists a point $s \in S$ within ϵ of s, denoted as $d(x, s) < \epsilon$.)
- $\lambda =$ the minimum positive distance between M and its medial axis,
- $B_r(x) =$ the 3-ball of radius r centered at x, with $r \in (0, \infty)$,
- np to abbreviate the nearest point mapping over a domain to be stated.

The pseduocode now follows.

Pseudocode:

Input: S

Choose ρ such that $\rho \in (2\epsilon, \lambda)$;

For each $x \in S$, create $B_\rho(x)$;

Let $D = \bigcup_{x \in S} B_\rho(x)$;

Find ∂D as an approximation to $E_\rho(M)$;

Using S and approximations of normals for M, derive a sample set $\hat{S}$ for $E_\rho(M)$;

Use $\hat{S}$ as input to an algorithm for an approximation K of $E_\rho(M)$
    // Relies upon $np : M \to E_\rho(M)$ being a homeomorphism //
    // Set $K = np(M)$ //

Use K to obtain a PL approximation, L of M.

Output: L, a PL ambient isotopic approximation of M, as proven in [1].

Discussion: The value for ρ is, of course, estimated. When surface defintions are available, such as the widely-used splines, this can be done quite accurately via computation of curvature on C^2 surfaces together with standard numerical methods to estimate minimal point separations, where these values are fully explained in the companion theory paper [1]. Of course, there is some error associated with each of these numerical computations, but the expectation is that these computations will be much better conditioned and more stable than approximations of the medial axis. Hence, the promise is to replace dependancy upon medial axis algorithms with these other numerical techniques. In the pseudocode, all the steps until the last can be done merely by writing new code to create $\hat{S}$, whereupon $\hat{S}$ can be used as input to an existing implementation, such as the Power Crust algorithm. This was exactly the approach taken on the examples given here. The last steps lie in the construction of L, since $np(M)$ is guaranteed to be ambient isotopic to M, but is not necessarily PL. Indeed, if one were satisfied with non-PL approximations, then the algorithm could terminate upon computation of $np(M)$. The component of the medial axis of $E_\rho(M)$ that lies in the bounded portion of $\mathbf{R}^3$ relative to $E_\rho(M)$ is M. Hence, the approximation of this component of the medial axis of $E_\rho(M)$ that is created by the Power Crust was used to generate the approximations here. There remains more work to study their topological properties versus those of M, but these experimental results are promising case studies toward resolving any disparities that may arise between these practical computations and the associated theory.

Notes:

- The sampling densities for S and $\hat{S}$ are related and this remains the subject of further investigation.
- Previously, the surface M had been assumed to be C^2 to ensure that λ was positive. The companion theory paper shows that weaker conditions on M are sufficient, as well as how to compute bounds on λ. The choice of ρ should be sufficiently small to provide an appropriate sampling density, but large enough to yield acceptable performance. Work continues on balancing those criteria.
- The creation of an accurate $\hat{S}$ is dependant upon the approximation of surface normals and that relation is continuing to be explored.

4.2. **Simple Example: Cylinder.** A cylinder is chosen as a simple example of a 2-manifold with boundary to illustrate the techniques of our reconstruction process.

First, in Figure 1, there are four images that depict the process of constructing an envelope. The top left is a tesselation of the quadratic NURBS cylinder. The top right presents a graphical display of an approximation of the envelope by balls centered at each vertex on the tesselation. The bottom left displays points selected along the surface normals of the cylinder at a distance equivalent to the radius of the spheres. The normals and tangents of the surface are used to define sample points on the envelope around the boundaries. The bottom right shows a point cloud set of the extent for the cylinder envelope and completes our pre-process for refining input data for the Power Crust.

Second, in Figure 2, there are two images, providing a comparison between a direct Power Crust reconstruction of points sampled directly from the cylinder surface (left), and our reconstruction of the cylinder relying upon input from sampled points from the envelope of the cylinder (right). It is of particular interest to note, that the modest additional code can provide the crisp image seen on the right, with no visibly apparent artifacts along the boundaries. We remark that since the Power Crust is designed to produce a closed surface, its output on the left is not surprising.

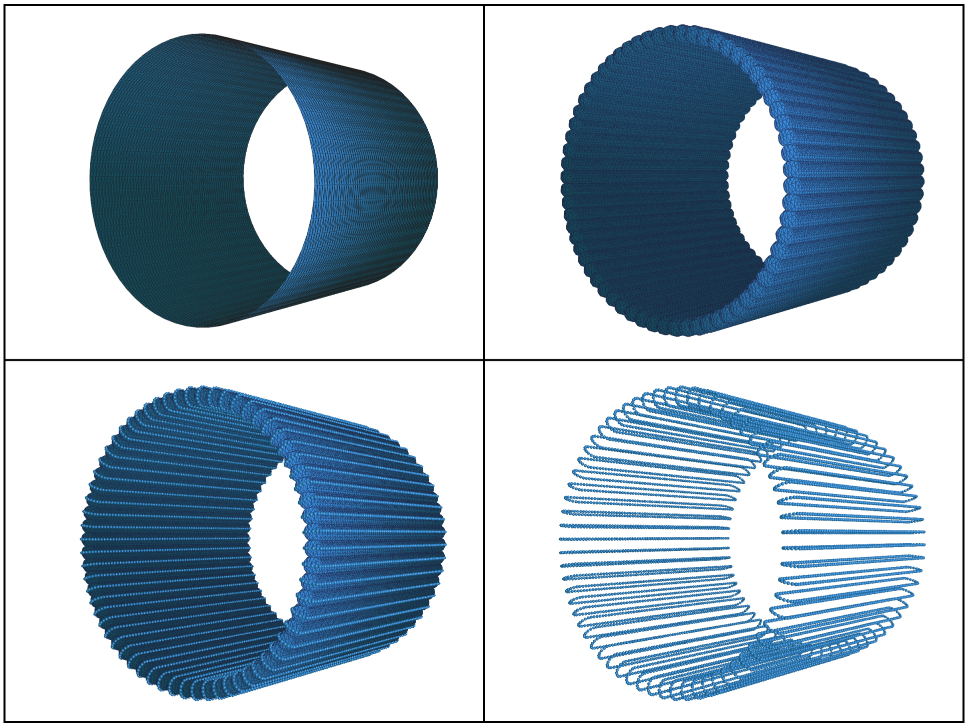

FIGURE 1. Cylinder

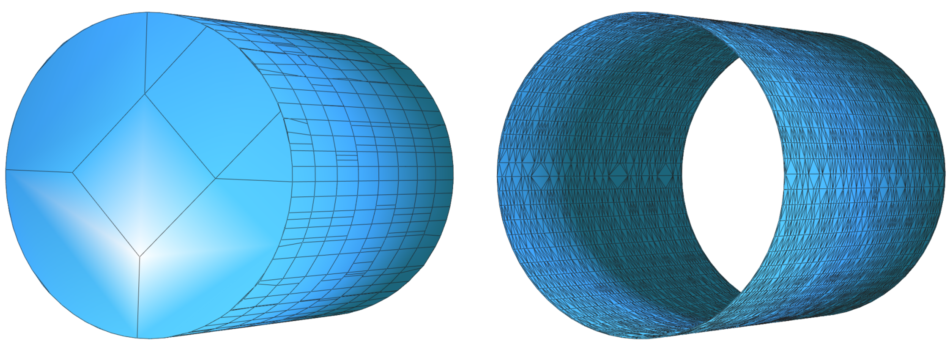

FIGURE 2. Comparison of Methods: Cylinder

4.3. **Difficult Example: Foot Data.** The example presented here is a challenging one already seen in the literature, where one heuristic approach was created to respect the boundary [5] and an alternate method was presented to close off that boundary [13]. Here, no surface definitions were known in advance (in contrast to the other examples presented) and the point cloud data was provided by the previously cited author [13].

Figure 3 has two images. On the left is a direct reconstruction of the foot from the sample points provided using the Power Crust algorithm. This image also has an enlarged view of the boundary region near the ankle, where many artifacts are clearly visible which close the surface. The right image shows a reconstruction of the foot using the same original sample points as input to the pre-process that builds the envelope of this data. Again, there is a closer view of the boundary region near the ankle, showing that the boundary is more faithfully preserved. It is our conjecture that these results could be further improved by better approximations of the normals.

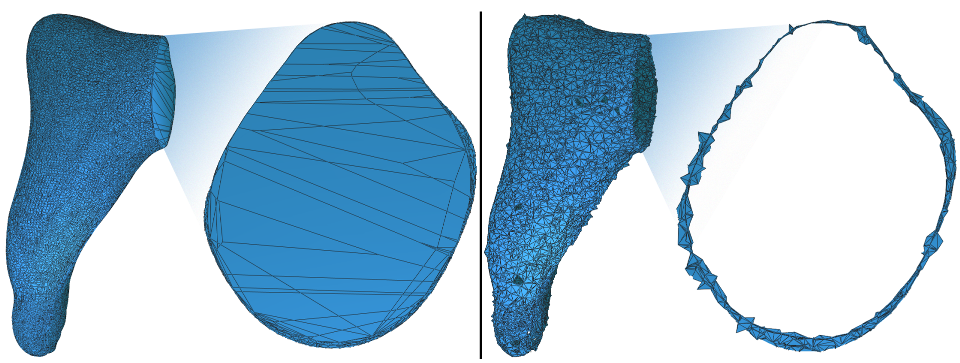

FIGURE 3. Comparison of Methods: Foot Data

Figure 4 has four images. The top left shows the original sample points for the foot, where these points were measured by a laser scan of the actual foot, and then their (x, y, z) co-ordinates were recorded in a text file. The top right shows the polar balls produced by the Power Crust, representing the radial field of the approximated medial axis of the point cloud, where noise is evident near the toes. The bottom left shows a sampling of poles determined from the Power Crust algorithm. The poles approximate normals to the original surface. The bottom right shows a point cloud representation for the envelope enclosure for the original point cloud. Since surface normals are central to the definition of the envelope and none are explicitly available here, envelope points are determined along the poles at a distance from the medial axis that is equivalent to the radius of the polar balls and offset in both directions. This foot envelope was constructed adaptively, where the radius used to construct the envelope varies with location of the sample point. This results in a tighter envelope around the toes and a slightly more generous envelope around the ankle. Along the boundary, additional points are created with user specified normals and tangents, appropriate to the envelope construction. This aspect remains within the judgement of the user. The careful investigation of 2-manifolds with boundary that appears

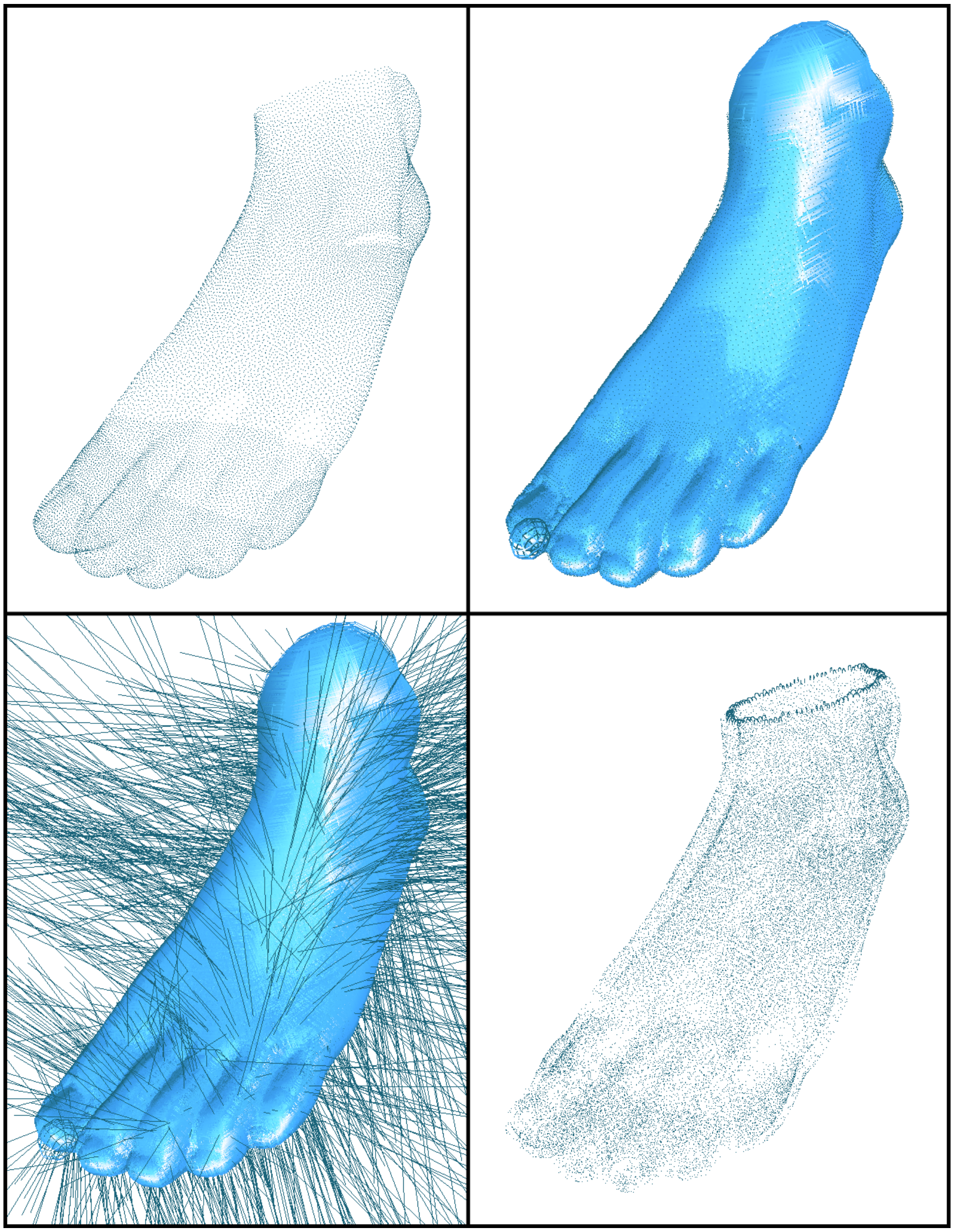

Figure 4. Stages of Method: Foot Data

in the companion paper [1] and the resultant experiments reported here provided crucial guidance in our refinement of these normal estimates, leading to the topology improvements shown in the boundary.

4.4. **Nonorientable Example: Möbius Strip.** The Möbius strip is the classical example of a non-orientable surface with boundary that can be embedded in $\mathbf{R}^3$. We know of no other technique that can produce such PL approximations of the Möbius strip, within error bounds chosen by the user. Our techniques readily handle this case.

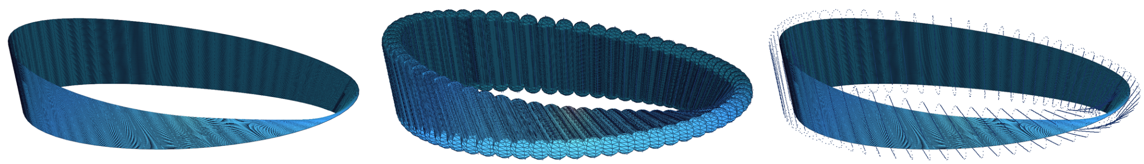

FIGURE 5. Möbius Strip

The images warrant some explanation. First, in Figure 5 there are three images. The left one shows the original surface as a tessellated graphics display of a Möbius strip. The middle image is a graphical display of many balls, centered at each vertex of the Möbius strip tesselation, to give a visual representation of the surface envelope. The right shows a point cloud of the extent of the envelope of the Möbius strip.

Next, in Figure 6, there are four images. The top left is a reconstruction of the Möbius strip reconstructed merely by feeding the sampled point cloud data directly to the Power Crust algorithm. Although not designed to directly accept such input (as already mentioned), we judged this to be valuable to include as a basis of comparison. The top right image shows the approximation of the medial axis of the Möbius strip that is generated by the Power Crust algorithm. The lower left shows the envelope of the Möbius strip, as reconstructed by the Power Crust. The lower right shows the final reconstruction of this non-orientable surface as the internal medial axis of the reconstructed envelope.

4.5. **Discussion of Input Needed and Final Output.** Throughout, it was necessary to make judicious choices about the radii for the balls used. In the case of the cylinder, this is trivial, as the only variable that needed to be considered is the radius of the cylinder. The foot was more complex, but the minimum feature size (in the sense of Amenta et al [2]) has been discussed in the literature as being related to the minimum separation between the toes. Hence, an estimate was made of this value. For the Möbius strip, an estimate was easily obtained. The simplicity of the cylinder and Möbius strip examples was specifically chosen to demonstrate the key processes involved, while the significantly more complex foot example shows the power of this method and its advantage over existing techniques. Our use of estimates regarding the point cloud data is consistent with many discussions at a recent DIMACS conference [15] on surface reconstruction which expressed that practical implementations also used pragmatic estimates of feature sizes.

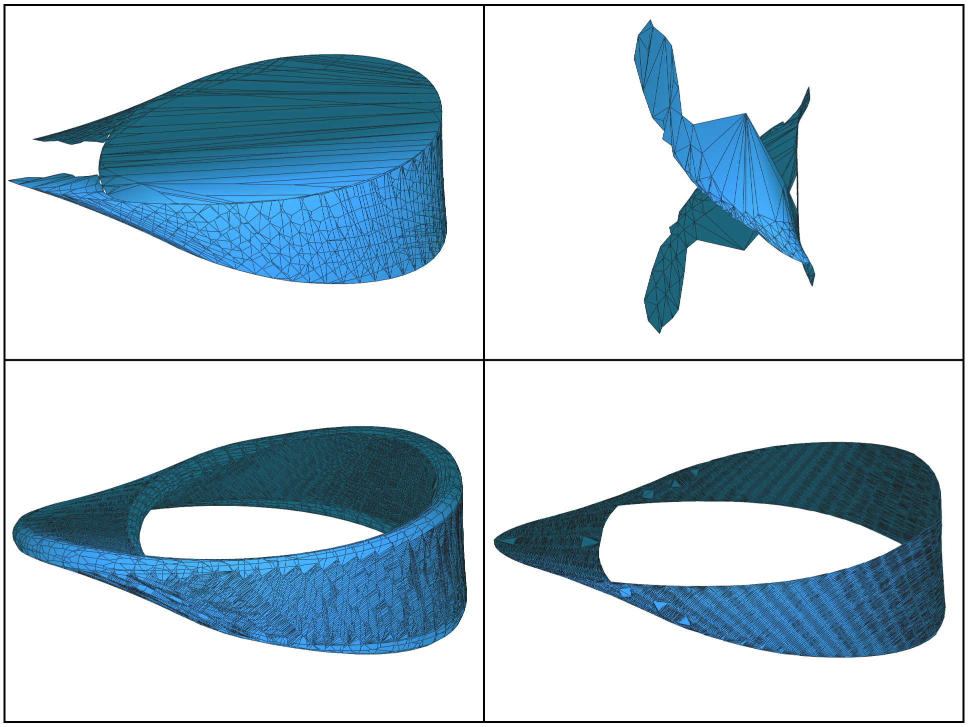

Figure 6. Comparison of Methods: Möbius Strip

The companion theory paper [1], presents a guarantee of an ambient isotopic approximation to the original manifold with boundary by extraction of a subset from PL approximations which are homeomorphic to M via the least distance map. The implementation of the supporting code for this extraction remains an ongoing task, but the promising results achieved here were by an effective expedient that did *not* depend upon the existence of this post-processing code. Namely, the component of the medial axis of an envelope surface that is contained in the interior of the envelope was selected as an approximation to the original manifold. An approximation to this component is already produced by the Power Crust algorithm and the results presented here show this to be a good approximation in practice. Further theoretical work is needed to integrate these empirical findings with the conditions needed to guarantee ambient isotopy of the approximation [1].

As previously noted, M is equal to the interior component of the medial axis of $E_\rho(M)$. Hence, in principle, the expedient used here is well-founded, but there remain some issues for further investigation. Namely, the Power Crust necessarily produces an approximation of the medial axis, so, if there are any deviations of this approximation from the true medial axis, then no formal topological guarantees can be given for the examples presented here. This remains the subject of futher investigation. Furthermore, since previously published theory [8] proving an ambient isotopy approximation relied only upon a positive distance from the medial axis, it is expected that the corresponding implementation similarly does not need the previously stipulated condition of the surface being C^2, where, again, these results corroborate the correctness of that assumption.

Initial 'proof-of-concept' experiments were performed on several simple NURBS surfaces, some of which are illustrated in Subsections 4.2 and 4.4. This permited a controlled environment to analyze the results obtain by the envelope technique. All the information necessary to produce an envelope may be found analytically in a NURBS surface representation. The normals, partial derivatives and maximum curvature can be readily obtained to produce a precise envelope. This information, together with an estimate of the minimum feature size [2], then can guide the sampling rate to guarantee an ambient isotopic approximation similar to techniques already discussed in the literature [8] which are extended in our companion theory paper [1]. The examples presented here show that an accurate envelope construction will yield a faithful and desirable reconstruction.

The foot example is, of course, more challenging, already having been used in the literature as an important test case [5, 13]. Our reconstruction presented compares well with that done by heurisitc methods [5], where our advantage is a reliance upon provable techniques and a well-defined class of permissible input surfaces. (The other primary approach to this problem [13] is not directly comparable, as it eliminated this boundary, whereas we preserve it.) However, the results of Subsection 4.3 can be compared to the other subsections to see that our reconstruction still suffers in the absence of critical geometric data, which we attribute primarily to the need to approximate normals from the foot point cloud data to use in our envelope construction technique. Those approximated normals are shown in Figure 4 in addition to the polar balls which indicate the accuracy of the medial axis approximation. Artifacts in the foot reconstruction appear in the form of holes and local maximum/minimum that are inconsistent with the original geometry. Hence, we stress the need for further investigation of techniques for approximating surface information from point cloud data to support accurate construction of envelopes for manifolds with boundary.

5. Concluding Remarks

In this paper we present a method for establishing surfaces which can approximate a given C^2 compact manifold M, where our fundamental contribution is that M can be with boundary, thereby extending past results which applied only to manifolds without boundary. The results presented here start from considerations of curvature. While curvature and the medial axis are closely related, there will be advantages to our input requirements, as it is known that it is not always possible to create a topologically correct approximation of the medial axis, if only sampled point data is available. Hence, these theoretical results will be useful in extending existing surface reconstruction algorithms, particularly as our experimental observations indicate that sampling density is not the only factor regarding current limitations on reconstruction of surfaces with boundaries.

The theory is sufficiently general for ambient isotopic approximation that need not be piecewise linear, although the examples presented here rely on the Power Crust software to produce piecewise linear approximations. However, higher order approximations may be particularly useful in engineering applications where spline geometry dominates, so this option merits further investigation. Furthermore, we have defined the envelope of a C^2 surface with boundary. In our companion paper we show that this envelope is a surface without boundary, which sufficiently smooth to permit topologically correct surface reconstruction. (The envelope will not be C^2.) This led to our prototype code that extends the Power Crust software to accept even surfaces with boundary. We show examples of how that modification can be effective and discuss the further integration and testing that should be undertaken.

As an unexpected side-effect of these investigations into surfaces with boundary, we have also discovered that previous hypotheses on differentiability for surfaces without boundary were overly restrictive in their focus upon C^2 surfaces. This observation now affords the opportunity to consider reconstruction of surfaces that are commonly used in engineering design, where fillets and blends are functionally crucial but preclude C^2 continuity. More details are provided in the companion theory paper.

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