Math 1132 Worksheet 11.4 Name: ________________________ Discussion Section: _____

11.4 The Comparison Tests

Comparison Test. For series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with positive terms,

(i) if $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$ for all n then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \geq b_n$ for all n then $\sum_{n=1}^{\infty} a_n$ diverges.

If either inequality in the Comparison Test has a finite number of exceptions, then the conclusion of the test still works. So it isn't important for the series to start at $n = 1$.

Tip for checking inequalities: for $x > 0$ and $y > 0$, you can increase $\frac{x}{y}$ by increasing x or decreasing y, and you can decrease $\frac{x}{y}$ by decreasing x or increasing y.

Limit Comparison Test. For series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with positive terms, if $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$ then both series converge or both series diverge.

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$ converges or diverges.

Thinking about the problem:

Set $a_n = \frac{n^3}{n^4+1}$. To apply the Comparison or Limit Comparison Tests we seek b_n to compare to a_n with convergence or divergence of $\sum_{n=1}^{\infty} b_n$ being known. Looking at leading terms of a_n, take $b_n = \frac{n^3}{n^4} = \frac{1}{n}$. Then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, which diverges.

We will try to use the Comparison and Limit Comparison Tests to see which ones work.

Doing the problem:

We want to compare $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$ with $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges, so to apply the Comparison Test with a divergent series we'd want to have $\frac{n^3}{n^4+1} \geq \frac{1}{n}$. However, the inequality goes the other way: $\frac{n^3}{n^4+1} < \frac{1}{n}$. Nevertheless, we can make this work by comparing to $\frac{1}{2n}$:

$$\frac{n^3}{n^4+1} \geq \frac{1}{2n} \iff 2n^4 \geq n^4 + 1 \iff n^4 \geq 1 \ \checkmark.$$

Then divergence of $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ implies divergence of $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$.

It is easier here to use the Limit Comparison Test. When $a_n = \frac{n^3}{n^4+1}$ and $b_n = \frac{1}{n}$, we have $\frac{a_n}{b_n} = \frac{n^4}{n^4+1} = \frac{1}{1+1/n^4} \to 1$, so divergence of $\sum_{n=1}^{\infty} b_n$ implies divergence of $\sum_{n=1}^{\infty} a_n$.

1. Determine whether the series $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ converges or diverges by using either the Comparison Test or Limit Comparison Test (indicate which test you use).

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$ converges or diverges by using either the Comparison Test or Limit Comparison Test (indicate which test you use).

3. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}2^n}$ converges or diverges by using either the Comparison Test or Limit Comparison Test (indicate which test you use).

4. T/F (with justification) The series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the Comparison Test with $\sum_{n=2}^{\infty} \frac{1}{n}$ since $\frac{1}{n \ln n} > \frac{1}{n}$ for all $n \geq 2$.

5. T/F (with justification) If $0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.