COMPUTATIONAL TOPOLOGY FOR RECONSTRUCTION OF SURFACES WITH BOUNDARY, PART II: MATHEMATICAL FOUNDATIONS

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Abstract. This paper presents new mathematical foundations for topologically correct surface reconstruction techniques that are applicable to 2-manifolds with boundary, where provable techniques previously had been limited to surfaces without boundary. This is done by an intermediate construction of the *envelope* (as defined herein) of the original surface. For any C^2 manifold it is then shown that its envelope is $C^{1,1}$ and this envelope can be reconstructed with topological guarantees. The proof is then completed by defining functions which permit the mapping of a subset of the reconstruction of the envelope and it is further shown that the image of this mapping is a piecewise linear ambient isotopic approximation of the original manifold. The emphasis of this paper is upon proof of the new mathematical insights needed for these extensions, where more practical applications and examples are presented in a companion paper.

Keywords: Ambient isotopy; computational topology; computer graphics; surface approximation; topology methods for shape understanding and visualization.

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1. Introduction and Motivation

Several recent approaches to topology-preserving surface approximation have been restricted to those 2-manifolds without boundary which are also C^2 [4, 6, 8, 19, 42]. The proofs presented here provide two directions of generalization so that reconstruction can now be applied to:

(1) C^2 2-manifolds with boundary, and
(2) 2-manifolds without boundary that are merely $C^{1,1}$.

This goal of provable surface reconstruction techniques for surfaces with boundary has been presented as an open challenge within previously published literature [19]. We see this theory as being completely responsive, even as we note in the companion application paper [1], that some pragmatic refinements have been made in our prototpye implementation. The prototype is now supporting experimental research to resolve an acceptable balance between theory and practice for a comprehensive surface reconstruction system.

The paper is organized as follows: In Section 2, we summarize related work. Section 3 provides background material in differential geometry. Section 4 presents the definition of the envelope and its properties, as supporting new material for the proofs that follow. Section 5 demonstrates families of ambient isotopic surfaces of the envelope. Section 6 presents our lemma demonstrating a minimum positive distance between a $C^{1,1}$ surface and its medial axis, generalizing well-known results for C^2 surfaces. Section 7 contains the primary results to support new approaches to ambient isotopic surface approximation and reconstruction. It consists of three subsections. The first describes the intutitive ideas behind construction of a PL ambient isotopic approximation, as is utilized in the companion application paper [1]. The second subsection gives the details of the required proofs. The third subsection indicates where the theory is used to support specific aspects of the construction algorithm used in the companion application paper [1]. Concluding remarks are presented in the last section.

Figure 1, below, also appears in the application paper and it is reproduced here to illustrate the value of this work. In this simple case the improvement of our techniques is obvious, where the left image shows a reconstruction of a cylinder with boundary that was not created with our methods, whereas the one on the right has clean boundaries via our method.

2. Related Work

There are several recent publications [4, 6, 8, 19] with an emphasis upon topological guarantees for surface reconstruction. This paper presents significant theortical extensions beyond that cited literature, as noted in the previous section. Furthermore, examples showing the power of these theoretical extensions is given in a companion

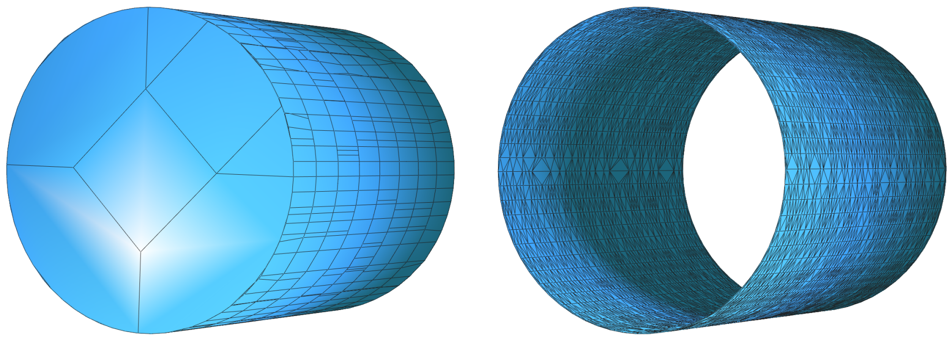

FIGURE 1. Cylinder

paper [1], where the application context is discussed in detail. Hence, the reader is referred to that companion paper for further application details, in order to keep the presentation here focused upon the theory to support reconstruction of surfaces with boundary. The theoretical concerns in providing topological guarantees for surface approximations near boundaries have been presented in the literature [5, 19, 25] within the context of approximants created during surface reconstruction.

The value in preferring ambient isotopy for topological equivalence [8, 42] versus the more traditional equivalence by homeomorphism [47] has previously been presented [8, 42] and the interested reader is referred to those papers for formal definitions.

Since the primary focus here is the supporting theory and mathematics we merely indicate that the most central references for those proofs are in standard mathematical texts [27, 28]. These serve as the primary references for the proofs presented here.

3. PRELIMINARIES

The proofs that are presented here rely heavily upon basic notions from differential geometry and the relevant aspects are summarized here. For the reader who is already well versed in those topics, it may be sufficient to use this section primarily as a reference for the notation that follows in the rest of the paper.

Throughout this paper, the following terminology will be assumed.

Remark 3.1. *All surfaces will be assumed to be compact (orientable[7]) manifolds within* $\mathbf{R}^3$.

[7]For the context of this paper, we are considering compact 2-manifolds embedded in $\mathbf{R}^3$. Since closed non-orientable 2-manifolds without boundary can only be embedded in $\mathbf{R}^n$, for $n \geq 4$ [28, Theorem 4.7], the additional assumption of orientability leads to no loss of generalization in the present proofs. For surfaces with boundary, however, the orientability condition is crucial; otherwise the envelope construction used here results in a double cover of the original surface, which will no longer be diffeomorphic to the original surface under the end point map.

Remark 3.2. *A function f from a compact manifold M into R^3 is an embedding $f : M \to R^3$ if the following are true*

- *f is continuous and injective,*
- *the Jacobian map of f is of full rank, and*
- *f preserves the induced subspace topology taken from R^3.*

In this article we present theoretical foundations for our work with computational models of curves and surfaces. We begin by defining the elements of differential geometry required to state and prove our results. Good treatments of this elementary material can be found the texts [27, 14].

Although the concepts and properties we describe below in this section extend to any dimension and any degree of differentiability greater than two, we restrict our attention to the curves and surfaces in three dimensional Euclidean space for the sake of simplicity and our current needs. Hereafter we assume that all differentiable objects are C^2, as defined below, unless otherwise stated (Plesae see [14]).

Definition 3.1. *A Hausdorff topological space M satisfying the second countability axiom is called a C^2* differentiable manifold of dimension two (without boundary) *if it satisfies the following:*

(1) *For any point $x \in M$, there exists a pair (U, ϕ_U), where U is an open neighborhood of x in M, and $\phi_U : U \to A \subset \mathbf{R}^2$ is a homeomorphism of U with an open set of $\mathbf{R}^2$. The neighborhood U is called a coordinate neighborhood (or patch) of x and the function ϕ_U is called a coordinate function of x. The function ϕ_U introduces the* local coordinates *$\phi_U(x) = (u_1(x), u_2(x))$ for this patch. The pair (U, ϕ_U) is often referred to as a* coordinate patch.
(2) *For any coordinates patches U, V with $U \cap V \neq \emptyset$, the map $\phi_V \circ (\phi_U)^{-1} : \phi_U(U \cap V) \to \phi_V(U \cap V)$ is C^2.*

Similarly, a C^2 differentiable manifold M of dimension two with boundary ∂M is defined as follows.

Definition 3.2.

(1) *If $x \in M - \partial M$, there is a coordinate pair as in (1) above. If $x \in \partial M$, there is a coordinate pair (U, ϕ_U) with a surjective homeomorphism $\phi_U : U \to H^2$, where H^2 is the half plane $\{(x_1, x_2) \in \mathbf{R}^2 : x_2 \geq 0\}$.*
(2) *Given two coordinates patches U, V with $U \cap V \neq \emptyset$, the function $\phi_V \circ (\phi_U)^{-1} : \phi_U(U \cap V) \to \phi_V(U \cap V)$ is C^2 in the usual sense if $U \cap V$ contains no point in ∂M. Otherwise, the map $\phi_V \circ (\phi_U)^{-1}$ can be extended to a C^2 homeomorphism in a open subset of $\mathbf{R}^2$ that contains the domain $\phi_U(U \cap V)$.*

If M is compact, ∂M is a disjoint union of finite closed curves, each of which is diffeomorphic [39] to the unit circle.

Let M be a two dimensional manifold with or without boundary. A function $f : M \to \mathbf{R}^3$ is said to be a C^2 differentiable map if for any point $x \in M$, there is a coordinate patch (U, ϕ_U) about x so that the composition $f \circ (\phi_U)^{-1} : \phi(U) \to \mathbf{R}^3$ is C^2.

Definition 3.3. *A C^2 submanifold of dimension two in $\mathbf{R}^3$ is a pair (M, f) of a manifold M of dimension two and an injective C^2 differentiable map $f : M \to \mathbf{R}^3$ such that the rank of the Jacobian map of $f \circ (\phi_U)^{-1} : \phi(U) \to \mathbf{R}^3$ is two for all coordinate patches (U, ϕ_U).*

What we see as a surface in $\mathbf{R}^3$ in the conventional sense is the image of M in R^3 under f. In the case when M is a submanifold of $\mathbf{R}^3$, we often identify M with $f(M)$ if there is no risk of confusion. The map f is also called the parametrization of the surface. However, as is in the cases to follow, we often need to distinguish M and its image.

Since the Jacobian map $f_*(x)$ of f at $x \in M$ is of full rank 2, it gives rise to an injective linear map of the tangent space[8] TM_x into the tangent space $T\mathbf{R}^3_{f(x)}$, which is identified with $\mathbf{R}^3$ in the conventional way.

The tangent space TM_x is identified with $\mathbf{R}^2$ with the standard coordinates (u_1, u_2) under the coordinate map ϕ_U. In terms of these coordinate systems, the matrix representation of $f_*(x)$ is the following three by two matrix:

$$\begin{pmatrix} \dfrac{\partial x_1}{\partial u_1} & \dfrac{\partial x_1}{\partial u_2} \\ \dfrac{\partial x_2}{\partial u_1} & \dfrac{\partial x_2}{\partial u_2} \\ \dfrac{\partial x_3}{\partial u_1} & \dfrac{\partial x_3}{\partial u_2} \end{pmatrix},$$

where $x_i(u_1, u_2) = f_i(u_1, u_2)$, $i = 1, 2$, are the coordinate functions of f.

The image $f_*(x)(TM_x)$ is a plane passing through $f(x)$ in $\mathbf{R}^3$ and is called the *tangent plane* to the surface $f(M)$ at $f(x)$, but also referred to as the tangent plane to M at x. The ordinary dot product in $\mathbf{R}^3$ induces an inner product in the tangent plane. The induced inner product gives rise to the induced Riemannian metric in M. When we say a *surface* in $\mathbf{R}^3$, we implicitly imply the triple consisting the manifold M, the embedding f and the induced Riemannian metric.

Let (M, f) be an embedded surface in $\mathbf{R}^3$. Denote by $\mathbf{n} = \mathbf{n}_x$ a (local) unit normal field along $f(M)$. Given a tangent vector X to M at x, $D_{f_*(X)}\mathbf{n}$ denotes the directional derivative of $\mathbf{n}$ in the direction of $f_*(X)$ in $\mathbf{R}^3$, where f_* is the Jacobian map of f at

[8]The tangent space is an abstraction of the standard notion of a plane of tangent vectors for each point of a differentiable manifold in $\mathbf{R}^3$.

x. The derivative $D_{f_*(X)}\mathbf{n}$ is tangential to $f(M)$ at $f(x)$. By setting

$$D_{f_*(X)}\mathbf{n} = -f_*(AX),$$

one can obtain a linear operator A of the tangent space TM_x, see [27]. The map A determines the local geometric shape of the embedded surface $f(M)$. A is a symmetric linear operator with respect to the induced Riemannian metric; hence A can be represented by a 2×2 symmetric matrix with respect to any orthonormal basis for TM_x.

Definition 3.4. *The linear operator $A = A_x$ is called the* shape operator *(or the second fundamental form) of the surface (M, f). The eigenvalues of A are the* principal curvatures *of the surface at the point x (see, e.g.,* [27]*).*

Definition 3.5. *A point $x \in M$ is said to be a* critical *point of a C^2 function $g : M \to \mathbf{R}$ if the differential $dg = \dfrac{\partial g}{\partial u_1} du_1 + \dfrac{\partial g}{\partial u_2} du_2 = 0$ at x, where (u_1, u_2) is a coordinate system about x in M. A critical point is called* nondegenerate *if its Hessian $Hg(x) = \left(\dfrac{\partial^2 g}{\partial u_i \partial u_j}\right)$ is invertible; otherwise it is called* degenerate.

For our purposes, it is convenient to characterize the critical points of a function defined in M in the context of submanifolds, namely, in the extrinsic setting. Let g be as above. We state the following proposition without proof.

Proposition 3.1. *The point $x \in M$ is a critical point of g if there is an open neighborhood U of $f(x)$ in $\mathbf{R}^3$ and a C^2 function $\tilde{g} : U \to \mathbf{R}$ with $\tilde{g} = g \circ f^{-1}$ restricted to $f(M) \cap U$ such that the gradient $\nabla \tilde{g}$ in $\mathbf{R}^3$ is normal to the tangent plane to $f(M)$ at $f(x)$. Furthermore, such an (local) extention $\tilde{g}$ always exists.*

We now define the (global) energy function for a manifold with boundary.

$$G : M \times M \to \mathbf{R}, \quad G(x, y) = \|x - y\|^2, \tag{1}$$

where $\|x - y\|^2$ is the square of the ordinary distance function on $\mathbf{R}^6$.

We need to identify the critical points of G. In the intrinsic sense, a critical point is a pair $(x, y) \in M \times M$ such that $dG(x, y) = 0$ as we defined above. Extrinsically, recall that that M is embedded by f into $\mathbf{R}^3$; hence, $M \times M$ is canonically embedded into $\mathbf{R}^6 = \mathbf{R}^3 \times \mathbf{R}^3$ under $f \times f : M \times M \to \mathbf{R}^3 \times \mathbf{R}^3$. Note also that the function G can naturally be extended in the entire $\mathbf{R}^3 \times \mathbf{R}^3$. Therefore, we may redefine, by Proposition 3.1, a critical point of $G : M \times M \to \mathbf{R}$ to be a point $(x, y) \in M \times M$ where the gradient field $\nabla G(x, y)$ is normal to $(f \times f)(M \times M)$ at $(f \times f)(x, y)$.

Proposition 3.1. *Let G be defined as in Equation 1. Then, there exists a minimal positive critical value of G in $M \times M$.*

Proof: Obviously, $G(x, y) > 0$, for $x \neq y$. Second, note that G has a critical value $r > 0$, for example, the maximal value, since $M \times M$ is compact. The gradient of G in $\mathbf{R}^3 \times \mathbf{R}^3$ is given by

$$\nabla G = 2(x - y, -(x - y))$$

where $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3)$ are the standard Euclidean coordinates of x, y, respectively.

On the other hand, the tangent plane to $f(M)$ at $f(p)$, $p \in M$ in $\mathbf{R}^3$ is spanned by two vectors $\dfrac{\partial f}{\partial u_i}$, $i = 1, 2$. Hence, the tangent space of $(f \times f)(M \times M)$ at $(x, y) = (f(p).f(q))$ in $\mathbf{R}^3 \times \mathbf{R}^3$ is the 4-space spanned by four vectors $\dfrac{\partial f}{\partial u_i}(p)$, $i = 1, 2$ and $\dfrac{\partial f}{\partial v_i}(q)$, $i = 1, 2$, where, as before, (u_1, u_2), (v_1, v_2) denote local coordinates about p, q, respectively. The gradient ∇G is normal to the tangent space of $(f \times f)(M \times M)$ at $(f \times f)(p, q)$ if and only if

$$\textstyle\sum_{k=1}^3 (f_k(p) - f_k(q)) \dfrac{\partial f_k}{\partial u_i}(p) = 0,\ i = 1, 2,$$

$$\textstyle\sum_{k=1}^3 -(f_k(p) - f_k(q)) \dfrac{\partial f_k}{\partial v_i}(p) = 0,\ i = 1, 2.$$

If M has no boundary, this immediately tells us that (p, q) is a critical point of G if and only if either the line segment connecting $f(p), f(q)$ is normal to the tangent planes to $f(M)$ at $f(p)$ and $f(q)$ in $\mathbf{R}^3$, or $f(p) = f(q)$. We claim that if

$$c = \inf\{r > 0 \mid r \text{ is a critical value of } G\} \tag{2}$$

then c is positive. An elementary proof of c being positive for compact surfaces without boundary is found in [42, 43].

When M has a non-empty boundary, the situation is slightly more complicated. There will be three possible cases for critical points to occur. (1) (x, y) is a critical point of G and x and y both lie in the interior of M; (2) (x, y) is a critical point G and one of them lies in the interior of M and the other lies in ∂M; (3) (x, y) is a critical point of G and x and y both lie in ∂M. In any of these cases, a slightly modified proof for the case without boundary also works. ■

4. The Definition of the Envelope

The purpose of this section is to define a new surface that can be created from M, which we call the envelope of M. Some properties of the envelope are then proven. These proofs rely upon the use of boundary collars [28] as well as upon the value of ρ

to ensure that the resulting envelope will not be self-intersecting or degenerate. Let M be an abstract surface with boundary. Then we have from the definition:

(1) ∂M is a disjoint union of closed curves $c_1, \cdots, c_l$, each of which is diffeomorphic to the unit circle S^1.
(2) Along each $c_j, 1 \leq j \leq l$, we can attach a collar of the form $c_j \times [0, 2\epsilon_j)$, for some positive number ϵ_j, so that the topological space $M_j = M \cup \{c_j \times [0, 2\epsilon_j)\}$ (where M_j is defined under the quotient map that identifies c_j and $c_j \times 0$ in the natural way) is a surface with the same differentiable structure as M and the same degree of differentiability as M. The surface M_j contains M and M_j now has the previous boundary component in its interior. Thus, successive attachments of collars along all boundary components produce an open surface
$$\tilde{M} = M \cup (\cup_{1 \leq j \leq l} \{c_j \times [0, 2\epsilon_j)\} \ .$$
$\tilde{M}$ contains M as a proper subset and $\partial \tilde{M} = \emptyset$. Furthermore, we can choose ϵ_j, $1 \leq j \leq l$, in such a way that the embedding f of M can be extended to an embedding $\tilde{f}$ of $\tilde{M}$. This means the pair $(\tilde{M}, \tilde{f})$ is a surface in $\mathbf{R}^3$ which extends the original surface (M, f).

For purely technical reasons, we introduce a new surface $\hat{M}$ with boundary $\partial \hat{M}$ given by $\hat{M} = \tilde{M} - \cup_{j=1}^{l} (c_j \times (\epsilon_j, 2\epsilon_j))$. We note here that the minimal positive critical values of G defined in $\hat{M}$ is less than or equal to that in M.

With respect to the induced metric in $\tilde{M}$ from R^3, consider a unit normal field ξ to $\tilde{M}$. The shape operator A_ξ of $\tilde{M}$ is given as the tangential component of the directional derivative of ξ; namely, $\tilde{f}_*(A_\xi(X)) = -D_X \xi$, which is the directional derivative of ξ in the X-direction. The operator D is also called the covariant derivative in differential geometry (or often as the standard Riemannian connection).

Let $\hat{c}$ denote the smallest positive critical value of $\hat{G}$, the natural extension of G to $\hat{M} \times \hat{M}$. Then we see that $\hat{c} \leq c$. Also, denote by $\kappa = \max_{x \in M} \{|K_1(x)|, |K_2(x)|\}$, where $K_i(x)$, $i = 1, 2$ are the principal curvatures at $x \in M$. Now denote by $\hat{\kappa}$ the number defined to be $\max_{x \in M'} \{|\hat{K}_1(x)|, |\hat{K}_2(x)|\}$, where $\hat{K}_i(x)$, $i = 1, 2$ are the principal curvatures at $x \in \hat{M}$. As noted before these are at least continuous in M and $\hat{M}$, respectively. Then $\kappa \leq \hat{\kappa}$. Since $\hat{M}$ is compact, the absolute values of these quantities attain the absolute extrema.

Definition 4.1. *Set* $\hat{\delta} = \dfrac{1}{2} \min\{\hat{c}, \dfrac{1}{\hat{\kappa}}\}$.

Note here that we use the convention $1/\kappa = +\infty$ when $\kappa = 0$ without loss of generality. Also note that it is well known that M is a part of a plane if the principal curvatures are zero everywhere in M. We may then exclude this case since an ambient isotopy of such a set can readily be constructed. Hence, we assume $\hat{\delta}$ to be a finite positive number.

Let M be a surface with boundary in general. We introduce a compact closed surface called the r-envelope of M as follows. Let c_i, $1 \leq i \leq n$ be the boundary curves of M. We first define a surface $P_r(c_i)$ about c_i, $1 \leq i \leq n$ (which are called the pipe surfaces [38]). A specific parmetrization of these surfaces are also given for later use.

Let $c = c(t),\ t \in [0, l]$ be a regular closed space curve in R^3. Further assume that the curve has no self-intersection and that it is parmetrized by its arc length; hence, l is the total arc length of the curve. For a sufficiently small $r > 0$,

$$P_r(s,t) = c(t) + r\xi(t)\cos s + r\eta(t)\sin s,\ 0 \leq t < l,\ 0 \leq s < 2\pi$$

gives rise to a closed surface in R^3 parametrized by (s,t), where $\xi(t)$ and $\eta(t)$ form an orthonormal frame normal to the curve. For example, they can be the pair consisting of the normal and binormal of the curve [39]. We have $P_r(s,t) = P_r(c_i)$ when $c = c_i$. One may consider (t,s) as its coordinates (see the remark below). The tangent plane to this surface at (t,s) is spanned by the following two tangent vectors:

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial(c(t) + r\xi(t)\cos s + r\eta(t)\sin s)}{\partial t} &&= \frac{dc(t)}{dt} + r\frac{d\xi}{dt}\cos s + r\frac{d\eta}{dt}\sin s \\
\frac{\partial}{\partial s} &= \frac{(\partial c(t) + r\xi(t)\cos s + r\eta(t)\sin s)}{\partial s} &&= -r\xi(t)\sin s + r\eta(t)\cos s.
\end{aligned}$$

One can readily see from the above expressions that these tangent vectors are linearly independent for sufficiently small r, hence, the resulting surface is indeed an embedded surface in R^3. The surface $E_r(c)$ for each sufficiently small r is called the r-pipe surface [38]. It is the well-known embedded r circle bundle of the curve. The radial vectors emanating from $c(t)$ are the radial vectors of the circles. Hence, they are given by $r\xi(t)\cos s + r\eta(t)\sin s,\ 0 \leq t < l,\ 0 \leq s \leq 2\pi$. We show that these radial vectors are, indeed normal to the surface at each (t,s). First note the following.

(1) $\dfrac{dc(t)}{dt} \cdot \xi = \dfrac{dc(t)}{dt} \cdot \eta = 0,$
where $X \cdot Y$ means the dot product between X & Y.

(2) $\dfrac{d\xi}{dt} \cdot \xi = \dfrac{d\eta}{dt} \cdot \eta = 0$, since ξ and η are unit vectors.

Using (1) and (2), one can easily compute that the dot products between $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial s}$ and the radial vectors are 0; hence, the radial vectors are normal to the surface $E_r(c)$.

In [28], it is shown that there is a certain positive number δ_c such that the map given by $(s,t,r) \mapsto c(t) + r\xi(t)\cos s + r\eta(t)\sin s,\ 0 \leq t < l,\ 0 \leq s < 2\pi\ 0 \leq r < \delta$ is an embedding into R^3. This is an embedding for sufficiently small r, typically called the *r-tubular neighborhood*. Then $E_r(c)$ also gives a special case of what we call an envelope as defined below.

Let x be a point in ∂M. We may assume that x belongs to a C^2-regular space curve $c_i = c_i(t)$, $0 \leq t < l_i$ with $c_i(0) = x$. We may even assume c_i is parametrized by its arc length without loss of generality. This implies $|dc_i/dt| \equiv 1$ for all t and that l_i equals the arc length of c_i. Let ξ be a unit normal to M. Denote by $\xi(t)$ and $\eta(t)$ the restriction of ξ to c_i and the unit outward normal at $c_i(t)$, respectively, so chosen that $\frac{dc}{dt}, \xi(t), \eta(t)$ form the right hand system with respect to the standard orientation of R^3. Here the outward normal means the unit vector that is perpendicular to the plane spanned by dc_i/dt and ξ and that points away from M at $c_i(t)$. Since M is C^2, These vectors are at least C^1 along $c_i(t)$.

For any $r > 0$, define $E_r(M)$ by

$$E_r(M) = \{x \pm r\xi,\ x \in M\} \cup c_i(t) + r\xi(t)\cos s + r\eta(t)\sin s,\ 0 \leq t < l_i,\ 0 \leq s < \pi\}.$$

Definition 4.2. *$E_r(M)$ is called the r-envelope of M.*

Note that $E_r(M)$ is not even a topological manifold for some r, but it is readily seen[9] that for a sufficiently small r, $E_r(M)$ is at least C^1 everywhere but in a finite number of curves where it is at least G^1. We now give an explicit description of those curves for the future use. Set $S_i(r,t) = c_i(t) + r\xi(t)$, $|r| < \hat{\delta}$, $0 \leq i \leq n$, where c_i's are the boundary components and ξ is the unit normal to M along those components. Note that at this point S_i may not be a regular surface, but it is the union of open line segments of length $2\hat{\delta}$ centered at the points in $c_i(t)$. In fact, they are ruled surfaces built on the boundary curves with $\xi(t)$ as the direction of the rulings. The set $E_r(M) \cap S_i(r,t)$ gives rise to a curve in $E_r(M)$ for each fixed r. Denote such a curve by $S_{i,r}$ for each i. In fact, we will show later that $E_r(M)$, for a certain range of r to be specified later, is C^2 everywhere but along $S_{i,r}$'s where it is at least $C^{1,1}$.

Now set

$$\delta = \min\{\hat{\delta}, \delta(c_i),\ 1 \leq i \leq n\}, \tag{3}$$

where c_i, $1 \leq i \leq n$ is a boundary curve of M and $\delta(c_i)$ is the maximal radius of the regularly embedded pipe surfaces $E_r(c_i)$ [34].

Let r_0 be a sufficiently small positive number so that $E_{r_0}(M)$ is well defined and C^1 except for along the curves S_{1,r_0}'s where it is G^1. Define a map

$$F_{r_0} : E_{r_0}(M) \times (-r_0, \hat{\delta} - r_0) \to \mathbf{R}^3$$

by

[9] Wolter [48] constructed the envelope of a spline surface parametrized in $\mathbf{R}^3$ by $[0,1] \times [0,1]$, although he did not call it an envelope. He states without proof that this envelope is a $C^{1,1}$-surface which is $C^{1,1}$-diffeomorphic to the unit 2-dimensional sphere for sufficiently small r. Strictly speaking, our proof is not applicable to his case since $[0,1] \times [0,1]$ is not a surface with boundary according to our Definition 3.2.

$$F_{r_0}(x,r) = x + r\mathbf{n},\ (x,r) \in E_{r_0}(M) \times (-r_0, \hat{\delta} - r_0), \tag{4}$$

where $\mathbf{n}$ is the unit normal field to $E_{r_0}(M)$ which points away from M at each point of $E_{r_0}(M)$. Such a choice of a normal is possible because of the definition of the envelope.

Lemma 4.1. *$F_{r_0}(x,r)$ is globally injective.*

Proof: First, we clearly see that $F_{r_0}(x,r)$ is a globally injective C^1 diffeomorphism when it is restricted to the pipe surface portions of the envelope by the choice of $\hat{\delta}$. Furthermore, the implicit function theorem yield $E_r(M)$ is a C^1 surface in the neighborhood of the points in the pipe surface portions. For any point $x \in E_r(M)$ given by the expression $\{x \pm r\xi,\ x \in M - \partial M,\ 0 < r < \hat{\delta}\}$, we need somewhat more elaborate and lengthy (but more or less elementary) arguments, for which we only give an outline here to save space. First we enlarge the set to $\{x \pm r\xi,\ x \in \hat{M} - \partial\hat{M},\ 0 < r < \hat{\delta}\}$. Now define a map $F : (\hat{M} - \partial\hat{M}) \times (-\hat{\delta}, \hat{\delta}) \to R^3$ by

$$F(x,r) = x + rn,\ r \in (-\hat{\delta}, \hat{\delta}), \tag{5}$$

where $n = n_x$ is a unit normal to $\hat{M}$ at x. Then it is well known that the Jacobian map F_* of F at (x,r) is the symmetric linear map whose eigen values are given by $\dfrac{K_i}{1 - rK_i}$ and 1. where $K_i,\ i = 1,2$ are the principal curvatures of $(\hat{M}, f)$. Consequently, F is non-singular as long as $|r| < \hat{\delta}$. This implies that F is locally a C^1 diffeomorphism since $\hat{M}$ is a C^2 surface. Hence, $F_{r_0}(x,r)$ is locally injective near every $(x,r) \in E_{r_0}(M) \times (-r_0, \hat{\delta} - r_0) - P_\delta$, where $P_\delta = \cup_{r<\delta,\, 1\le i\le n} P_{i,r}(s,t)$, with each $P_{i,r}(s,t)$ being the previously defined set $P_r(s,t)$ specific to the curve c_i. Note that $F_{r_0}(x,r)$ is basically defined by restricting F to this set.

Finally along the $S_i = c_i(t) + r\xi(t),\ |r| < \hat{\delta},\ 0 \le i \le n$, it is not hard to see that the envelope is G^1, i.e. the tangent planes vary continuously, and that $F_{r_0}(x,r)$ is locally injective along S_i's by the definition of the envelope and the local injective property of F_{r_0} off S_i's as described above.

To show that F_{r_0} is globally injective, first note that F_{r_0} is globally injective in $E_{r_0}(M) \times (-r_0, 0]$ by the choice of r_0. Let $\epsilon_0 = \inf \epsilon$ such that $F_{r_0}(x,r)$ fails to be globally injective in $E_{r_0}(M) \times (-r_0, \epsilon)$. It can be shown then that the existence of such an ϵ_0 less than $\delta - r_0$ presents a contradiction to the choice of δ, using arguments similar to ones previously published [42, 43], now applied to $E_{r_0}(M)$ in place of the compact closed surface M. Note that $E_{r_0}(M)$ is a compact closed surface. Although $E_{r_0}(M)$ is not C^2 as assumed in [8, 43], the basic arguments still applies to $E_{r_0}(M)$ with slight modifcations. ∎

5. Isotopies of the Envelope

Given a point $x \in R^3$, define a real valued function ρ_M by $\rho_M(x) =$ the ordinary distance function from x to M. Since M is compact, there is a point $m_x \in M$ such that $\rho_M(x) = |x - m_x|$. Since M is a C^2 surface (with or without boundary), the line joining x and m_x meets M perpendicularly. Thus, m_x is the foot of the perpendicular projection of x onto M. From Lemma 4.1 above, m_x is uniquely determined if x lies in the (connected) component of $R^3 - E_{\hat{\delta}}(M)$ which contains M. The component is an open neighborhood of M. Denote it by U_δ. This tells us that ρ_M is well-defined in U_δ. We know in general such a ρ_M is Lipschitz continuous.

Definition 5.1. *A function f is $C^{1,1}$ if its gradient, ∇f, is everywhere Lipschitz continuous.*

Since we have assumed that all of our manifolds are embedded in R^3, it is natural to adapt the previous definition to manifolds.

Definition 5.2. *A manifold M is $C^{1,1}$ if its gradient, ∇f, is everywhere Lipschitz continuous, where f is the function embedding M within R^3.*

We remark that there are natural C^1 manifolds that are not $C^{1,1}$ and natural $C^{1,1}$ manifolds that are not C^2. The manifold M_1 generated by rotating the graph of $y = (x-1)^{3/2}, x \geq 1$ about the y-axis is C^1, but not $C^{1,1}$. However, the lack of $C^{1,1}$ continuity is unlikely to be detected by visual inspection, as an image of M_1 looks virtually indistinguishable from that of the manifold M_2 which is generated by rotating the graph of $y = (x-1)^2, x \geq 1$ about the y-axis. Then, the well-known the stadium curves are $C^{1,1}$ but not C^2 and these easily generalize to surface examples.

We remark that this is strictly weaker then a manifold being C^2 as well as being strictly stronger than a manifold being C^1. The well known stadium curves provide the typical example that $C^{1,1}$ is strictly weaker than C^2.

This relaxation to $C^{1,1}$ is characterized by linear boundary segments being joined to arcs. In engineering applications, such curves could be the boundaries of slots or other machined cut-outs in manufactured parts. A typical stadium curve is shown in Figure 2.

Figure 2. Stadium Curve

Theorem 5.1. *The distance function ρ_M is a C^2 function in $U_\delta - M = \cup_{0<r<\delta} E_r(M)$ except along a finite number of surfaces S_i, $1 \leq i \leq n$, where it is $C^{1,1}$. The envelope*

$E_r(M)$, $0 < r < \delta$ is C^2 everywhere except along the curves

$$S_{i,r} = c_i(t) \pm r\xi(t),\ 0 < r < \delta,\ 1 \leq r \leq n,$$

and it is at least $C^{1,1}$ along those curves.

Proof: First we restrict the distance function ρ_M to the following two subsets;

$$\nu_\delta = \{x \pm rn(x),\ x \in M - \partial M,\ 0 < r < \delta\}\ ,$$

where $n(x)$ is the normal to M at x and $\xi = n$ along ∂M and

$$B_\delta = \{c_i(t) + r\xi(t)\cos s + r\eta(t)\sin s,\ 0 \leq t < l_i,\ 0 < s < \pi,\ r < \delta\}\ .$$

We first show that the distance function ρ defined in these sets are C^2-functions. $F(x, r) = x + rn(x)$, $|r| < \delta$ is locally diffeomorphic at $x \in M - \partial M$ by the choice of δ. It is not hard to see that this diffeomorphism is actually a C^1-diffeomorphism, since the Jacobian map of F is locally given in terms of the shape operator of the r level set $F(x, r)$, where $r \in [0, \delta)$ is fixed to be a constant. Note that the shape operators (or their eigen values) are at least continuous [28]. Thus we may consider F as giving a C^1-local coordinate chart about every point in ν_δ. With this coordinate system, it is easy to see that the gradient field $\nabla\rho$ of the distance function ρ is the unit tangential field to the normal rays emanating from M. The normal rays are generated by the normal field n to M and n is at least C^1, since M is assumed to be C^2. Hence, the tangential field is C^1. This implies that the gradient field $\nabla\rho$ is a C^1-field; consequently, ρ is a C^2-function in $\nu_\delta = \{x \pm rn(x),\ x \in M - \partial M,\ 0 < r < \delta\}$. Applying the implicit function theorem, the level sets of the distance function are also C^2 in ν_δ. Similarly, we see that the gradient field of the distance function in B_δ is the unit C^1 field generated by the radial rays emanating from the boundary of M. This is an easy consequence of our choice of δ [21, 34]. One can, in fact, show that the map $F : (0, \delta) \times R^2(s, t) \to R^3(x, y, z)$ defined by

$$F(r, s, t) = \{c_i(t) + r\xi(t)\cos s + r\eta(t)\sin s,\ 0 \leq t < l_i,\ 0 \leq s \leq 2\pi,\ r < \delta\} \tag{6}$$

is a C^1 diffeomorphism. This, in turn, yields that the gradient field $\nabla\rho$ of the distance function $\rho(F(r, s, t)) = r$ coincides with the radial unit normal which is defined to be the field of the unit tangent vectors to the radial rays that emanate from each point of c_i into the normal directions to the curve c_i at the point; hence, the desired result. Once again, one can show that the radial normal field is at least C^1. Thus, the distance function ρ_M is a C^2-function in B_δ. The implicit function theorem again yields the desired result that the level surfaces of the distance function ρ_M are C^2-surfaces except at $r = 0$, where it degenerates to be the boundary curves.

We now construct a specific C^1-local coordinate chart $(\tilde{U}_m, \psi_m)$ in R^3 about every point m in the surface $S_i(r, t) = c_i(t) + r\xi(t)$, $0 < r < \delta$, $0 \leq t < l_i$. Let $\eta_i(r, t)$ be the outward unit normal field to the $S_i(r, t)$. Then $\eta_i(r, t)$ is a local C^1-vector field along $S_i(r, t)$ and it is tangent to $E_r(M)$. Note that the surfaces S_i's are actually at least

C^1 surfaces. This can be verified by realizing that these surfaces occur in interior of the solid pipes over the boundary components, or can be regarded as surfaces in $\{x \pm rn(x),\ x \in \hat{M} - \partial\hat{M},\ 0 < r < \delta\}$, where $n(x)$ is the normal to $\hat{M}$ at x and $\xi = n$ along $\partial\hat{M}$. Define a new vector field $\tilde{\eta}_i$ along $S_i(r,t)$ by $\tilde{\eta}_i(r,t) = r\eta(r,t)$. $\tilde{\eta}_i(r,t)$ is also a C^1-vector field along the surface, since r is clearly a C^1-function there. Extend $\tilde{\eta}_i(r,t)$ to a non zero C^1 vector field in a neighborhood V_m of m and denote it by the same letter $\tilde{\eta}$ for convenience. Then $\tilde{\eta}$ can be regarded as a C^1 map from $R^1(t) \times V_m \subset R^4(t,u,v,w)$ into $R^3(u,v,w)$ by setting $\tilde{\eta}(x) = (\tilde{\eta}_1(x), \tilde{\eta}_2(x), \tilde{\eta}_3(x))$. Consider the system ordinary differential equations

$$\frac{dx_i}{dt} = -\tilde{\eta}_i,\ 1 \le i \le 3. \tag{7}$$

By the existence and uniqueness theorem for ordinary differential equations [14] there is a unique solution $x(t) = (x_1(t), x_2(t), x_3(t))$ to this system for a given initial condition in a sufficiently small neighborhood U_m of m, satisfying $x(0) = x_0$, $\frac{dx}{dt}(0) = \tilde{\eta}(x_0)$. The theorem also states that the map $x : (-t_0, t_0) \times U_m \to V_m$ defined by the solutions $x(t)$ is a C^1-map for a sufficiently small $t_0 > 0$. We choose the set of initial conditions to be the pair $(x, \tilde{\eta}(x)),\ x \in S_i \cap U_m$ and restrict the above map to $(-t_0, t_0) \times S_i \cap U_m$. It is easy to see that this restricted map has a non-degenerate Jacobian map at $(0, m)$. Hence, by the inverse function theorem, this restriction map is a C^1 diffeomorphism in a small neighborhood of m. Denote the diffeomorphism by $\tilde{\psi}_m$ and the neighborhood by $\tilde{U}_m$. The C^1-local coordinate system of the pair $(\tilde{U}_m, \tilde{\psi}_m)$ is denoted by (u, v, w) with $(0,0,0)$ representing m. Now define ψ_m by

$$\psi_m(u,v,w) = \begin{cases} \tilde{\psi}(u,v,w) & \text{if } w \in (-t_0, 0), \\ c_i(u) + (r_0 + v)\xi(c_i(u))\cos w + & \\ \quad (r_0 + v)\eta(c_i(u))\sin w, & \text{if } w \in [0, t_0). \end{cases} \tag{8}$$

Note that r_0 above corresponds to the radius of the pipe surface that contains m.

$\psi(u,v,w)$ is clearly C^1 except possibly along $w = 0$. $\frac{\partial\psi_m}{\partial u}$, $\frac{\partial\psi_m}{\partial v}$ are continuous even along the surface defined by $w = 0$, hence, they are continuous everywhere. We need to show that $\frac{\partial\psi_m}{\partial w}$ is also continuous along $w = 0$. $\frac{\partial\psi_m}{\partial w}$ is given by tangent vectors of the solutions to the above system of differential equations when $w \le 0$ and it converges to $\hat{\eta}$ as the points approach the surface $w = 0$ from the negative side of w. On the other hand, $\frac{\partial\psi_m}{\partial w}$ is given by the $\frac{\partial F}{\partial s}$ on the positive side of w. $\frac{\partial F}{\partial s}$ converges to $\hat{\eta}$ as $w \to 0$ from right. This shows that $\frac{\partial\psi_m}{\partial w}$ is continuous at the points in the surface $w = 0$. Hence, all first partials are continuous in the neighborhood of m. giving that the above coordinates are C^1. We are ready to show that the distance function ρ is $C^{1,1}$ along the surfaces S_i, $1 \le i \le n$. We already know that the distance function is C^2 off the surfaces S_i, $1 \le i \le n$. As before, let m be a point in one of S_i, $1 \le i \le n$. Denote by (x,y,z) the standard rectangular coordinates of R^3. Without loss of generality, we may assume that $(0,0,0)$ in this coordinates represents m. As we have seen, the gradient field $\nabla\rho$ of ρ is given as the unit tangential field of the normal rays everywhere off the surfaces S_i, $1 \leq i \leq n$. Since the coordinate transformation between two coordinate systems (x, y, z) and (u, v, w) around m is a C^1 transformation, the induced Jacobian transformation is continuous. From the particular choice of the coordinate system (u, v, w), we see that $\nabla\rho$ is continuous and it, indeed, is the unit tangential field to the normal radial ray emanating from the points in M. By the chain rule, we see that $\nabla\rho$ in terms of (x, y, z) is given as a continuous function of (u, v, w) off the surfaces S_i, $1 \leq i \leq n$. Since the coordinate transformation between them is C^1 diffeomorphism, $\nabla\rho$ in (x, y, z), as (x, y, z) approaches points in S_i, $1 \leq i \leq n$, must converge to the image of $\nabla\rho$ in terms of (u, v, w) under the Jacobian transformation. Since $\nabla\rho$ in the (x, y, z) coordinates is the unit tangential field to the normal radial ray off the surfaces S_i, it converges to the unit tangential field of the normal rays emanating from the boundary curves c_i, $0 \leq i \leq n$. Thus, the unit tangential field to the normal radial rays must be gradient field even along the surfaces S_i, $1 \leq i \leq n$. Consequently, $\nabla\rho$ is C^1 off the surfaces S_i, $1 \leq i \leq n$, and continuous along those surfaces. We will see that $\nabla\rho$ is Lipschitz continuous along them. To this end, let B_m be a sufficiently small open ball in $R^3(x, y, z)$ centered at a point m in one of the surfaces S_i, $1 \leq i \leq n$, say, S_i. We can assume that S_i divides U into two subsets with the common boundary $B_m \cap S_i$. We can also assume that for any $p, q \in B_m$ which belong to the same side of the surface the Lipschitz condition $|\frac{\partial\rho}{\partial x}(q) - \frac{\partial\rho}{\partial x}(p)| < k|q-p|$ holds. This can be seen as follows. Since p, q belong to the same side of S_i, p, q belong to an open set where ρ is a C^2 function as seen before. $\nabla\rho$ is C^1, hence, Lipschitz. The same observation holds for other two partials, Now suppose that p, q belong to opposite sides of the surface in B_m. Join p, q by the line segment between them. Since B_m is convex, the entire line segment belongs to B_m. The line segment meets S_i at a point b in B_m. Then the triangle inequality yields

$$|\frac{\partial\rho}{\partial x}(q) - \frac{\partial\rho}{\partial x}(p)| \leq |\frac{\partial\rho}{\partial x}(q) - \frac{\partial\rho}{\partial x}(b)| + |\frac{\partial\rho}{\partial x}(b) - \frac{\partial\rho}{\partial x}(p)| < k|q-b| + k|b-p| = k|q-p|.$$

The same proof also works for other partials. This implies that $\nabla\rho$ is (locally) Lipschitz continuous along the surfaces S_i, $1 \leq i \leq n$; hence, ρ is $C^{1,1}$ there. In particular, applying the implicit function theorem to the distance function, one gets that each level surface is C^2 off S_i, $1 \leq i \leq n$ and $C^{1,1}$ along S_i, $1 \leq i \leq n$. ■

Corollary 5.2. *The envelope $E_r(M)$, $\hat{\delta} > r > 0$ is the r level surfaces of the distance function ρ. Furthermore, $E_r(M)$, $\hat{\delta} > r > 0$ form an ambient isotopic family.*

Proof: The first statement is clear from Theorem 5.1. For the second statement, let $0 < r_1 < r_2 < \hat{\delta}$ be any two levels. The gradient field of ρ is given by the unit normal field n. Let ϵ be a sufficiently small positive number such that $0 < r_1 - \epsilon < r_1 < r_2 < r_2 + \epsilon < \hat{\delta}$ holds. Let f be a positive C^∞ real-valued function satisfying

$$f(r) = \begin{cases} 1 & \text{if } r_1 \le r \le r_2, \\ 0 & \text{if } r \le r_1 - \epsilon \text{ or } r \ge r_2 + \epsilon. \end{cases} \tag{9}$$

Denote a new vector field $\tilde{n}(r, x)$ in $U_{\tilde{\delta}}$ is defined by $\tilde{n}(x) = f(r)n(x)$, $\forall x \in U_{\tilde{\delta}}$. Then $\tilde{n}$ gives rise to a C^∞ vector field in R^3 with compact support. It generates a one parameter family of diffeomorphisms of R^3 which deforms E_{r_1} onto E_{r_2} [36]. ■

Corollary 5.3. *Let M be a compact C^2 surface in R^3. Denote by ∂M its boundary, which could be empty. Denote by M_r the r offset surface of M. Then for all r, $|r| < \hat{\delta}$, the M_r's are mutually ambient isotopic and the isotopy is obtained through the flow generated by the normal field n to M.*

Proof: If M has no boundary, Corollary 5.3 is proven in [8]. Otherwise, consider $\hat{M}$ introduced earlier. $\hat{M}$ is a C^2 compact surface with boundary $\partial\hat{M}$. The existence of a tubular neighborhood for such a surface tells that there is a sufficiently small $r_0 > 0$ such that all $|r| < r_0$ offset surfaces are ambient isotopic to each other and the ambient isotopy is obtained by the normal flow. This can be seen as follows. Since $\hat{M}$ is compact there is a sufficiently small $r_0 > 0$ such that $F : \hat{M} \times [-2r_0, 2r_0] \to R^3$ defind by $F(x, r) = x + rn_x$, $|r| < 2r_0$ is an injective diffeomorphphism, where n_x is a fixed unit normal field to $\hat{M}$. Both $F(\hat{M} \times [-2r_0, 2r_0])$ and $F(\tilde{M} \times [-r_0, r_0])$ are compact in R^3 and $F(\hat{M} \times [-2r_0, 2r_0])$ contains $F(\tilde{M} \times [-r_0, r_0])$ as a proper subset. It is well-known then that there is a C^∞ $f : R^3 \to [0, 1]$ such that f is identically 0 outside $F(\hat{M} \times [-2r_0, 2r_0])$ and f is identically 1 inside $F(\tilde{M} \times [-r_0, r_0])$. Let n be the unit tangent vector field to the normal field in $F(\hat{M} \times [-2r_0, 2r_0])$. Then $f \cdot n$ gives rise to a C^1 vector field in R^3 with a compact support. This vector field creates a flow which is identical to the normal flow in $F(\tilde{M} \times [-r_0, r_0])$. Furthermore, the one parameter family of diffeomorphisms generates the desired ambient isotopy. Now combining this ambient isotopy with the ambient isotopy given in Corollary 5.2 yields the desired ambient isotopy. ■

6. Minimum Positive Distance from the Medial Axis

This section presents a lemma which may be of general interest regarding the relation between a surface and its medial axis. It extends the known proofs for the existence of a positive minimum distance between the surface and its medial axis to surfaces which are $C^{1,1}$, as defined, below.

One of the basic consequences from the definition is that M can be locally represented as the "graph" of a real valued function of 2 variables. In particular, we can assume that for a given point $m \in M$, there is an open neighborhood $U(x, y, z)$ of 0 in R^3 such that $m = 0$ and the graph of a function $z = f(x, y)$ with $\nabla f(0, 0) = 0$ represents M^2 in U, where ∇f denotes the gradient of f.

Lemma 6.1. *For a compact, $C^{1,1}$ manifold M, there exists a positive minimum distance between M and its medial axis.*

Proof: First note that

$$|\nabla f(x,y) - \nabla f(0,0)| = |\nabla f(x,y)| \leq k|(x,y) - (0,0)| = k|(x,y)|. \tag{10}$$

Hence, along any line given by $ax + by = 0$, or $(t, -a/bt)$, $-\delta < t < \delta$ for a small δ,

$$f(t,(-a/b)t) = \int_\alpha \nabla f(t, -a/bt) \leq \int_\alpha |\nabla f(t, -a/bt)| \leq k \int_\alpha \sqrt{1 + (a/b)^2} t, \tag{11}$$

where $\alpha = \alpha(t)$ is the space curve given by $\alpha(t) = (t, -a/bt, f(t, -a/bt))$ and the first equality follows from the Fundamental Theorem of Line Integrals. On the other hand, $k \int_\alpha \sqrt{1 + (a/b)^2} t = (k/2)\sqrt{1 + (a/b)^2} t^2$. This yields that

$$f(x,y) = f(t, -a/bt) \leq (k/2)\sqrt{1 + (a/b)^2} t^2,$$

for all a, b. Note if $b = 0$, just parametrize the y-axis in t. Since

$$(1/2)\sqrt{1 + (a/b)^2} \leq 1 + (a/b)^2,$$

the last inequality expressed in terms of x, y gives

$$f(x,y) \leq k(x^2 + y^2), \tag{12}$$

in a small neighborhood of $(0,0)$.

This shows that the graph, therefore the surface, lies below the paraboloid $z = k(x^2 + y^2)$. It is now clear that the curvature sphere of the paraboloid at (0,0,0) is tangent to the graph $z = f(x,y)$ at (0,0,0) and fits entirely above the graph. The equation of the curvature sphere is given by $x^2 + y^2 + (z - (1/2k))^2 = (1/2k)^2$. Applying this argument at every point in M and using the compactness hypothesis, we get a minimum radius λ of the spheres. Then, similar to previous proofs [42, 43] a minimum critical value c is defined. Although these previous proofs assumed that the manifold was C^2, the hypothesis here of $C^{1,1}$ is sufficient to derive this value of c. We then define ρ, as

$$\rho = \min \{ \lambda, c \}.$$

Then there are no points in the medial axis of M within any distance less than ρ. ∎

Remark 6.1. *This minimum distance lemma can be directly applied to previously presented theorems about C^2 manifolds* [8, 43]*, to extend them to compact, $C^{1,1}$ 2-manifolds without boundary.*

7. Isotopy of the Manifold with Boundary

This section presents the main theorem of this paper and its proof. This theorem forms the theoretical basis for the existence of a piecewise linear approximation of a compact (orientable or non-orientable) surface with boundary. The approximation is ambient isotopic to the original surface. Previously, there were only firm theoretical foundations for creation of piecewise linear approximations of manifolds without boundary. Those techniques all have relied upon the demonstration of a positive minimum distance between the surface and its medial axis. In order to prove our main theorem, it is valuable, to now present several supportive lemmas, the first of which extends the known proofs for the positive minimum distance between the surface and its medial axis to surfaces which are $C^{1,1}$, as defined, below.

We first outline the intended construction and then give the details of its proof.

7.1. **The Construction of the Approximation of M.** The construction proceeds by assuming the availability of a simplicial approximation for the envelope $E_r(M)$, where this simplicial approximation will be denoted as $K(r)$. Furthermore, we will assume that the minimum distance map from $K(r)$ to $E_r(M)$ is a surjective homeomorphism, such as previously proven [8]. We are then interested in the restriction of the inverse mapping to M. Then each normal emanating from a point $x \in M$ meets $K(r)$ at a unique point $\sigma(x)$. The correspondence $x \mapsto \sigma(x)$ gives rise to a homeomorphism between M and $\sigma(M)$ if σ is restricted to a specific unit normal direction n to M. (Note that there are two natural choices for a unit normal to M.) Denote it by σ_n for one of such unit normals n. Observe that $\sigma_n(M)$ is not necessarily a simplicial subcomplex of $K(r)$, so that the remainder of the process is to create an ambient isotopic PL approximation along the boundary. The formal proofs of this creation are given in the lemmas and theorem of the next section, but we will first illustrate the underlying intutitive ideas with two figures.

Figure 3 shows how a typical boundary component c of ∂M is mapped by σ_n into $K(r)$. This image $\sigma_n(c)$ is shown to intersect the edges of $K(r)$ in two endpoints denoted as k_j and k_{j+1}. Denoting the segment between k_j and k_{j+1} as $[k_j, k_{j+1}]$, it is then easy to observe that $\sigma_n(c) \cap [k_j, k_{j+1}]$ can be represented by finitely many points $x_0 = k_j < x_1 < \ldots < x_{m-1} < x_m = k_{j+1}$, as shown in Figure 3, specifically noting that, for some i the segment $[x_i, x_{i+1}]$ may be a subset of $\sigma_n(c)$. Otherwise, when this intersection is just the two endpoints, naemly for each i such that $[x_i, x_{i+1}] \cap \sigma_n(c) = \{x_i, x_{i+1}\}$ there is a compact topological 2-disc, denoted as D_i formed between $\sigma_n(c)$ and $[x_i, x_{i+1}]$. The intutitive notion is that these discs D_i should be isotopically mapped into a new simplicial complex which has a PL boundary curve. In Figure 3 the vertices of this PL boundary within a particular 2-simplex of $K(r)$ can be understood to be the points $x_0, x_1, \ldots, x_{m-1}, x_m$, even as we add the further technical note that it may be necessary to partition this set even more finely in order to make each segment of $\sigma_n(c)$ that begins at x_i and ends at x_{i+1} be differentiable and satisfy the hypotheses of Lemma 7.1. In Figure 3 points of intersection $x_0, \ldots, x_6$ are depicted, where the interval $[x_1, x_2]$ is a collinear segment of both the triangle and $\sigma_n(c)$ and the closed disc D_3 is illustrated.

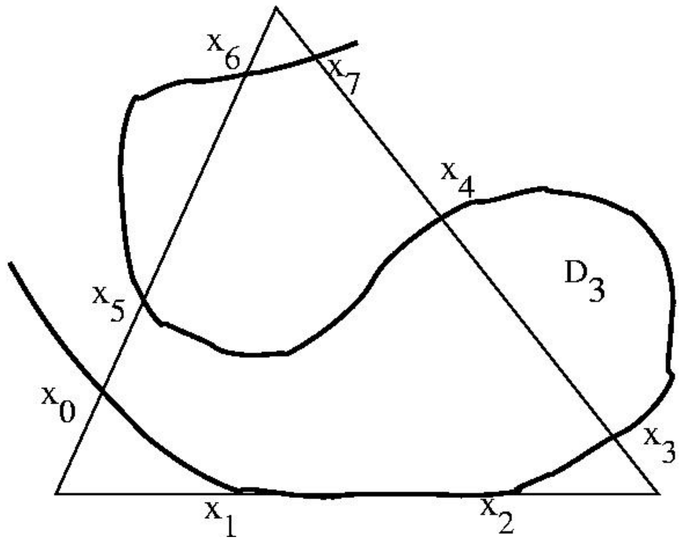

FIGURE 3. Polyline Boundary Isotopy

This construction is an isotopy only within the planar face of the given 2-simplex, similar to Bing's push [12]. A set of compact support is expressed in Lemma 7.1 for the deformation of each closed D_i represented in Figure 3. It is then easy to see how this set of compact support is a generalization of Bing's triangular set of compact support, where a push function in R^3 can then have compact support of a tetrahedron. A similar extenstion to R^3 is then undertaken in Lemma 7.2. This yields a polytope $PL(M)$ which is ambient isotopic to $\sigma_n(M)$. Then a simplicial complex that is ambient isotopic to M can be created by triangulating $\sigma_n(M)$, while noting that isotopy is an equivalence relation. Furthermore, it should be noted that the supporting lemmas allow an arbitrary choice of $\epsilon > 0$ for an upper bound on the distance between M and its ambient isotopic simplicial approximation $S(M)$.

7.2. **Proof of Main Theorem.** In order to prove the validity of the above construction, we consider the boundary segment $\sigma_n(c)$ as if it were the graph of a function over a closed interval. The following notation is appropriate to this consideration of $\sigma_n(c)$ as a graph of a function, as well as for the construction of a neighborhood of compact support for the isotopy being constructed.

Let $y = f(x)$, $a \leq x \leq b$ be a non-negative (or non-positive) real valued piecewise C^1 function such that $f(a) = f(b) = 0$. Let $y = g_i(x)$, $i = 1, 2$ be a real valued C^1 functions defined in $[a, b]$ satisfying

(1) $y = g_i(a) = g_i(b) = 0$, $i = 1, 2$,
(2) $g_2'(a) \geq g_1'(a) \geq f_+'(a)$ and $g_2'(b) \leq g'(b) \leq f_-'(b)$ and

(3) $f(x) < g_1(x) < g_2(x)$, $\forall x \in (a,b)$,
where $f'_+(a), f'_-(b)$ indicate the right and left derivative at a, b, respectively.

As notation for the next lemma, set $D = \{(x,y) : a < x < b; 0 < y < f(x)\}$ and let $\epsilon > 0$ be an arbitrary but sufficiently small positive number.

Lemma 7.1. *There is an isotopy Ψ_t, $0 \le t \le 1$ of R^2 such that*

(1) *$\Psi_0 = id$ and Ψ_1 maps the closure $\bar{D}$ of D onto the closure of the domain between the line $[a,b]$ and the graph of $y = -\epsilon f$. Furthermore, Ψ_1 maps the graph of $y = f(x)$ onto the closed interval $[a,b]$ and $[a,b]$ onto the graph of $y = -\epsilon f$, respectively.*

(2) *Ψ_1 maps the closure of the domain between the graphs of f and g_1 homeomorphically onto the closed domain bounded by $[a,b]$ and the graph of f in such a way that the graph of g_1 is mapped onto the graph of f.*

(3) *Ψ_1 maps the closure of the domain between $[a,b]$ and the graph of $y = -\epsilon f$ onto the closure of the domain between the graphs of $y = -\epsilon g_i$, $i = 1,2$ in such a way that $[a,b]$ and the graph of $y = -\epsilon f$ are mapped homeomorphically onto the graphs of he graphs of $y = -\epsilon g_i$, $i = 1,2$, respectively. (4) Ψ_t, $0 \le t \le 1$ keeps the complement of the closure of the domain between the graphs of $y = g_2$ and $y = -\epsilon g_2$ pointwise fixed.*

Proof: The proof of this lemma is elementary and uses ideas similar Bing's push [12]. Here, points are pushed along the vertical lines to obtain the desired deformation. The actual proof is left to the reader. ∎

Lemma 7.2. *Ψ_t can be extended to an isotopy $S\Psi_t$ of R^3.*

Proof: The general outline of this proof is to note that the isotopy Ψ_t is a generalization of Bing's push [12] which has a triangular set of compact support within $\mathbf{R}^2$. Then, it is easy to generalize Bing's push to $\mathbf{R}^3$, where the set of compact support becomes a tetrahedron. The remainder of the present proof is similar and the additional technical details are left to the reader. The only subtlety to note, particularly for applications as discussed in our companion paper [1], is that the size of the set of compact support can be controlled to be within ϵ of the original domain of Ψ_t, for any $\epsilon > 0$. Hence, for applications, the choice of this ϵ will be guided by values of curvature and separation distance. ∎

Lemma 7.3. *$PL(M)$ is ambient isotopic to $\sigma_n(M)$.*

Proof: This proof is left to the reader. ∎

Theorem 7.4. *There is a simplicial complex with boundary $S(M)$ which is ambient isotopic to M and can be made arbitrarily close to M.*

Proof: First we see from Corollary 5.3 that M is ambient isotopic to $\sigma_n(M)$. By Lemma 7.2, $\sigma_n(M)$ is ambient isotopic to $PL(M)$ which is a cell complex. Now we triangulate $PL(M)$ into $S(M)$ while preserving both the ambient isotopy relation and the upper bound on approximation. ∎

7.3. **Resultant Pseudocode for Practical Algorithms.** The pseudocode resulting from these theorems is now presented.

For a compact, C^2 manifold, denote as follows:

- $\lambda =$ the minimum positive distance between M and its medial axis,
- $B_r(x) =$ the 3-ball of radius r centered at x, with $r \in (0, \infty)$,
- $S =$ a set of sample points from M, meeting appropriate density criteria,
- np to abbreviate the nearest point mapping over a domain to be stated.

The pseduocode now follows.

Pseudocode:

Input: S

Choose ρ such that $\rho \in (0, \lambda)$;
// Depends upon Lemma 5.1. //

For each $x \in S$, create $B_\rho(x)$;

Let $D = \bigcup_{x \in S} B_\rho(x)$;

Find ∂D as an approximation to $E_\rho(M)$;
// Definition of Envelope. //

Using S and approximations of normals for M, derive a sample set $\hat{S}$ for $E_\rho(M)$;

Use $\hat{S}$ as input to an algorithm for an approximation K of $E_\rho(M)$
// Relies upon $np : M \to E_\rho(M)$ being a homeomorphism //
// Set $K = np(M)$ //
// Depends upon Lemma 7.2. //

Use K to obtain a PL approximation, L of M.
// Depends upon Theorem 7.4

Output: L**, a PL ambient isotopic approximation of** M**.**

8. Conclusion: Integration of Theory and Practice

This paper presents new theory for reconstruction and approximation of C^2 surfaces with boundary with guarantees for topological equivalence under ambient isotopy. This led to experimentation on practical examples, which are presented in the companion application paper. The main integrative like between the theory and practice is the pseudocode that appears in both papers, but here there are specific added annotations to show where various lemmas and theorems are crucial to the algorithm design.

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