Math 1132 Worksheet 3 Name ____________________ Discussion Section ______

Methods of Integration

Solutions to these problems should show all of your work, not just a single final answer.

Part 1: **Integration by parts**. Do each problem as follows: (1) specify u and dv, (2) compute du and v, (3) use integration by parts with your choice of u and dv. (4) If you need integration by parts more than once, each time go through steps 1, 2, and 3 again.

Example. Compute $\int x^2 e^x \, dx$.

Solution.

(1) Set $u = x^2$ and $dv = e^x \, dx$.

(2) We have $du = 2x \, dx$ and $v = e^x$.

(3) Now $\int x^2 e^x \, dx = \int u \, dv = uv - \int v \, du = x^2 e^x - \int e^x (2x) \, dx = x^2 e^x - 2 \int x e^x \, dx$.

(4) To find $\int x e^x \, dx$, set $u = x$ and $dv = e^x \, dx$, so $du = dx$ and $v = e^x$. Then $\int x e^x \, dx = \int u \, dv = uv - \int v \, du = x e^x - \int e^x \, dx = x e^x - e^x$.

(5) Substituting the result of (4) into (3),

$$\int x^2 e^x \, dx = x^2 e^x - 2(x e^x - e^x) + C = (x^2 - 2x + 2)e^x + C.$$

1. Compute $\int x \cos(5x) \, dx$.

2. Compute $\int x^2 2^x \, dx$. (Hint: You can find an antiderivative of 2^x by recalling how to differentiate 2^x.)

Part 2: **Integration of rational functions**.

Example. Compute $\int \frac{2x+1}{x^2-4}\,dx$ using partial fractions.

Solution. Write $\frac{2x+1}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$ for some A and B. Clearing the denominator, $2x+1 = A(x-2)+B(x+2)$. Setting $x=2$ we get $5=4B$, so $B=5/4$. Setting $x=-2$ we get $-3=-4A$, so $A=3/4$. Thus $\frac{2x+1}{x^2-4} = \frac{3/4}{x+2} + \frac{5/4}{x-2}$, so

$$\int \frac{2x+1}{x^2-4}\,dx = \int \left(\frac{3/4}{x+2} + \frac{5/4}{x-2}\right) dx = \frac{3}{5}\ln|x+2| + \frac{5}{4}\ln|x-2| + C.$$

3. Compute $\int \frac{10}{x^3-x^2-6x}\,dx$ using partial fractions.

4. Compute $\int \frac{x^2+x+1}{x(x^2+4)}\,dx$ using partial fractions.

Part 3: **Approximate Integration**.

Example. (a) Compute the trapezoid approximation to $\int_1^3 \sqrt{x}\,dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point.

(b) Use the error bound for the trapezoid rule to determine an n such that the trapezoid approximation is *guaranteed* by the error bound to be with .01 of the value of the integral.

Solution.
(a) The trapezoid approximation with $n = 4$ is

$$\begin{aligned}\frac{b-a}{2n}(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)) &= \frac{2}{8}(f(1)+2f(1.5)+2f(2)+\\ &\qquad 2f(2.5)+f(3))\\ &\approx 2.79306.\end{aligned}$$

(b) An upper bound on the error from the trapezoid rule with n intervals is $\frac{K(b-a)}{12}(\Delta x)^2$, where $\Delta x = (b-a)/n$ and K is an upper bound on $|f''(x)|$ for all x in $[a, b]$. In our problem, $f(x) = \sqrt{x}$, so $f''(x) = -\frac{1}{4}x^{-3/2}$. For $1 \leq x \leq 3$, we have $x^{-3/2} \leq 1$, so $|f''(x)| \leq 1/4$ when $1 \leq x \leq 3$. Thus we can use $K = 1/4$, so the trapezoid error bound is $\frac{(1/4)(3-1)}{12}(\frac{2}{n})^2 = \frac{1}{6n^2}$. Having the error be less than .01 means

$$\frac{1}{6n^2} < .01 \iff n^2 > \frac{1}{6(.01)} \iff n > \sqrt{\frac{1}{.06}} \approx 4.082,$$

so for $n \geq 5$ the trapezoid approximation will be within .01 of the integral.

5. (a) Compute the trapezoid approximation to $\int_2^3 x \sin x\,dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point. (Remember to set your calculator to radian mode for trigonometric functions.)

   (b) Use the error bound for the trapezoid rule to determine an n such that the trapezoid approximation is *guaranteed* by the error bound to be with .01 of the value of the integral.

6. (a) Compute the Simpson's rule approximation to $\int_1^2 \sqrt{x}\, dx$ using $n = 4$ subintervals, rounding your approximation to 5 digits after the decimal point.

   (b) Use the error bound for Simpson's rule to determine an n such that the Simpson's rule approximation is *guaranteed* by the error bound to be with 10^{-6} of the value of the integral. (Remember n must be even.)

Part 4: **Improper Integrals**.

7. For $a > 0$, compute the improper integrals $\int_0^\infty e^{-ax}\,dx$ and $\int_0^\infty xe^{-ax}\,dx$. Your answer will be in terms of a.

8. Compute the improper integral $\int_0^\infty \frac{dx}{(x+2)(x+5)}$ using partial fractions.

9. Decide if the improper integral $\int_0^\infty \frac{x}{x^2+1}\,dx$ is convergent or divergent. If it is convergent, evaluate it.

Optional Question.

10. Vibrations show up in many places: civil engineering (oscillations in a bridge or the reaction of a building to an earthquake), music (sound is a vibration of pressure waves), and ski design (smaller vibrations make a smoother ride). The following computation is *fundamental* in any mathematical study of vibrations: for all positive integers m and n, use integration by parts to show

$$\int_0^{2\pi} \sin(mx)\cos(nx)\,dx = 0.$$

(Hint: Use the bounds of integration during the integration by parts, and treat $m = n$ and $m \neq n$ separately. It may help to first try this for specific m and n, such as $m = 2$ and $n = 3$, and then $m = 5$ and $n = 5$.) There are two other integral formulas related the one above, with products of two sines and two cosines:

$$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi, & \text{if } m = n, \\ 0, & \text{if } m \neq n \end{cases} \qquad \int_0^{2\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} \pi, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

Here too m and n are positive integers. The optional question is only the first formula.