Section 3.8: Exponential Growth

(1) In this section, we learn how to find the solution to the equation of the form $\dfrac{dy}{dt} = ky$. Finish the following sentence:

> When solving the equation $\dfrac{dy}{dt} = ky$, we are looking for a function y whose derivative is a constant (k) times itself (y).

(2) What is the solution to this differential equation? Take the derivative of it and see that it satisfies the rule.

> The solution to this differential equation is $y(t) = Ce^{kt}$ where C is a constant. Taking the derivative of $y(t)$ gives $y^{'}(t) = C(ke^{kt}) = k(Ce^{kt}) = ky(t)$, which is a constant times $y(t)$ itself.

(3) In the differential equation $\frac{dy}{dt} = ky$ or its solution $y = Ce^{kt}$, what does the k represent? What is the significance of k being positive or negative? Explain both using the differential equation and its solution.

> In the differential equation $\frac{dy}{dt} = \frac{d}{dt}y(t) = y^{'}(t) = ky(t)$ where k is constant, we say that the $y^{'}(t)$ grows or decays at a rate proportional to $y(t)$. k is the constant rate of growth ($k > 0$) or decay ($k > 0$).
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> If $k > 0$, it is called law of natural growth. For instance, in the context of population growth, we have $\frac{dP}{dt} = kP$ or $\frac{1}{P}\frac{dP}{dt} = k$ where $P(t)$ is the size of a population at time t and $P(0)$ is the population at time 0 or the initial population. The solution to the differential equation $\frac{dP}{dt} = kP$ is $P(t) = P(0)\,e^{kt}$. In other words, the population grows exponentially with a constant relative growth rate k and the population will be $P(t)$ after time t. Another way to see this is by considering $\frac{dy}{dt} = ky$. Since k is positive, the overall derivative is positive, so we get that the function is increasing.
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> If $k > 0$, it is called law of natural decay. We can see that it is decay since $y = e^{kt}$ with $k < 0$ gives us decay. From the differential equation, $\frac{dy}{dt} = ky$. Since k is negative, the overall derivative is negative, so we get that the function is decreasing. For example, in nuclear physics the mass of a radioactive substance decays at a rate proportional to the mass, which will be detailed explained in (4).

(4) Explain how you can use the half-life of a substance to determine its differential equation and solution to the differential equation.

The half-life of a substance is the time it takes for half the substance to decay. So, if we assume we start with some initial amount C_0, we know that after the amount of time equal to the half-life, we will have $\frac{C_0}{2}$ left. Plugging this in to our function (the population function, not its differential equation) will allow us to solve for k.

(5) Newton's Law of Heating and Cooling tells us that the rate of change in the temperature of an object is proportional to the difference in the objects temperature and the temperature of the surrounding area. What is the differential equation we get from Newton's Law? What is the solution to this differential equation?

The differential equation we get from Newton's Law is

$$\frac{dT}{dt} = k(T - T_s)$$

where T(t) is the temperature of the object at time t and T_s is the temperature of the surroundings.
Let $y(t) = T(t) - T_s$. Since T_s is constant, we have $y^{'}(t) = T^{'}(t)$ and the differential equation becomes $\frac{dy}{dt} = ky$. Hence, the solution to this differential equation is $y = Ce^{kt}$ where $y = T - T_s$. We can rewrite this as

$$T = Ce^{kt} + T_s.$$

It is important to remember that C is not the initial temperature of the object but it the initial difference in temperature between the object and its surroundings.

(6) How does compounded continuous interest rate compare to interest compounded n times a year? What is the differential equation and what is its solution for interest being compounded continuously?

In general, if an amount A_0 is invested at an interest rate r, then after t years it is worth $A_0(1+r)^t$.

However, if the interest is compounded n times a year, then in each compounding period the interest rate is $\frac{r}{n}$ and there are compounding periods in t years, so the value of the investment after t years is $A_0(1+\frac{r}{n})^{nt}$.

If we let $n \to \infty$, then we will be compounding the interest continuously and the value of the investment will be $A_t = \lim_{n\to\infty} A_0(1+\frac{r}{n})^{nt} = \lim_{n\to\infty} A_0\left[(1+\frac{r}{n})^{\frac{n}{r}}\right]^{rt} = A_0 \lim_{n\to\infty}\left[(1+\frac{r}{n})^{\frac{n}{r}}\right]^{rt} = A_0\left[\lim_{m\to\infty}(1+\frac{1}{m})^{m}\right]^{rt}$ where $(m=\frac{n}{r})$. So with continuous compounding of interest at interest rate r, the amount after t years is

$$A(t) = A(0)e^{rt}$$

and its differential equation is

$$\frac{dA}{dt} = rA_0e^{rt} = rA(t).$$

Extra Practice in Book: 3.8: 3, 5, 9, 14, 15, 21