7.7 Approximate Integration

For the Midpoint Rule, Trapezoidal Rule, and Simpson's Rule to approximate $\int_a^b f(x)\,dx$, we summarize here the approximation and an error bound. We always set $\Delta x = \frac{b-a}{n}$.

Midpoint Rule: $\int_a^b f(x)\,dx \approx M_n = (f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n))\Delta x$, where

$$\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1}, x_i].$$

The error bound in the Midpoint Rule is

$$|E_M| \leq \frac{K(b-a)^3}{24n^2},$$

where K is chosen so that $|f''(x)| \leq K$ for $a \leq x \leq b$.

Trapezoidal Rule: $\int_a^b f(x)\,dx \approx T_n = (f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n))\frac{\Delta x}{2}$, where

$$x_i = a + i\Delta x.$$

The error bound in the Trapezoidal Rule is

$$|E_T| \leq \frac{K(b-a)^3}{12n^2},$$

where K is chosen so that $|f''(x)| \leq K$ for $a \leq x \leq b$.

Simpson's Rule: $\int_a^b f(x)\,dx \approx S_n$, where

$$S_n = (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))\frac{\Delta x}{3}$$

for *even* n. The error bound in Simpson's rule is

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

where K is chosen so that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$.

Example:

(a) Apply the Trapezoidal Rule to $\int_1^3 \sqrt{x}\,dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point.

(b) Use the bound on $|E_T|$ to determine an n so that the error bound for the Trapezoidal Rule in this case will be at most .01.

Thinking about the problem:

Using $n = 4$ subintervals within the interval $[1, 3]$ we have $\Delta x = \dfrac{b-a}{n} = \dfrac{3-1}{4} = \dfrac{1}{2}$, which is used to find the endpoints of the trapezoids under the curve $f(x) = \sqrt{x}$ in part (a).

For part (b), we want the error $|E_T|$ to be at most .01. Since $|E_T| \leq \dfrac{K(b-a)^3}{12n^2}$, to ensure $|E_T| < .01$, we will find n such that $\dfrac{K(b-a)^3}{12n^2} \leq .01$ after we figure out what K can be.

Doing the Problem:

For part (a), $\Delta x = \dfrac{b-a}{n} = \dfrac{3-1}{4} = \dfrac{1}{2}$. From $x_i = a + i\Delta x$ we get the following tables for x_i and then $f(x_i)$ rounded to 5 digits after the decimal point.

x_0	x_1	x_2	x_3	x_4
1	1.5	2	2.5	3

$f(1)$	$f(1.5)$	$f(2)$	$f(2.5)$	$f(3)$
1	1.22474	1.41421	1.58113	1.73205

Thus the Trapezoidal Rule approximation to $\int_1^3 \sqrt{x}\,dx$ with $n = 4$ is

$$T_4 \approx (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))\frac{\Delta x}{2}$$

$$= (f(1) + 2f(1.5) + 2f(1) + 2f(2.5) + f(3))\frac{.5}{2}$$

$$\approx 2.79305.$$

This answers part (a). Note: If we use more accurate values for the square roots, we would find T_4 to be 2.79306, which is a better answer and illustrates the accumulated effect of round-off error in the middle of a calculation.

For part (b), to find an n such that the error bound is less than .01, we seek n making

$$\frac{K(3-1)^3}{12n^2} \le .01,$$

where $|f''(x)| \le K$ for $1 \le x \le 3$. From $f(x) = \sqrt{x}$ we have $f''(x) = -\frac{1}{4x^{3/2}}$. For $1 \le x \le 3$, we have $1/x^{3/2} \le 1$, so $|f''(x)| = \frac{1}{4x^{3/2}} \le \frac{1}{4}$. Use $K = 1/4$:

$$\frac{\frac{1}{4}(3-1)^3}{12n^2} \le .01 \Leftrightarrow \frac{2}{12n^2} \le .01 \Leftrightarrow \frac{1}{6n^2} \le .01 \Leftrightarrow n^2 \ge \frac{1}{6(.01)} \Leftrightarrow n \ge \frac{1}{\sqrt{.06}} \approx 4.082.$$

Since n is an integer, we get $n \ge 5$, so when estimating $\int_1^3 \sqrt{x}\,dx$ by the Trapezoidal Rule we have $|E_T| \le .01$ using $n \ge 5$, which answers (b). (This doesn't mean $|E_T|$ can't be $\le .01$ for smaller n, but the error bound says for $n \ge 5$ the error is definitely at most .01. It turns out that the Trapezoidal Rule estimates at $n = 3$ and $n = 4$ in this case are both within .01 of the integral.)

Solutions should show all of your work, not just a single final answer.

1. (a) Apply the Trapezoidal Rule to $\int_0^2 e^{-x^2}\,dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point. (Don't confuse $e^{-x^2} = e^{-(x^2)}$ and $(e^{-x})^2 = e^{-2x}$.)

   (b) To three digits after the decimal point find the bound on $|E_T|$ applied to $\int_0^2 e^{-x^2}\,dx$ using $n = 4$ subintervals. (First find K by seeing where $|f''(x)|$ is maximized on $[0, 2]$. You may use a graph.)

   (c) Set up the bound for $|E_T|$ applied to $\int_0^2 e^{-x^2}\,dx$ using n subintervals for general n.

   (d) Use the bound for $|E_T|$ to determine an n such that $|E_T|$ is at most .001.

2. (a) Apply Simpson's Rule to $\int_1^2 \sqrt{x}\,dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point.

(b) Use the bound for $|E_S|$ to determine an n such that Simpson's Rule for $\int_1^2 \sqrt{x}\,dx$ is within 10^{-6} of the value of the integral. (Remember n must be even.)

3. T/F (with justification) The Trapezoidal Rule for $\int_a^b f(x)\,dx$ has no error if $f(x)$ is linear.