Math 1132

Solutions to Practice Exam 2

1(A)

If the n^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 1 + \frac{n}{3^n}$ then $a_n = \frac{2-n}{3^n}$ for $n > 1$ (a) T F

Solution: ________ FALSE ...

1. $s_n = a_1 + a_2 + a_3 + \ldots a_{n-1} + a_n = s_{n-1} + a_n$
2. $s_n = s_{n-1} + a_n \implies a_n = s_n - s_{n-1}$
3. $a_n = (1 + \frac{n}{3^n}) - (1 + \frac{n-1}{3^{n-1}}) = \frac{n}{3^n} - \frac{n-1}{3^{n-1}}$
4. $a_n = \frac{n}{3^n} - \frac{3(n-1)}{3^n} = \frac{n-3(n-1)}{3^n} = \frac{3-2n}{3^n} \neq \frac{2-n}{3^n}$

1(b)

The geometric series $\sum_{n=4}^{\infty}(\frac{1}{3})^n$ converges to $\frac{3}{2}$. (b) T F

______________________________ FALSE

Solution 1:

1. $s_n = (\frac{1}{3})^4 + (\frac{1}{3})^5 + \ldots + (\frac{1}{3})^n$
2. $(\frac{1}{3})s_n = \quad (\frac{1}{3})^5 + \ldots + (\frac{1}{3})^n + (\frac{1}{3})^{n+1} \ldots$
3. $(\frac{2}{3})s_n = (\frac{1}{3})^4 - (\frac{1}{3})^{n+1} \implies s_n = (\frac{3}{2})\frac{1}{3^4} - (\frac{3}{2})(\frac{1}{3})^{n+1}$
4. $\lim_{n\to\infty} s_n = (\frac{3}{2})\frac{1}{3^4} - (\frac{3}{2})\lim_{n\to\infty}(\frac{1}{3})^{n+1} = \frac{1}{2\cdot 3^3} = \frac{1}{54}$

Solution 2:

If $|r| < 1$ then $\sum_{k=0}^{\infty} a\, r^k = \frac{a}{1-r}$. If $|r| \geq 1$ the series diverges.

1. $$\sum_{n=0}^{\infty} (\frac{1}{3})^n = 1 + \frac{1}{3} + (\frac{1}{3})^2 + (\frac{1}{3})^3 + \sum_{n=4}^{\infty} (\frac{1}{3})^n$$
2. $$\frac{1}{2/3} = \frac{40}{27} + \sum_{n=4}^{\infty} (\frac{1}{3})^n$$ $r = 1/3$
3. $$\sum_{n=4}^{\infty} (\frac{1}{3})^n = \frac{3}{2} - \frac{40}{27} = \frac{1}{54}$$

1(c)

If $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges. (c) T F

Solution: __________ FALSE

A counter example

1. $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series $(p = 1)$
2. $\lim_{n\to\infty} \frac{1}{n} = 0$

1(d)

The series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ converges conditionally. (d) T F

Solution: ____________ FALSE

1. $\displaystyle\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^3} \right| = \sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series $(p = 3 > 1)$
2. The given series is absolutely convergent, not conditionally convergent.

1(e)

If $\sum_{n=1}^{\infty} |a_n|$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges. ____________ FALSE

1. $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series $(p = 1)$
2. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is a convergent series. The answer is: FALSE

What is true:

- If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} |a_n|$ diverges.

1(f)

The sequence $a_n = \dfrac{\ln(2n)}{\ln(n)}$ converges to 1. (f) T F

Solution: __________ TRUE ...

1. $$\lim_{n\to\infty} \frac{\ln(2n)}{\ln(n)} = \lim_{n\to\infty} \frac{\ln(2) + \ln(n)}{\ln(n)}$$

2. $$\lim_{n\to\infty} \frac{\dfrac{\ln 2}{\ln n} + 1}{1} = 1$$

Or, use L'Hospital's Rule on $\dfrac{\ln(2x)}{\ln(x)}$

1(g)

If the power series $\sum_{k=0}^{\infty} a_k (x-4)^k$ has a radius of convergence equal to 2 then $\sum_{k=0}^{\infty} a_k$ diverges. (g) T F

Solution: __________ FALSE ...

1. The center is $x = 4$ so the interval of convergence is one of the following:
   - $(2,6),\ [2,6],\ [2,6),\ (2,6]$ All include $x = 5$
2. When $x = 5$ the series looks like $\sum_{k=0}^{\infty} a_k (x-4)^k = \sum_{k=0}^{\infty} a_k$
3. When $x = 5$ the series converges since 5 is inside all these intervals

2(a)

Which of the following sequences is both bounded and monotonic?

(i) $a_n = n^2$ (ii) $a_n = \dfrac{n}{n+1}$ (iii) $a_n = \dfrac{\sin(\pi n)}{n}$ (iv) $a_n = \dfrac{n}{\sqrt{n+1}}$

(i) $a_n = n^2$ not bounded: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^2 = \infty$

(ii) $a_n = \dfrac{n}{n+1}$ is bounded and monotonic

- Bounded: $0 < \dfrac{n}{n+1} < \dfrac{n+1}{n+1} = 1$ bounded
- Monotonic:
  - $f(x) = \dfrac{x}{x+1} \implies f'(x) = \dfrac{x+1-x}{(x+1)^2} = \dfrac{1}{(x+1)^2} > 0$
  - f increasing $\implies a_n = f(n) = \dfrac{n}{n+1}$ is increasing

(iii) $a_n = \dfrac{\sin(\pi n)}{n}$ bounded and monotonic: Every term is 0

(iv) $a_n = \dfrac{n}{\sqrt{n+1}}$: Monotonic but not bounded

- $$a_n = \frac{\sqrt{n^2}}{\sqrt{n+1}} = \sqrt{\frac{n^2}{n+1}} = \sqrt{\frac{n}{1+1/n}} \longrightarrow \infty \text{ not bounded}$$

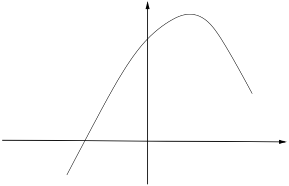
- $$f(x) = \frac{x}{(x+1)^{1/2}} \Longrightarrow f'(x) = \frac{(x+1)^{1/2} - (x/2)(x+1)^{-1/2}}{x+1}$$

$$f'(x) = \frac{x+1-(x/2)}{(x+1)^{3/2}} = \frac{1+(x/2)}{(x+1)^{3/2}} > 0$$

$f(x)$ is increasing, hence $f(n) = a_n$ is monotonic

2(b)

The function $f(x)$, whose graph is shown, has the Taylor polynomial of degree 2 centered at $x = 0$ given by $p_2(x) = a + bx + cx^2$. What can you say about a, b, c?



Solution: ______________________________

$$f(x) \approx p_2(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2$$

1. $a = f(0) > 0$ positive
2. $b = f'(0) > 0$ f is increasing
3. $c = \dfrac{f''(0)}{2} < 0$ concave down

2c: $\sum_{k=1}^{\infty}(\frac{1}{k}-\frac{1}{k+1})$

$$S_4=\sum_{k=1}^{4}(\frac{1}{k}-\frac{1}{k+1})=(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})-(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5})$$

$$S_4=\frac{1}{1}+(-\frac{1}{2}+\frac{1}{2})+(-\frac{1}{3}-\frac{1}{3})+(-\frac{1}{4}+\frac{1}{4})-\frac{1}{5}=1-\frac{1}{5}$$

$$S_n=(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})-(\frac{1}{3}-\frac{1}{4})+\ldots+(\frac{1}{n}-\frac{1}{n+1})$$

$$S_n=1+(-\frac{1}{2}+\frac{1}{2})+(-\frac{1}{3}-\frac{1}{3})+\ldots+(-\frac{1}{n}+\frac{1}{n})-\frac{1}{n+1}$$

$$S_n=1-\frac{1}{n+1}$$

$$\sum_{k=1}^{\infty}(\frac{1}{k}-\frac{1}{k+1})=\lim_{n\to\infty}S_n=\lim_{n\to\infty}(1-\frac{1}{n+1})=1$$

3. Consider the following series, all of which converge. For which of these series do you get a conclusive answer when using the **Ratio Test** to check for convergence? Write the letters of all possible answers. If no series satisfies this condition, write "none". You do not need to show your work.

A $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{2k^5 + k^2 + 1}$ **B** $\displaystyle\sum_{k=1}^{\infty} \frac{k^6}{k!}$ **C** $\displaystyle\sum_{k=1}^{\infty} (3k+4)^{-k}$

D $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ **E** $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{2}{5^k}$

To use the Ratio Test on $\displaystyle\sum_{k=1}^{\infty} a_k$ we compute the limit

$$r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}| \qquad \text{(Must have all positive terms)}$$

3A $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{2k^5+k^2+1}$ $\quad r = 1$ **ratio test gives no conclusion**

$$\lim_{k\to\infty} \frac{\dfrac{(k+1)^3}{2(k+1)^5+(k+1)^2+1}}{\dfrac{k^3}{2k^5+k^2+1}} = \lim_{k\to\infty} \frac{(k+1)^3(2k^5+k^2+1)}{k^3(2(k+1)^5+(k+1)^2+1)}$$

$$= \lim_{k\to\infty} \frac{(k+1)^3}{k^3} \quad \lim_{n\to\infty} \frac{2k^5+k^2+1}{2(k+1)^5+(k+1)^2+1}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^3 \quad \lim_{k\to\infty} \frac{2+1/k^3+1/k^5}{2(k+1)^5/k^5+(k+1)^2/k^5+1/k^5} = 1$$

3B $\displaystyle\sum_{k=1}^{\infty} \frac{k^6}{k!}$ $\quad r = 0 < 1$ **Converges by the ratio test**

$$r = \lim_{k\to\infty} \frac{\dfrac{(k+1)^6}{(k+1)!}}{\dfrac{k^6}{k!}} = \lim_{k\to\infty} \frac{(k+1)^6 k!}{k^6 (k+1) k!} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^6 \frac{1}{k+1} = 0$$

3C $\displaystyle\sum_{k=1}^{\infty} (3k+4)^{-k}$ $\quad r = 0 < 1$ **Converges by the ratio test**

$$r = \lim_{k\to\infty} \frac{(3(k+1)+4)^{-(k+1)}}{(3k+4)^{-k}} = \lim_{k\to\infty} \frac{(3k+4)^k}{(3k+7)^{(k+1)}}$$

$$= \lim_{k\to\infty} \frac{(3k+4)^k}{(3k+7)^k} \frac{1}{(3k+7)} = \lim_{k\to\infty} \frac{1}{(3k+7)} = 0 < 1$$

3D $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ $\quad r = 1$ **No conclusion from ratio test**

$$\lim_{k\to\infty} \frac{\dfrac{\ln(k+1)}{(k+1)^2}}{\dfrac{\ln k}{k^2}} = \lim_{k\to\infty} \frac{k^2 \ln(k+1)}{(k+1)^2 \ln k} = \lim_{k\to\infty} \frac{k^2}{(k^2+2k+1)} \frac{\ln(k+1)}{\ln k}$$

$$= \lim_{k\to\infty} \frac{1}{1+2/k+1/k^2} \frac{\ln(k+1)}{\ln k} = \lim_{k\to\infty} \frac{\ln(k+1)}{\ln k}$$

$$\lim_{k\to\infty} \frac{\ln(k+1)}{\ln k} = \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} = \lim_{x\to\infty} \frac{\dfrac{1}{x+1}}{\dfrac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} = 1$$

$r = 1$ No conclusion

3E $\sum_{k=1}^{\infty}(-1)^k \frac{2}{5^k}$ **Absolutely convergent**

Apply ratio test to $\sum_{k=1}^{\infty}|(-1)^k \frac{2}{5^k}| = \sum_{k=1}^{\infty}\frac{2}{5^k}$

- $r = \lim_{k\to\infty} \frac{\frac{2}{5^{(k+1)}}}{\frac{2}{5^k}} = \lim_{k\to\infty} \frac{2 \cdot 5^k}{2 \cdot 5^{(k+1)}} = \lim_{k\to\infty} \frac{1}{5} < 1 = \frac{1}{5}$

4(i) $a_n = \left(\dfrac{1-2n}{n+1}\right)^2$ bounded?, increasing?, convergent?

$$a_n = \frac{1-4n+4n^2}{n^2+2n+1} = \frac{1/n^2+4/n+4}{1+2/n+1/n^2} \longrightarrow 4 \quad \text{as} \quad n \to \infty$$

This sequence is convergent, hence also bounded.

To see if it is increasing, we consider $f(x) = \left(\dfrac{1-2x}{x+1}\right)^2$ ...

$$f'(x) = 2\left(\frac{1-2x}{x+1}\right)\frac{-3}{(x+1)^2} = 2\left(\frac{2x-1}{x+1}\right)\left(\frac{3}{(x+1)^2}\right)$$

$f'(x) > 0$ for $x \geq 1$

$f(x)$ is increasing hence $f(k) = a_k$ is also increasing

4 (ii), (iii)

(ii) $b_n = 3^{n+5}2^{-n}$

$$3^{n+5}2^{-n} = \frac{3^{n+5}}{2^n} = 3^5\ \frac{3^n}{2^n} = 3^5\ \left(\frac{3}{2}\right)^n$$

This is an unbounded, divergent geometric sequence that is increasing (each term is $\frac{3}{2}$ times the previous term)

(iii) $c_n = \dfrac{(-5)^{n+1}}{(3)^n}$

$$\frac{(-5)^{n+1}}{(3)^n} = (-1)^{n+1}5\frac{5^n}{3^n} = (-1)^{n+1}5\ \left(\frac{5}{3}\right)^n$$

An unbounded, divergent geometric sequence oscillating between positive and negative values. It is not increasing.

4(iv) $S_n = \sum_{k=2}^{n} \frac{k}{k^3 - 2}$

$$S_{n+1} = S_n + \frac{n+1}{(n+1)^3 - 2} \quad \Longrightarrow \quad S_n \quad \text{is increasing}$$

- Use Limit Comparison Test on $\sum_{k=2}^{\infty} \frac{k}{k^3 - 2}$
- $\frac{k}{k^3 - 2} \sim \frac{1}{k^2} \Rightarrow$ Compare to the convergent p-series $\sum_{k=2}^{\infty} \frac{1}{k^2}$
- $\lim_{k\to\infty} \frac{\frac{1}{k^2}}{\frac{k}{k^3-2}} = \lim_{k\to\infty} \frac{k^3 - 2}{k^3} = \lim_{k\to\infty} (1 - \frac{2}{k^3}) = 1$

$\lim_{n\to\infty} S_n = \sum_{k=2}^{\infty} \frac{k}{k^3 - 2}$ converges by the Limit Comparison Test and hence is also bounded.

5. (i) Find the Taylor polynomial of order 3 centered at a.
(ii) Use the 3rd order Taylor polynomial to approximate $f(x)$.
(iii) Find an upper bound for the absolute error in this approximation.

Taylor Polynomial of degree n at $x = a$

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}}{3!}(a)(x-a)^3 + \cdots + \frac{f^{(n)}}{n!}(a)(x-a)^n$$

Taylor's Inequality

Suppose $|f^{(n+1)}(c)| \leq M$ for all c between a and x. Then

$$|\text{error}| = |T_n(x) - f(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

5a: $T_3(x)$ for $f(x)=\sqrt{x}$ centered at 16

$T_3(x) =$
$$f(16)+f'(16)(x-16)+\frac{f''(16)}{2}(x-16)^2+\frac{f^{(3)}(16)}{3!}(x-16)^3$$

- $f(x)=\sqrt{x} \quad \Longrightarrow \quad f(16)=4$
- $f'(x)=\frac{1}{2}x^{-1/2} \quad \Longrightarrow \quad f'(16)=\frac{1}{2}16^{-1/2}=\frac{1}{2\sqrt{16}}=\frac{1}{8}$
- $f''(x)=-\frac{1}{4}x^{-3/2} \quad \Longrightarrow \quad f''(16)=-\frac{1}{4}16^{-3/2}=-\frac{1}{256}$
- $f^{(3)}(x)=\frac{3}{8}x^{-5/2} \quad \Longrightarrow \quad f^{(3)}(16)=\frac{3}{8}16^{-5/2}=\frac{3}{8192}$

$$T_3(x)=4+\frac{1}{8}(x-16)+\frac{-1/256}{2}(x-16)^2+\frac{3}{(8192)\cdot 3!}(x-16)^3$$

5A: (II) $T_3(x) \approx \sqrt{x}$ FOR x NEAR $x = 16$

$$T_3(x) = 4 + \frac{1}{8}(x-16) + \frac{-1/256}{2}(x-16)^2 + \frac{3}{(8192)\cdot 3!}(x-16)^3$$

$$\sqrt{16.2} = f(16.2) \approx T_3(16.2)$$

$$\sqrt{16.2} \approx 4 + \frac{1}{8}(0.2) + \frac{-1/256}{2}(0.2)^2 + \frac{3}{(8192)\cdot 3!}(16)(0.2)^3$$

5a: (iii) Find a bound for the error

$$|\text{error}| \le \frac{M}{4!}(x-16)^4 \text{ when } |f^{(4)}(c)| \le M, \quad 16 \le c \le 16.2$$

$$f^{(3)}(x) = \frac{3}{8}x^{-5/2} \Rightarrow f^{(4)}(x) = -\frac{15}{16}x^{-7/2} \Rightarrow |f^{(4)}(x)| = \frac{15}{16}x^{-7/2}$$

$|f^{(4)}(x)|$ is decreasing

$$\Longrightarrow |f^{(4)}(c)| \le |f^{(4)}(16)| = M \quad \text{for } 16 \le c \le 16.2$$

$$|\text{error}| \le \frac{M}{4!}(16.2-16)^4 = |\frac{15}{16\cdot 4^7\cdot 4!}(16.2-16)^4| = (\frac{15}{16})\frac{(0.2)^4}{4!\,4^7}$$

$$|\text{error}| \le (\frac{15}{16})\frac{(0.2)^4}{4!\,4^7}$$

5b: $T_3(x)$ for $f(x) = \cos x$ centered at $\frac{\pi}{4}$

$$f(x) = \cos x; \quad f'(x) = -\sin x; \quad f''(x) = -\cos x; \quad f^{(3)}(x) = \sin x$$

$$f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}; \quad f'(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}; \quad f''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}; \quad f^{(3)}(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

$$T_3(x) = f(\frac{\pi}{4}) + f'(\frac{\pi}{4})(x - \frac{\pi}{4}) + \frac{f''(\frac{\pi}{4})}{2}(x - \frac{\pi}{4})^2 + \frac{f^{(3)}(\frac{\pi}{4})}{3!}(x - \frac{\pi}{4})^3$$

$$T_3(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) - \frac{1}{2\sqrt{2}}(x - \frac{\pi}{4})^2 + \frac{1}{3!\sqrt{2}}(x - \frac{\pi}{4})^3$$

(ii) $T_3(\frac{3\pi}{16}) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(-\frac{\pi}{16}) - \frac{1}{2\sqrt{2}}(-\frac{\pi}{16})^2 + \frac{1}{3!\sqrt{2}}(-\frac{\pi}{16})^3$

$$\cos(\frac{3\pi}{16}) \approx \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(\frac{\pi}{16}) - \frac{1}{2\sqrt{2}}(\frac{\pi}{16})^2 - \frac{1}{3!\sqrt{2}}(\frac{\pi}{16})^3$$

5B: (III) FIND A BOUND FOR THE ERROR

$$|\text{error}| \leq \frac{M}{4!}(x - \frac{\pi}{4})^4 \quad \text{when } |f^{(4)}(c)| \leq M, \quad \frac{3\pi}{16} \leq c \leq \frac{\pi}{4}$$

From $f^{(3)}(x) = \sin x$ we compute $f^{(4)}(x) = \cos x$

We choose $M = 1$ since $|f^{(4)}(x)| = |\cos x| \leq 1$

$$|\text{error}| \leq |\frac{M}{4!}(\frac{\pi}{16})^4| = \frac{1}{4!}(\frac{\pi}{16})^4$$

$$|\text{error}| \leq \frac{1}{4!}(\frac{\pi}{16})^4$$

5c: $T_3(x)$ for e^{-2x} centered at 0

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

- $f(x) = e^{-2x}$
- $f'(x) = -2e^{-2x}$
- $f''(x) = 4e^{-2x}$
- $f^{(3)}(x) = -8e^{-2x}$
- $f(0) = 1$
- $f'(0) = -2$
- $f''(0) = 4$
- $f^{(3)}(0) = -8$

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$T_3(x) = 1 - 2x + \frac{4}{2!}x^2 - \frac{8}{3!}x^3 = 1 - 2x + 2x^2 - \frac{4}{3}x^3$$

5c: $e^{-2x} \approx T_3(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3$

(ii) $e^{-0.2} = f(0.1) \approx T_3(0.1)$

$$= 1 - 2(\frac{1}{10}) + 2(\frac{1}{100}) - \frac{4}{3}\frac{1}{1000} = \frac{3000}{3000} - \frac{300}{3000} + \frac{60}{3000} - \frac{4}{3000}$$

$$= \frac{3000 - 300 + 60 - 4}{3000} = \frac{2756}{3000} = 0.918666\ldots$$

(iii) The error is bounded by

$$|\text{error}| \leq \frac{M}{4!}(0.1)^4 \text{ , where } |f^{(4)}(x)| \leq M \text{ on } [0.0.1]$$

$$f^{(3)}(x) = -8e^{-2x} \Longrightarrow f^{(4)}(x) = 16e^{-2x}$$

$$|f^{(4)}(x)| = 16e^{-2x} \text{ decreasing } \Longrightarrow |f^{(4)}(c)| \leq f^{(4)}(0) = 16.$$

Choose $M = 16$ $\qquad |\text{error}| \leq |\frac{M}{4!}(0.1)^4| \leq \frac{16}{4!}(0.1)^4$

Alternate approach for e^{-2x}

3rd-order Taylor polynomial at $x = 0$:

- For e^x: $\quad T_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$

- For e^{-2x}: Replace x by $(-2x)$

$$T_3(x) = 1 + (-2x) + \frac{1}{2!}(-2x)^2 + \frac{1}{3!}(-2x)^3$$

$$T_3(x) = 1 - 2x + \frac{4}{2!}x^2 - \frac{8}{3!}x^3 = 1 - 2x + 2x^2 - \frac{4}{3}x^3$$

However, this does not give $f^{(4)}(x) = 16e^{-2x}$ which is necessary for the error bound

6A: $T(x)$ FOR e^{-x^2} CENTERED AT 0

- First find Taylor Series for e^x:

$$f^{(n)}(x) = -e^{-x} \quad \Longrightarrow \quad f^{(n)}(0) = 1$$

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots + \frac{1}{n!}x^n + \ldots$$

- For e^{-x^2}: Replace x by $-x^2$

$$e^{-x^2} = 1 + (-x^2) + \frac{1}{2!}(-x^2)^2 + \frac{1}{3!}(-x^2)^3 + \ldots + \frac{1}{n!}(-x^2)^n + \ldots$$

$$e^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 \ldots + (-1)^n\frac{1}{n!}x^{2n} + \ldots$$

6b Integration of a power series

$$e^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 \ldots + (-1)^n \frac{1}{n!}x^{2n} + \ldots$$

$$\int e^{-x^2}\,dx = \int 1\,dx - \int x^2\,dx + \int \frac{1}{2!}x^4\,dx - \int \frac{1}{3!}x^6\,dx$$

$$+ \int \frac{1}{4!}x^8\,dx \ldots + \int (-1)^n \frac{1}{n!}x^{2n}\,dx \ldots + C$$

$$\int e^{-x^2}\,dx = x - \frac{1}{3}x^3 + \frac{1}{5\cdot 2!}x^8 - \frac{1}{7\cdot 3!}x^7 + \frac{1}{9\cdot 4!}x^9$$

$$\frac{1}{11\cdot 4!}x^{11} \ldots + (-1)^n \frac{1}{(2n+1)n!}x^{2n+1} \ldots + C$$

6c: Taylor series for $2xe^{-x^2}$

$$e^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 \ldots + (-1)^n \frac{1}{n!}x^{2n} + \ldots$$

Method 1: Multiply by $2x$

$$2xe^{-x^2} = 1(2x) - x^2(2x) + \frac{1}{2!}x^4(2x) - \frac{1}{3!}x^6(2x) + \frac{1}{4!}x^8(2x) \ldots + (-1$$

$$2xe^{-x^2} = 2x - 2x^3 + \frac{2}{2!}x^5 - \frac{2}{3!}x^7 + \frac{2}{4!}x^9 \ldots + (-1)^n \frac{2}{n!}x^{2n+1} + \ldots$$

Method 2: Differentiate term-by-term

$$2xe^{-x^2} = -\frac{d}{dx}e^{-x^2} = -\left[(1)' - (x^2)' + (\frac{1}{2!}x^4)' - (\frac{1}{3!}x^6)' \ldots + ((-1)^n \frac{1}{n!}x^{2n})' +\right.$$

$$2xe^{-x^2} = 2x - \frac{4}{2!}x^3 + \frac{6}{3!}x^5 - \frac{8}{4!}x^7 \ldots + (-1)^{n+1} \frac{2n}{n!}x^{2n-1} + \ldots$$

7A: $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{k}$ CONVERGES OR DIVERGES?

$\dfrac{\sqrt{k^2+1}}{k} \approx \dfrac{\sqrt{k^2}}{k} = \dfrac{k}{k} = 1$ for large k

Consider the Divergence Test: $\displaystyle\lim_{k\to\infty} \frac{\sqrt{k^2+1}}{k} = \lim_{k\to\infty} \frac{\sqrt{k^2+1}}{\sqrt{k^2}}$

$$= \lim_{k\to\infty} \sqrt{\frac{k^2+1}{k^2}} = \lim_{k\to\infty} \sqrt{\frac{1+1/k^2}{1}} = 1$$

Diverges by the Divergence Test

7B: $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ CONVERGES OR DIVERGES?

Use Integral Test with $f(x) = \dfrac{1}{x \ln x}$. f is continuous, positive

$f(x) = \dfrac{1}{x \ln x} \implies f'(x) = \dfrac{-(\ln x + 1)}{(x \ln x)^2} < 0.$ $\quad f(x)$ is decreasing

Let $u = \ln x.$ $\quad \displaystyle\int \frac{1}{x \ln x}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln|\ln x| + C$

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_2^{b} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \ln|\ln b| - \ln|\ln 2| = \infty$$

Conclusion: $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges by the Integral Test

7C: $\displaystyle\sum_{k=2}^{\infty} ke^{-2k^2} = \sum_{k=2}^{\infty} \frac{k}{e^{2k^2}}$

The Integral Test works but the Ratio Test is simpler

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\dfrac{k+1}{e^{2(k+1)^2}}}{\dfrac{k}{e^{2k^2}}} = \lim_{k\to\infty} \frac{(k+1)\ e^{2k^2}}{k\ e^{2k^2+4k+2}}$$

$$= \lim_{k\to\infty} (1 + \frac{1}{k})(\frac{1}{e^{4k+2}}) = 0 < 1$$

Conclusion: $\displaystyle\sum_{k=2}^{\infty} ke^{-2k^2}$ converges by the Ratio Test

7D: $\sum_{k=0}^{\infty} \frac{4+3^k}{4^k}$ Two choices:

- Limit Comparison: $\frac{4+3^k}{4^k} \approx (\frac{3}{4})^k$

  Compare to the convergent geometric series $\sum_{k=0}^{\infty} (\frac{3}{4})^k$

- Ratio Test:

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{4+3^{k+1}}{4^{k+1}}}{\frac{4+3^k}{4^k}} = \lim_{k\to\infty} \frac{4^k(4+3^{k+1})}{4^{k+1}(4+3^k)} = \frac{3}{4} < 1$$

Conclusion: $\sum_{k=0}^{\infty} \frac{4+3^k}{4^k}$ converges by the Ratio Test

7E: $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ INTEGRAL TEST

$f(x) = \frac{1}{x(\ln x)^3}$ is a positive, decreasing, continuous function

Consider $\int \frac{1}{x(\ln x)^3}\,dx$ and take $u = \ln x$

$$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{1}{u^3}\,du = \frac{-1}{2u^2} + C = \frac{-1}{2(\ln x)^2} + C$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty} \left(\frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln b)^2} \right)$$

Conclusion: $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ converges by the Integral Test

7F: $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k+1}{2k}\right)^k$

$$\left(\frac{k+1}{2k}\right)^k = \left(\frac{1+1/k}{2}\right)^k \approx \frac{1}{2^k}$$

Use the Limit Comparison Test

$$\lim_{k\to\infty}\frac{\left(\frac{1+1/k}{2}\right)^k}{\frac{1}{2^k}} = \lim_{k\to\infty} 2^k\left(\frac{(1+\frac{1}{k})^k}{2^k}\right) = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e$$

$\displaystyle\sum_{k=1}^{\infty}\left(\frac{k+1}{2k}\right)^k$ converges by the Limit Comparison Test

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = \lim_{k\to\infty} e^{k\ln(1+1/k)} = \lim_{k\to\infty} e^{\frac{\ln(1+1/k)}{1/k}} = \lim_{x\to 0} e^{\frac{\ln(1+x)}{x}} = e^1$$

7G: $\displaystyle\sum_{k=1}^{\infty} \frac{k^4}{e^{3k}}$

- Ratio Test:

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{(k+1)^4}{e^{3(k+1)}}}{\frac{k^4}{e^{3k}}} = \lim_{k\to\infty} \frac{(k+1)^4}{k^4} \frac{e^{3k}}{e^{3k+3}}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^4 \frac{1}{e^3} = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^4 \frac{1}{e^3} = \frac{1}{e^3} < 1$$

Conclusion: $\displaystyle\sum_{k=1}^{\infty} \frac{k^4}{e^{3k}}$ converges by the Ratio Test

7H: $\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2-1}$

- Use Limit Comparison Test on $\sum_{k=2}^{\infty} \left|\frac{(-1)^k}{k^2-1}\right| = \sum_{k=2}^{\infty} \frac{1}{k^2-1}$
- $\frac{1}{k^2-1} \sim \frac{1}{k^2} \Rightarrow$ Compare to the convergent p-series $\sum_{k=2}^{\infty} \frac{1}{k^2}$
- $\lim_{k\to\infty} \frac{\frac{1}{k^2}}{\frac{1}{k^2-1}} = \lim_{k\to\infty} \frac{k^2-1}{k^2} = \lim_{k\to\infty} \frac{1-1/k^2}{1} = 1$

$\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2-1}$ converges absolutely by Limit Comparison Test

Limit Comparison Test on $\sum_{k=1}^{\infty} a_k$

Choose $\sum_{k=1}^{\infty} b_k$ and compute $\lim_{k\to\infty} \frac{a_k}{b_k} = L$

1. $0 < L < \infty \longrightarrow$ both converge or both diverge
2. $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges $\longrightarrow \sum_{k=1}^{\infty} a_k$ converges
3. $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges $\longrightarrow \sum_{k=1}^{\infty} a_k$ diverges

(Both a_k and b_k must be positive)

Compares two series without having to worry about inequalities.

7I: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k^2+1}}$ Test for absolute convergence first

- Use Limit Comparison Test on $\sum_{k=1}^{\infty} |\frac{(-1)^{k+1}}{\sqrt{k^2+1}}| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$
- $\frac{1}{\sqrt{k^2+1}} \sim \frac{1}{\sqrt{k^2}} = \frac{1}{k} \Rightarrow$ Compare to the divergent $\sum_{k=1}^{\infty} \frac{1}{k}$
- $\lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{\sqrt{k^2+1}}} = \lim_{k\to\infty} \frac{\sqrt{k^2+1}}{k} = \lim_{k\to\infty} \frac{\sqrt{k^2+1}}{\sqrt{k^2}}$
  $= \lim_{k\to\infty} \sqrt{\frac{k^2+1}{k^2}} = \lim_{k\to\infty} \sqrt{\frac{1+1/k^2}{1}} = 1$
- $\sum_{k=1}^{\infty} |\frac{(-1)^{k+1}}{\sqrt{k^2+1}}|$ diverges by Limit Comparison Test

$$\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{\sqrt{k^2+1}}\right| \text{ diverges by Limit Comparison Test}$$

Now apply the **Alternating Series Test** to $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k^2+1}}$

1. The terms $(-1)^k \dfrac{1}{\sqrt{k^2+1}}$ alternate sign
2. $\displaystyle\lim_{k\to\infty} \frac{1}{\sqrt{k^2+1}} = 0.$
3. Let $f(x) = \dfrac{1}{\sqrt{x^2+1}} = (x^2+1)^{-1/2}$.

   $f'(x) = -\dfrac{1}{2}(x^2+1)^{-3/2}(2x) = -x(x^2+1)^{-3/2} < 0.$

   f is decreasing $\implies a_k = f(k) = \dfrac{1}{\sqrt{k^2+1}}$ decreasing

Conclusion: This series converges by the Alternating Series Test and thus is conditionally convergent.

8: $\sum_{k=1}^{\infty} \frac{1}{k2^k}(x-2)^k$: Find the Interval of convergence

Step 1. Ratio Test on $\sum_{k=0}^{\infty} \frac{1}{k2^k}|x-2|^k$

- $$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{1}{(k+1)2^{k+1}}|x-2|^{k+1}}{\frac{1}{k2^k}|x-2|^k}$$

$$= \lim_{k\to\infty} \frac{k\ 2^k}{(k+1)2^{k+1}}|x-2| = \lim_{k\to\infty} \frac{k}{k+1}\frac{|x-2|}{2} = \frac{|x-2|}{2}$$

- Absolute convergence when $\frac{|x-2|}{2} < 1$ or when $|x-2| < 2$

Absolute convergence when

$$|x-2| < 2 \Longrightarrow -2 < x-2 < 2 \Longrightarrow 0 < x < 4$$

Step 2. Test $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k2^k}(x-2)^k$ for convergence at $x = 0$ and 4

- Take $x = 0$: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k2^k}(-2)^k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k}$
  - Converges by the Alternating Series Test
- Take $x = 4$: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k2^k}(2)^k = \sum_{k=0}^{\infty} \frac{1}{k}$
  - Divergent p-series ($p = 1$)

- The **interval of convergence** is $[0, 4)$
- **Center** $x = 2$
- $R = 2$ is the **radius of convergence**

9: $\sum_{k=1}^{\infty} \frac{3^k}{k} x^k$: Interval of convergence

Step 1. Ratio Test on $\sum_{k=1}^{\infty} \frac{3^k}{k} |x|^k$

- $$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{3^{k+1}}{k+1}|x|^{k+1}}{\frac{3^k}{k}|x|^k}$$

  $$= \lim_{k\to\infty} \frac{3^{k+1}}{3^k} \frac{k}{k+1}|x| = \lim_{k\to\infty} \frac{1}{1+1/k} 3|x| = 3|x|$$

- Absolute convergence when $3|x| < 1$ or when $|x| < \frac{1}{3}$

Absolute convergence when $|x| < \frac{1}{3} \Longrightarrow -\frac{1}{3} < x < \frac{1}{3}$

Step 2. Test $\sum_{n=1}^{\infty} \frac{3^k}{k} x^k$ for convergence at $x = \pm\frac{1}{3}$

- Take $x = -\frac{1}{3}$: $\quad \sum_{k=1}^{\infty} \frac{3^k}{k} (-\frac{1}{3})^k = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$
    - Converges by the Alternating Series Test
- Take $x = \frac{1}{3}$: $\quad \sum_{k=1}^{\infty} \frac{3^k}{k} (\frac{1}{3})^k = \sum_{k=1}^{\infty} \frac{1}{k}$ Divergent p-series

- The **interval of convergence** is $[-\frac{1}{3}, \frac{1}{3})$
- $R = \frac{1}{3}$ is the **radius of convergence**

10. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}(x-1)^n$ has a radius of convergence $R=2$.

(a) Explain carefully what the "radius of convergence" tells us about the series.

- The radius of convergence $R=2$ gives the radius of the interval of convergence centered at $x=1$. The series converges whenever $|x-1|<2$. It may or may not converge when $|x-1|=2$ (at the endpoints of the interval of convergence). The series diverges if $|x-1|>2$.

(b) Find the interval of convergence.

- Possible answers: $(-1,3)$, $(-1,3]$, $[-1,3)$, $[-1,3]$
- We need to test the endpoints, when $x=-1$ and $x=3$.

$\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{2^n}(x-1)^n$: Testing endpoints

1. Let $x = -1$ $\quad \sum_{n=1}^{\infty}(-1)^n \frac{n^2}{2^n}(-2)^n$
   - $\sum_{n=1}^{\infty}(-1)^n(-1)^n \frac{n^2}{2^n}(2)^n = \sum_{n=1}^{\infty} n^2$
   - Diverges by the divergence test since $\lim_{n\to\infty} n^2 \neq 0$
2. Let $x = 3$ $\quad \sum_{n=1}^{\infty}(-1)^n \frac{n^2}{2^n}(2)^n$
   - $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{2^n}(2)^n = \sum_{n=1}^{\infty}(-1)^n n^2$
   - Diverges by the divergence test since $\lim_{n\to\infty} (-1)^n n^2 \neq 0$
3. Interval of convergence: $(-1, 3)$

11. How many terms of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{7n+5}$ do we need to add in order to approximate the series with $|\text{error}| < 0.00001$?

The approximation $\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{7k+5}$ has an error less than the absolute value of the first neglected term, namely $\frac{1}{7(n+1)+5}$.

We want an n that makes $\frac{1}{7(n+1)+5} < 0.00001 = \frac{1}{10^5}$

$$7n + 12 > 10^5 \quad \Longrightarrow \quad n > \frac{10^5 - 12}{7} = 14284 \text{ will suffice}$$

We need 14, 284 terms.

$$\sum_{k=1}^{14,284}(-1)^{k+1}\frac{1}{7k+5} \approx \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{7k+5} \text{ has error} < 0.00001$$

12. Approximate $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2+1}$ correct to 3 decimal places.

The approximation $\sum_{k=1}^{n-1}(-1)^{k+1}\frac{1}{k^2+1}$ has an error less than the absolute value of the first neglected term, namely $\frac{1}{n^2+1}$.

We want an n that makes $\frac{1}{n^2+1} < 0.0001 = \frac{1}{10^4}$

$n^2+1 > 10^4$

$n^2 \geq 10^4 > 10^4 - 1 \quad \implies \quad n = 100$ will suffice

$$\sum_{k=1}^{99}(-1)^{k+1}\frac{1}{k^2+1} \approx \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2+1} \text{ correct to 3 decimals.}$$