University of Connecticut
Department of Mathematics

Math 1132 — Practice Exam 1 Solutions — Fall 2014

Name: ____________________

Instructor Name: ____________________ Section: ________

TA Name: ____________________ Discussion Section: ________

Read This First!

- Please read each question carefully. Show **ALL** work clearly in the space provided. In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question. Please cross out or fully erase any work that you do not want graded. The point value of each question is indicated after its statement. No books or other references are permitted.
- Calculators are allowed but you must show all your work to receive credit on the problem.
- Except for problems about approximate integration or probability, give any numerical answers in exact form, **not as approximations**. For example, one-third is $\frac{1}{3}$, not .33 or .33333. And one-half of π is $\frac{1}{2}\pi$, not 1.57 or 1.57079.
- **Note**: This is only a sample exam. It illustrates the kinds of topics that will be on the midterm, but is no substitute for a thorough review (including your homework and worksheet problems). In particular, problems on the actual exam as well as the point distribution among those problems will not be identical to what you see in the sample exam.

Grading - For Administrative Use Only

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	12	12	10	8	8	12	10	10	10	10	102
Score:											

1. If the statement is always true, circle the printed capital T. If the statement is sometimes false, circle the printed capital F. In each case, write a careful and clear justification or a counterexample.

(a) A sphere of radius 3 has volume $\int_{-3}^{3} \pi(9 - x^2)\,dx$. (a) T [2]

Justification: Revolve $y = \sqrt{9 - x^2}$ around x-axis for $-3 \leq x \leq 3$.

(b) Under Hooke's law, the work required to stretch a spring 2 inches beyond its natural length is twice that required to stretch it 1 inch beyond its natural length (b) F [2]

Justification: $F(x) = kx \Rightarrow W(b) = \int_0^b kx\,dx = kb^2/2$, so $W(2) = 2k$ and $W(1) = k/2$.

(c) The trapezoid rule with $n = 10$ for $\int_0^4 x^3\,dx$ will be an underestimate of the integral's value. (c) F [2]

Justification: Trapezoids lie on or above graph throughout.

(d) If $\int_0^1 f(x)\,dx$ is an improper integral, then so is $\int_1^2 f(x)\,dx$. (d) F [2]

Justification: Try $f(x) = 1/x^p$ for any $p > 0$.

(e) If $f(x)$ is a probability density function, then $0 \leq f(x) \leq 1$ for all x. (e) F [2]

Justification: Highly peaked bell curve. But $0 \leq \int_a^b f(x)\,dx \leq 1$.

(f) The length of the curve $x = \frac{2}{3}(y-1)^{3/2}$ over the interval $1 \leq y \leq 4$ by is given the definite integral $L = \int_1^4 \sqrt{y}\,dy$. (f) [2]

Justification:

$$L = \int_1^4 \sqrt{1 + (\frac{dx}{dy})^2}\,dy = \int_1^4 \sqrt{1 + ((y-1)^{1/2})^2}\,dy = \int_1^4 \sqrt{y}\,dy$$

2. Find the following general antiderivatives. Remember to add $+C$ to your final answer.

(a) $\int x^n \ln x\, dx$, where $n \neq -1$ [3]

$$u = \ln x, dv = x^n\, dx \implies \int x^n \ln x\, dx = x^n \ln x\, dx = \frac{x^{n+1}}{n+1}(\ln x) - \frac{x^{n+1}}{(n+1)^2} + C.$$

(b) $\int \frac{x}{\sqrt{x^2-1}}\, dx$ [3]

$$u = x^2 - 1 \implies du = 2x\, dx \implies \int \frac{x}{\sqrt{x^2-1}}\, dx = \int \frac{(1/2)du}{\sqrt{u}} = \sqrt{x^2-1} + C.$$

(c) $\int e^{2x} \cos(4x)\, dx$ [3]

$$u = \cos(4x)\, dx, dv = e^{2x} \implies \int e^{2x} \cos(4x)\, dx = \frac{1}{2}e^{2x}\cos(4x) + 2\int e^{2x}\sin(4x)\, dx$$

$$u = \sin(4x)\, dx, dv = e^{2x} \implies \int e^{2x} \sin(4x)\, dx = \frac{1}{2}e^{2x}\sin(4x) - 2\int e^{2x}\cos(4x)\, dx$$

Thus

$$I = \frac{1}{2}e^{2x}\cos(4x) + 2\left(\frac{1}{2}e^{2x}\sin(4x) - 2I\right) = \frac{1}{2}e^{2x}\cos(4x) + e^{2x}\sin(4x) - 4I,$$

so

$$I = \frac{1}{10}e^{2x}\cos(4x) + \frac{1}{5}e^{2x}\sin(4x) + C.$$

(d) $\int \frac{x^2-2}{x^3+4x}\, dx$ [3]

$$\frac{x^2-2}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \implies x^2 - 2 = A(x^2+4) + (Bx+C)x.$$

Set $x = 0$ to see $A = -1/2$ and set $x = 1$ and $x = -1$ to see $B+C = 3/2$ and $B-C = 3/2$, so $B = 3/2$ and $C = 0$.

$$\int \frac{x^2-2}{x^3+4x}\, dx = -\frac{1}{2}\int \frac{dx}{x} + \frac{3}{2}\int \frac{x}{x^2+4}\, dx = -\frac{1}{2}\ln|x| + \frac{3}{4}\ln|x^2+4| + C.$$

3. Determine whether each of the following improper integrals is convergent or divergent. For those that are convergent, give their exact value (not decimal approximations). Those determined to be divergent must have an explanation for their divergence.

   (a) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ [5]

   $$\lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}} = \lim_{t\to\infty} 2\sqrt{x}\Big|_1^t = \lim_{t\to\infty} 2\sqrt{t}-2=\infty.$$

   (b) $\displaystyle\int_0^\infty \frac{dx}{(2x+1)(x+4)}$ [5]

   $$\frac{1}{(2x+1)(x+4)} = \frac{A}{2x+1}+\frac{B}{x+4} \Longrightarrow 1 = A(x+4)+B(2x+1).$$

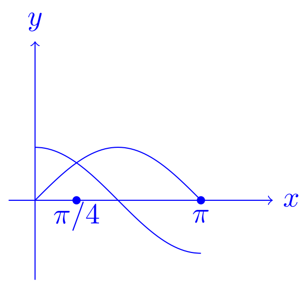
   Let $x=-1/2$ and $x=-4$ to see $A=2/7$ and $B=-1/7$. Thus

   $$\int_0^\infty \frac{dx}{(2x+1)(x+4)} = \lim_{t\to\infty}\int_0^t \frac{2}{7(2x+1)}\,dx - \int_0^t \frac{1}{7(x+4)}\,dx = \lim_{t\to\infty} \frac{1}{7}\ln|2x+1|\Big|_0^t - \frac{1}{7}\ln|x+4|\Big|_0^t$$

   $$= \lim_{t\to\infty}\frac{1}{7}\ln\left|\frac{2t+1}{t+4}\right| - \frac{1}{7}\ln(1)+\frac{1}{7}\ln|4| = \frac{1}{7}\ln 2+\frac{1}{7}\ln 4 = \frac{3}{7}\ln 2$$

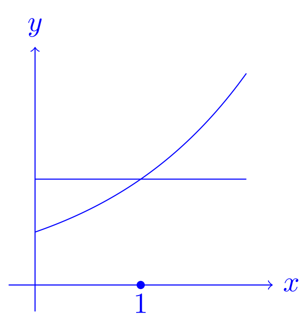
4. Set up, but do **NOT** evaluate, integrals or sums of integrals for the area of the region bounded by the following curves. Do not use absolute value signs in your final answer. On the set of axes in each part draw a picture of the curves and region between them.

   (a) $y=\sin x$ and $y=\cos x$ for $0\le x\le\pi$. [4]



   The area is $\displaystyle\int_0^{\pi/4}(\cos x-\sin x)\,dx+\int_{\pi/4}^{\pi}(\sin x-\cos x)\,dx$

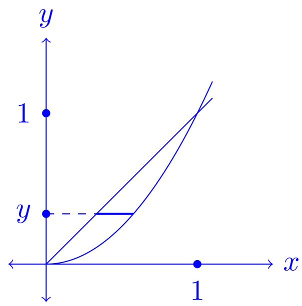
   (b) $y=2^x$ and $y=2$ in the first quadrant. [4]



   The area is $\displaystyle\int_0^1 (2-2^x)\,dx$

5. Set up, but do **NOT** evaluate, integrals equal to the volumes of the following solid regions. On the set of axes in each part draw a picture of the curves and region between them.

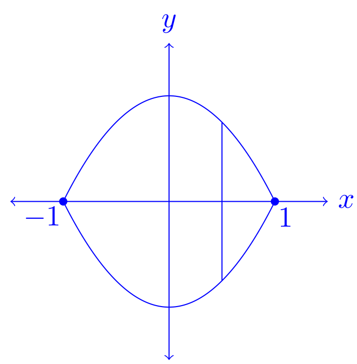
(a) Revolve the region between $y = x^2$ and $y = x$ in the first quadrant around the y-axis. [4]



On the y-slice, the outer curve has radius $x = \sqrt{y}$ and the inner curve has radius $x = y$, so the volume is

$$\int_0^1 \pi(\sqrt{y}^2 - y^2)\,dy = \int_0^1 \pi(y - y^2)\,dy.$$

(b) The solid whose base is bounded by $y = x^2 - 1$ and $y = 1 - x^2$ with $-1 \le x \le 1$ and whose cross-sections parallel to the y-axis are equilateral triangles. [4]



At the x-slice, the triangle's side length is $2((1 - x^2) - (x^2 - 1)) = 4(1 - x^2)$, and the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$, so the volume is

$$\int_{-1}^1 \frac{\sqrt{3}}{4}(4(1 - x^2))^2\,dx.$$

6. (a) A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done to stretch the spring from 12 cm to 20 cm? [4]

Let $F(x) = kx$ for this spring, so $30 = k(15/100 - 12/100)$, which implies $k = 1000$. Thus $W = \int_0^{8/100} 1000x\,dx = 500x^2\Big|_0^{8/100} = 500(8/100)^2 = .4$ Newtons.

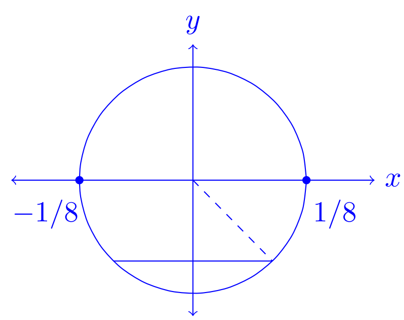
(b) A cylindrical keg of root beer with height 1 m and diameter $\frac{1}{4}$ m is *half-filled* and is standing upright (on one of its circular bases). The density of the root beer is 2000 kg/m^3. Set up, but do **NOT** evaluate, an integral that is equal to the work required to pump the root beer out of the top of the keg. [4]

The root beer fills the keg up to 1/2 meter from the bottom. Measuring height from the bottom, where $y = 0$, we care about $0 \le y \le 1/2$ and the y-slice has to be moved up by $1 - y$ meters. (If we measured from the top, with $1/2 \le y \le 1$, the y-slice has to be moved up y meters.) The y-slice has volume $\pi r^2 dy = \pi(1/8)^2\,dy$ and therefore mass $(2000)(\pi/64)dy$ kg. So the work to move the root beer to the top is

$$\int_0^{1/2} \frac{2000\pi}{64}(1-y)\,dy.$$

(c) Consider the same half-full cylindrical keg of root beer with height 1 m and diameter $\frac{1}{4}$ m as in part b, but set the keg down on its side. If a hole is made along the (new) top of the rotated keg, set up but do **NOT** evaluate an integral equal to the work required to pump the root beer out. (Hint: Start by drawing a good picture.) [4]

Below is a side view of the keg, whose horizontal slices are rectangles.

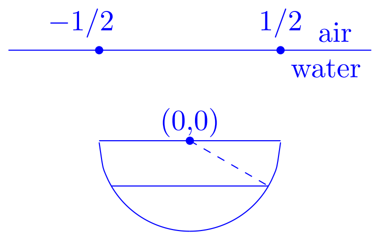


The y-slice has width depending on the rectangle's height above the bottom. Using the Pythagorean theorem, the endpoints of the solid line at height y are $\pm\sqrt{r^2 - y^2} = \pm\sqrt{1/64 - y^2}$. Therefore the width is $2\sqrt{1/64 - y^2}$, so the volume of the y-slice is $(1)(2\sqrt{1/64 - y^2})dy = 2\sqrt{1/64 - y^2}dy$. Its mass is the volume multiplied by 2000, and the y-slice has to be lifted up $1/4 - y$ meters, and the range of slices is $-1/8 \le y \le 0$, so the work involved in pumping out the root beer is (pay attention to the bounds of integration!)

$$\int_{-1/8}^{0} 2000(2)\sqrt{1/64 - y^2}\left(\frac{1}{4} - y\right)\,dy.$$

Remark: Computing the integrals, the work in part b is 36.8 J, and in part c it is 14.9 J, so the sideways keg in this problem needs over 50% less work to get out the root beer than the upright keg.

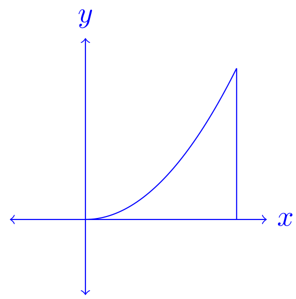
7. A semi-circular plate with *diameter* 1 m is submerged vertically in water with the diameter being the top of the plate, as in the diagram below. Set up but do **NOT** evaluate an integral equal to the hydrostatic force on the plate, in Newtons, if the diameter lies 1 meter below the surface of the water. The density of water is 1000 $\mathrm{kg/m^3}$. [10]



Let $(0,0)$ be the top center of the plate. Its radius is $1/2$, so the boundary of the plate is $y = -\sqrt{1/4 - x^2}$. The pressure along the y-slice is constant, so the hydrostatic force along the y-slice is the pressure at that depth times the area of the y-slice.
To find the area of the y-slice we first find its length. By the Pythagorean theorem, the length is $2\sqrt{1/4 - y^2}$ (as in the solution to problem 6c). Its height is dy, so the area of the y-slice is $A(y) = 2\sqrt{1/4 - y^2}\,dy$.
The depth of the y-slice below water level is $1 - y$ (note $y < 0$!), and the pressure at that depth, in metric units, is $P(y) = 1000g(1 - y) = 9800(1 - y)$.
Thus the hydrostatic force on the y-slice is $F(y) = P(y)A(y) = 9800(1 - y)(2\sqrt{1/4 - y^2}\,dy$. To compute the total hydrostatic force on the plate, integrate over the y-values covering the plate: $-1/2 \leq y \leq 0$, so the hydrostatic force on the plate is

$$\int_{-1/2}^{0} 9800(1 - y)\sqrt{1/4 - y^2}\,dy.$$

8. For $n > 0$ determine the coordinates of the centroid for the region in the first quadrant bounded by $y = x^n$, the x-axis, and the line $x = 1$. Your answer will depend on n. [10]



The coordinates of the centroid are

$$\overline{x} = \frac{\int_0^1 x f(x)\,dx}{A}, \quad \overline{y} = \frac{\int_0^1 \frac{1}{2} f(x)^2\,dx}{A},$$

where $f(x) = x^n$ and $A = \displaystyle\int_0^1 f(x)\,dx$ is the area of the region.
We calculate each relevant integral:

$$A = \int_0^1 x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1},$$

$$\int_0^1 x f(x)\,dx = \int_0^1 x^{n+1}\,dx = \frac{1}{n+2},$$

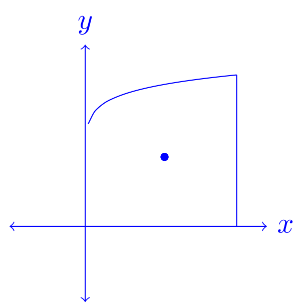
and

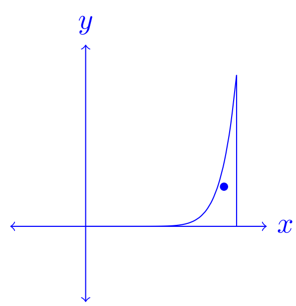
$$\int_0^1 \frac{1}{2} f(x)^2\,dx = \frac{1}{2}\int_0^1 x^{2n}\,dx = \frac{1}{2(2n+1)}.$$

Therefore

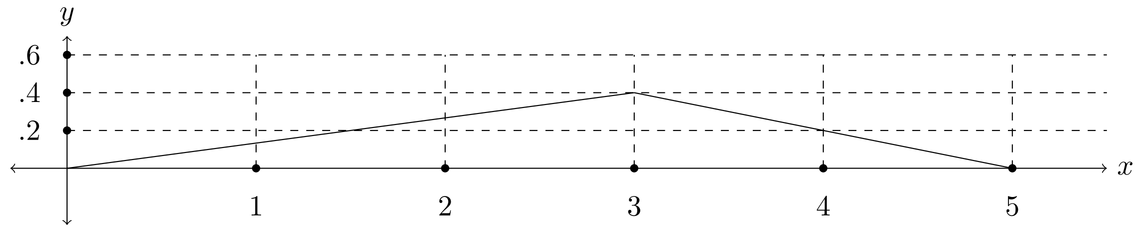
$$\overline{x} = \frac{1/(n+2)}{1/(n+1)} = \frac{n+1}{n+2} \quad\text{and}\quad \overline{y} = \frac{1/(2(2n+1))}{1/(n+1)} = \frac{n+1}{2(2n+1)},$$

so the centroid is $(\frac{n+1}{n+2}, \frac{n+1}{2(2n+1)})$.
Remark: If $n \to 0^+$ then $\overline{x} \to 1/2$ and $\overline{y} \to 1/2$, which makes sense since the graph is close to $y = 1$ and the region is nearly a unit square. See below left for $n = .1$. If $n \to \infty$ then $\overline{x} \to 1$ and $\overline{y} \to 1/4$. See below right for $n = 10$. For large n the bulk of the graph is concentrated near the line segment from $(1, 0)$ to $(1, 1)$, so it's no surprise $\overline{x}$ is near 1, but surprisingly $\overline{y}$ is not close to $1/2$ – is there an error?





9. (a) The graph below is a probability density function for a certain random variable X. Determine the probability that $1 \le X \le 3$ to the nearest hundredth. [4]



The region under the curve described by $1 \le X \le 3$ is a trapezoid (or difference of triangles). The right edge has height .4. The left edge is the y-value over the point where $x = 1$, which lies on the line from $(0,0)$ to $(3,.4)$, whose slope is $.4/3$. Therefore the line segment from $(0,0)$ to $(3,.4)$ is $y = (.4/3)x$, so $x = 1 \Rightarrow y = .4/3$. Thus the probability is

$$\frac{1}{2}h(b_1+b_2) = \frac{1}{2}(3-1)(\frac{.4}{3}+.4) = \frac{.4}{3}+.4 = \frac{1.6}{3} = .5333\cdots \approx .53.$$

Alternatively, computing the probability as $P(X \le 3) - P(X \le 1)$ it is the difference of areas of triangles:

$$\frac{1}{2}(3)(.4) - \frac{1}{2}(1)\frac{.4}{3} = \frac{1.2}{2} - \frac{.4}{6} = \frac{3.6}{6} - \frac{.4}{6} = \frac{3.2}{6} = \frac{1.6}{3}.$$

(b) The probability density function $f(x)$ that models how long someone has to wait on a customer service line is exponential: [6]

$$f(t) = \begin{cases} 0, & \text{if } t < 0, \\ ce^{-ct}, & \text{if } t > 0 \end{cases}$$

for some $c > 0$, where t is measured in minutes.

If the average time a customer waits to speak to someone is 10 minutes, determine the probability that a customer will wait for at most 5 minutes to the nearest hundredth. (Hint: first find c.)

The average of an exponential distribution is $1/c$, so $c = 1/10$. Therefore the probability of waiting at most 5 minutes is

$$\int_0^5 ce^{-ct}\,dt = \left. -e^{-ct}\right|_0^5 = -e^{-5c}+1 = 1-e^{-5c} = 1-e^{-1/2} = .3934\cdots \approx .39.$$

10. (a) Use the Trapezoid Rule with $n = 6$ to estimate $\int_0^6 x^2 + 1\,dx$ [10]

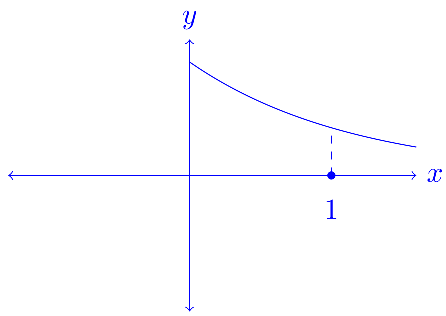
$x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6 \qquad \Delta x = 1$

$f(0) = 1, f(1) = 2, f(2) = 5, f(3) = 10, f(4) = 17, f(5) = 26, f(6) = 37$

$$T(6) = [\frac{f(x_0)}{2} + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + \frac{f(x_6)}{2}]\Delta x$$

$$T(6) = [\frac{1}{2} + 2 + 5 + 10 + 17 + 26 + \frac{37}{2}] \cdot 1 = 79$$

(b) Use the error bound formulas on the last page to determine an n such that the trapezoid rule with n subintervals approximates $\int_0^1 \frac{1}{3^x}\,dx$ to within .001.



The error bound is

$$\frac{K(1-0)^3}{12n^2} = \frac{K}{12n^2}$$

where K is an upper bound on the second derivative of $1/3^x$ for $0 \leq x \leq 1$. We have

$$(3^{-x})' = 3^{-x}(\ln 3)(-1) \Longrightarrow (3^{-x})'' = 3^{-x}(\ln 3)^2.$$

For $0 \leq x \leq 1$, $y''(x)$ is decreasing and positive, so $|y''(x)| \leq y(0) = (\ln 3)^2 \approx 1.206$: we can use $K = 1.3$ (or any larger value, such as 2). Thus, by the error bound for the trapezoid rule,

$$|E_T| \leq \frac{1.3}{12n^2}.$$

To guarantee that $|E_T| \leq .001$, make $\frac{1.3}{12n^2} \leq .001$, so $n \geq \sqrt{\frac{1.3}{12(.001)}} \approx 10.408$, so we can use $n \geq 11$. (**Note**: If we had used $K = 2$ then we'd want $n \geq \sqrt{2/(12(.001))} \approx 12.9$, so $n \geq 13$.)
For what it's worth, the least n for which T_n is within .001 of the integral is $n = 8$: the integral is $.60682\ldots$, while $T_8 = .6077\ldots$ is barely within .001 and $T_7 = .60807\ldots$.