Final Exam Review Sheet Solutions
Math 109

1. Give, in slope-intercept form, the equation of the line that:
   (a) Passes through the two points (3, 5) and (−2, 4).
   \[ y = \frac{1}{5}x + \frac{22}{5} \]
   (b) Passes through the point (2, 3) and is parallel to the line \( y = 2x - 5 \).
   \[ y = 2x - 1 \]
   (c) Passes through the point (1, 4) and is parallel to the line \( 2x + 3y = 6 \).
   \[ y = -\frac{2}{3}x + \frac{14}{3} \]
   (d) Passes through the point (−3, −1) and is perpendicular to the line \( y = 3x + 5 \).
   \[ y = -\frac{1}{3}x - 2 \]
   (e) Passes through the point (4, 7) and is perpendicular to the line \( y = 5 \).
   \[ x = 4 \]

2. (a) \( 2x - 5 \geq 3 \) \[ [4, \infty) \]
   (b) \( 3x + 4 < 5 \) \[ (-\infty, \frac{1}{3}) \]
   (c) \( |1 - 2x| \leq 5 \) \[ [-2, 3] \]
   (d) \( |1 - 2x| > 5 \) \[ (-\infty, -2) \cup (3, \infty) \]
   (e) \( x^2 - 5x + 6 \geq 0 \) \[ (-\infty, 2] \cup [3, \infty) \]

3. (a) Find the distance between the points (3, 4) and (1, 2).
   \[ \sqrt{(3-1)^2 + (4-2)^2} = \sqrt{8} = 2\sqrt{2} \]
   (b) Find the equation of the circle with center at (3, 4) that passes through the point (1, 2).
   Write the equation in standard form.
   \[ (x - 3)^2 + (y - 4)^2 = 8 \]
(c) Is the point (4, 2) on the circle in Part (b)?

No: \((4 - 3)^2 + (2 - 4)^2 = 5 \neq 8\)

4. Let \(f(x) = \sqrt{x^2 - 3x - 4}\). Calculate the following. Simplify, if possible.
(a) \(f(4) = 0\)
(b) \(f(-2) = \sqrt{6}\)
(c) \(f(-x) = \sqrt{x^2 + 3x - 4}\)
(d) \(f(x + 2) = \sqrt{x^2 + x - 6}\)
(e) Find the domain of the function \(f\).
\((-\infty, -1] \cup [4, \infty)\)

5. Consider the function \(f(x) = x^2 - 9\).
(a) Find the \(y\)-intercept(s) of \(f\). Write each intercept as an ordered pair \((x,y)\). If there are no \(y\)-intercepts, write “NONE.”
\((0, -9)\)
(b) Find the \(x\)-intercept(s) of \(f\). Write each intercept as an ordered pair \((x,y)\). If there are no \(x\)-intercepts, write “NONE.”
\((3, 0), (-3, 0)\)

6. Consider the function \(g(x) = 3x^2 + 2x - 1\).
(a) Find the \(y\)-intercept(s) of \(f\). Write each intercept as an ordered pair \((x,y)\). If there are no \(y\)-intercepts, write “NONE.”
\((0, -1)\)
(b) Find the \(x\)-intercept(s) of \(f\). Write each intercept as an ordered pair \((x,y)\). If there are no \(x\)-intercepts, write “NONE.”
\((\frac{1}{3}, 0), (-1, 0)\)

7. Consider the function \(f(x) = x^2 + 4\).
(a) Find the \(y\)-intercept(s) of \(f\). Write each intercept as an ordered pair \((x,y)\). If there are no \(y\)-intercepts, write “NONE.”
\((0, 4)\)
(b) Find the \(x\)-intercept(s) of \(f\). Write each intercept as an ordered pair \((x,y)\). If there are no \(x\)-intercepts, write “NONE.”
NONE

8. Below is the graph of the function \(g\).
(a) State the domain and range of the function \(g\).
Domain: \((-\infty, \infty)\) Range: \([1, \infty)\)
(b) Give a one or two sentence explanation of why \( g \) is a function.

It passes the *vertical* line test.

(c) Does \( g \) have any of the three symmetries discussed in class? If so, which?

No, it does not posses \( x \)-axis, \( y \)-axis, nor origin symmetry.

(d) Sketch the graph of the equation \( y = -g(x - 3) + 2 \) on the same coordinate plane as \( y = g(x) \).

See the graph above.

9. Let \( f(x) = 3 - 5x^2 \) and \( g(x) = \sqrt{2 - x} \). Calculate the following.

(a) \( (f - g)(-2) = \)

\(-17 - 2 = -19\)

(b) \( (f \cdot g)(1) = \)

\(-2)(1) = -2\)

(c) \( \left(\frac{f}{g}\right)(x) = \)

\(3 - 5x^2 \over \sqrt{2 - x}\)

(d) \( (f \circ g)(x) = \)

\(3 - 5\left(\sqrt{2 - x}\right)^2 = -7 + 5x\)

(e) What is the domain of the function \( f \circ g \)?

\(\{x : x \leq 2\} = (-\infty, 2]\)

(f) \( (f \circ g)(3) = \)

Undefined: 3 is not in the domain of \( f \circ g \).
10. Let \( h(x) = \frac{x^3}{3x^2 - 4} \) and \( k(x) = 3x^2 - 4 \). Calculate the following.

(a) What is the domain of the function \( h \)?
\[
\{ x : x \neq \pm \frac{2}{\sqrt{3}} \} = (-\infty, -\frac{2}{\sqrt{3}}) \cup \left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \left(\frac{2}{\sqrt{3}}, \infty\right)
\]

(b) \((k + h)(2) = 8 + \frac{8}{8} = 9\)

(c) \((h \cdot k)(x) = \left(\frac{x^3}{3x^2 - 4}\right) \cdot (3x^2 - 4) = x^3\)

(d) What is the domain of the function \( h \cdot k \)?
\[
\{ x : x \neq \pm \frac{2}{\sqrt{3}} \} = (-\infty, -\frac{2}{\sqrt{3}}) \cup \left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \cup \left(\frac{2}{\sqrt{3}}, \infty\right)
\]

(e) \((h \cdot k)\left(\frac{2}{\sqrt{3}}\right) = \text{Undefined}\)

(f) \((h \circ k)\left(\frac{2}{\sqrt{3}}\right) = h\left(k\left(\frac{2}{\sqrt{3}}\right)\right) = h(0) = 0\)

(g) If \( f(x) = \sqrt{x^2 + 4} \), then what is the domain of the function \( h \circ f \)?
\[
3\left(\sqrt{x^2 + 4}\right)^2 - 4 = 3x^2 + 8 > 0, \quad \text{so the domain is } (-\infty, \infty)
\]

11. Let \( P(x) = x^3 - 2x^2 - x + 2 \).

(a) Explain why \( x - 2 \) is a factor of \( P(x) \).
\[
P(2) = 0
\]

(b) \( P(x) \) has three rational zeros. Factor \( P(x) \) completely.
\[
P(x) = (x - 2)(x - 1)(x + 1)
\]

12. Let \( Q(x) = 3x^8 - 12x^6 + 23x^5 + 15x^3 - 12x - 10 \).

(a) What is the degree of \( Q(x) \)?
\[
8
\]

(b) What is the greatest number of zeros that \( Q(x) \) can have? Why?

Eight. The number of zeros cannot be more than the degree.
(c) Use the Rational Zero (Root) Test to list all possible rational zeros of $Q(x)$.

$$\pm 1, \pm 2, \pm 5, \pm 10, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{5}{3}, \pm \frac{10}{3}$$

(d) What is the leading coefficient of $Q(x)$?

3

(e) What is the constant term of $Q(x)$?

$-10$

13. Let

$$P(x) = 2x^3 - x^2 - 3x - 1.$$  

(a) Use the Rational Zero (Root) Test to list all possible rational zeros of $P(x)$.

$$\pm 1, \pm \frac{1}{2}$$

(b) Find all the zeros of $P(x)$.

$$-\frac{1}{2}, \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2},$$

(c) Factor $P(x)$ completely.

$$P(x) = 2 \left( x + \frac{1}{2} \right) \left( x - \frac{1 + \sqrt{5}}{2} \right) \left( x - \frac{1 - \sqrt{5}}{2} \right)$$

14. Let $f(x) = \sqrt{2 - x}$.

(a) Sketch the graph of $y = f(x)$.

See graph below

(b) State the domain of $f(x)$.

$$(-\infty, 2]$$

(c) State the range of $f(x)$.

$$[0, \infty)$$

(d) The inverse of $f(x)$ exists. Give a one to two sentence explanation of why this is true.

$f$ passes the horizontal line test, so is one-to-one.

(e) Determine an expression for $f^{-1}(x)$.

$$f^{-1}(x) = 2 - x^2$$
15. Let

\[ R(x) = \frac{x^2 - 3x + 2}{x^2 + 3x + 2}. \]

(a) State the domain of \( R(x) \).

\( \{x : x \neq -1, -2\} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty) \)

(b) Find the \( x \)-intercept(s) of \( R(x) \). Write each intercept as an ordered pair, or write “none” if there are no \( x \)-intercepts.

\( (1, 0), \quad (2, 0) \)

(c) Find the \( y \)-intercept(s) of \( R(x) \). Write each intercept as an ordered pair, or write “none” if there are no \( y \)-intercepts.

\( (0, 1) \)

(d) Find the vertical asymptote(s) of \( R(x) \). Give the equation of each asymptote, or write “none” if there are no vertical asymptotes.

\( x = -1, \quad x = -2 \)

(e) Find the horizontal asymptote(s) of \( R(x) \). Give the equation of each asymptote, or write “none” if there are no horizontal asymptotes.

\( y = 1 \)

16. Let \( R(x) = \frac{x^2}{3x^4 + 5x^2 + 1} \). Find the horizontal asymptote(s) of \( R(x) \).

\( y = 0 \)
17. Let \( R(x) = \frac{x^2}{3x^2 + 5x + 1} \). Find the horizontal asymptote(s) of \( R(x) \).

\[ y = \frac{1}{3} \]

18. The graph of \( y = g(x) \) is below.

(a) Sketch the graph of \( y = g(x - 2) + 2 \) on the same axes.

This is not done here. Translate right 2 and up 2.

(b) If \( g(x) \) is a polynomial, what is the smallest possible degree of \( g(x) \)?

3

(c) If \( g(x) \) is a polynomial, what are the factors of \( g(x) \)?

\[ x + 3, \ x - 2 \]

(d) If \( g(x) \) is a polynomial, what might be its equation?

\[ g(x) = \frac{1}{6}(x + 3)^2(x - 2) \]

19. Let \( t = 20^\circ \). Give the exact value for each of the following questions.

(a) Find the measurement of \( t \) in radians. Simplify your result.

\( \pi/9 \)

(b) In a circle of radius 5, what is the length of an arc with angle of \( t \) radians \((20^\circ)\)?

\( 5\pi/9 \)
20. Let $t = 120^\circ$. Give the **exact** value for each of the following questions.

(a) Find the measurement of $t$ in radians. Simplify your result.

\[
\frac{2\pi}{3}
\]

(b) In a circle of radius 4, what is the area of a circular sector with angle of $t$ radians ($120^\circ$)?

\[
\frac{16\pi}{3}
\]

21. Suppose $\sin \theta = \frac{2}{3}$ where $0 < \theta < \frac{\pi}{2}$. Calculate the following.

(a) $\cos \theta = \frac{\sqrt{5}}{3}$

(b) $\tan \theta = \frac{2}{\sqrt{5}}$

(c) $\sec \theta = \frac{3}{\sqrt{5}}$

(d) $\csc \theta = \frac{3}{2}$

(e) $\cot \theta = \frac{\sqrt{5}}{2}$

22. Suppose $\sin \theta = \frac{1}{4}$ where $\frac{\pi}{2} < \theta < \pi$. Calculate the following.

(a) $\cos \theta = -\frac{\sqrt{15}}{4}$

(b) $\sin(-\theta) = -\sin \theta = -\frac{1}{4}$

(c) $\cos(-\theta) = \cos \theta = -\frac{\sqrt{15}}{4}$

(d) $\sin(\theta + 2\pi) = \sin \theta = \frac{1}{4}$

(e) $\sin(\theta + 6\pi) = \sin \theta = \frac{1}{4}$

(f) $\sin(\theta + \pi) = -\sin \theta = -\frac{1}{4}$
(g) \( \sin(\theta + \frac{\pi}{2}) = \cos \theta = -\frac{\sqrt{15}}{4} \)

23. Determine the exact value of the following using a trigonometric identity.

(a) \( \sin \left( \frac{7\pi}{12} \right) = \frac{\sqrt{2} + \sqrt{6}}{4} = \frac{\sqrt{2} + \sqrt{3}}{2} \)

(b) \( \cos \left( \frac{\pi}{12} \right) = \frac{\sqrt{6} + \sqrt{2}}{4} = \frac{\sqrt{2} + \sqrt{3}}{2} \)

(c) \( \sin \left( \frac{5\pi}{12} \right) = \frac{\sqrt{2} + \sqrt{3}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4} \)

(d) \( \cos \left( \frac{5\pi}{12} \right) = \frac{\sqrt{6} - \sqrt{2}}{4} = \frac{\sqrt{2} - \sqrt{3}}{2} \)

(e) \( \sin \left( -\frac{7\pi}{12} \right) = -\sin \left( \frac{7\pi}{12} \right) = -\frac{\sqrt{2} + \sqrt{6}}{4} = -\frac{\sqrt{2} + \sqrt{3}}{2} \)

(f) \( \cos \left( \frac{13\pi}{12} \right) = -\cos \left( \frac{\pi}{12} \right) = -\frac{\sqrt{6} + \sqrt{2}}{4} = -\frac{\sqrt{2} + \sqrt{3}}{2} \)

(g) \( \cos \left( \frac{25\pi}{12} \right) = \cos \left( \frac{\pi}{12} \right) = \frac{\sqrt{6} + \sqrt{2}}{4} = \frac{\sqrt{2} + \sqrt{3}}{2} \)

24. Consider the function

\[ f(t) = 3 \sin \left( 5t + \frac{\pi}{4} \right) \]

(a) Find the amplitude and period of \( f \).

\[ A = 3, \quad P = \frac{2\pi}{5} \]

(b) Write \( f \) as a cosine function.

\[ f(t) = 3 \cos \left( 5t - \frac{\pi}{4} \right) \]
25. Consider the function 

\[ g(t) = 2 \cos(\pi t) + 3. \]

(a) Find the amplitude and period of \( g \).

\[ A = 2, \quad P = 2 \]

(b) Write \( g \) as a sine function.

\[ g(t) = 2 \sin\left(\pi t + \frac{\pi}{2}\right) + 3 \]

26. Below is the graph of \( y = f(x) \). Give all answers as exact values.

(a) State the domain and range of \( f \).

\[ D : (-\infty, \infty), \quad R : [-2, 4] \]

(b) State the period and amplitude of \( f \).

\[ A = 3, \quad P = 4\pi \]

(c) Write an equation for \( f \).

\[ f(x) = 3 \sin\left(\frac{x}{2}\right) + 1 \]

27. Use a sum or difference formula to show the following:

(a) \( \cos(t + 2\pi) = \cos(t) \)

\[ \cos(t + 2\pi) = \cos(t) \cos(2\pi) - \sin(t) \sin(2\pi) = \cos(t)(1) - \sin(t)(0) = \cos(t) \]

(b) \( \sin(t + \pi/2) = \cos(t) \)

\[ \sin(t + \pi/2) = \sin(t) \cos(\pi/2) + \sin(2\pi) \cos(t) = \sin(t)(0) + (1) \cos(t) = \cos(t) \]
28. (a) Find two different values of \( t \) so that \( \sin(t) = \frac{\sqrt{3}}{2} \).

\[ t = \frac{\pi}{3}, \quad \frac{2\pi}{3}, \quad \frac{-4\pi}{3}, \quad \ldots \quad (\text{there are many such } t) \]

(b) Find the value of \( \arcsin\left(\frac{\sqrt{3}}{2}\right) \).

\[ t = \frac{\pi}{3} \quad (\text{there is only one such } t) \]

29. Find all values of \( x \) in the interval \([0, 2\pi]\) which make the following equation true:

\[ 2 \cos x - \sqrt{2} = 0 \]

\[ x = \frac{\pi}{4}, \quad \frac{7\pi}{4} \]

30. Calculate the following:

(a) \( \arccos(-1/2) = \)

(b) \( \arcsin(-1/2) = \)

(c) \( \arctan(\sqrt{3}) = \)

(d) \( \arcsin(3) = \quad \text{Undefined: 3 is not in the domain of arcsine.} \)

31. Calculate the following:

(a) \( \sin(\arccos(3/4)) = \)

(b) \( \tan(\arcsin(2/5)) = \)

(c) \( \cos(\arctan(3)) = \)

(d) \( \sin(\arctan(x)) = \)

\[ \frac{x}{\sqrt{1 + x^2}} \]
32. Use the figure below to answer the following questions. In the figure, $\alpha$ and $\beta$ are the angle measures.

\[ \begin{align*}
\alpha & = \frac{\pi}{6} \\
\beta & = \frac{\pi}{3}
\end{align*} \]

Calculate the following:

(a) $\sin \alpha = \frac{3}{4}$

(b) $\cos \alpha = \frac{\sqrt{7}}{4}$

(c) $\sin(2\alpha) = \frac{3\sqrt{7}}{8}$

(d) $\sin(\alpha/2) = \sqrt{\frac{4 - \sqrt{7}}{8}}$

(e) $C = \sqrt{7}$

(f) $\sin \beta = \sqrt{\frac{3}{7}}$

(g) $\cos \beta = \frac{2}{\sqrt{7}}$

(h) $\sin(\alpha + \beta) = \frac{3}{2\sqrt{7}} + \frac{\sqrt{3}}{4}$

(i) $\cos(\alpha + \beta) = \frac{1}{2} - \frac{3\sqrt{3}}{4\sqrt{7}}$
33. Let \( f(x) = 3^{x-2} + 1 \).
   
   (a) Sketch the graph of \( y = f(x) \). Label any 3 points and indicate any asymptotes.
   
   (b) What is the domain of \( f \)?
       \((-\infty, \infty)\)
   
   (c) What is the range of \( f \)?
       \([1, \infty)\)
   
   (d) Describe how you would sketch the graph of \( y = 2f(x) \).
       For every \( x \), multiply the corresponding \( y \) by 2.
   
   (e) Does the inverse of \( f \) exist? If so, calculate the formula for \( f^{-1} \).
       \[ f^{-1}(x) = \log_3(x) - 1 + 2 = \frac{\ln(x - 1)}{\ln 3} + 2 \]
   
   (f) Write \( f \) in terms of the natural base, \( e \).
       \[ f(x) = e^{(x-2)\ln 3} + 1 \]

34. Let \( g(x) = -\ln(x + 1) \).
   
   (a) Sketch the graph of \( y = g(x) \). Label any 3 points and indicate any asymptotes.
   
   (b) What is the domain of \( g \)?
       \((-1, \infty)\)
   
   (c) What is the range of \( g \)?
       \((-\infty, \infty)\)
   
   (d) Describe how you would sketch the graph of \( 2g(x) \).
       For every \( x \), multiply the corresponding \( y \) by 2.
   
   (e) Does the inverse of \( g \) exist? If so, calculate the formula for \( g^{-1} \).
       \[ g^{-1}(x) = e^{-x} - 1 \]

35. Rewrite each expression as a single logarithm.
   
   (a) \( 3\ln x - \frac{1}{2}\ln(x + 2) \)
       \[ \ln \left( \frac{x^3}{\sqrt{x + 2}} \right) \]
   
   (b) \( \ln 2 + 4\ln(x + 7) \)
       \[ \ln \left[ 2(x + 7)^4 \right] \]
   
   (c) \( \log_3 x + 2\log_3(x - 1) + 3\log_3(x + 1) \)
       \[ \log_3 \left[ x(x - 1)^2(x + 1)^3 \right] \]
36. Solve each equation for $x$.
   (a) $\ln x + \ln(x - 1) = \ln 2$
       $x = 2, \ x = -1$
   (b) $\log_4(10 - x) = 2$
       $x = -6$
   (c) $2 \ln x = \ln 2 + \ln(x + 4)$
       $x = 4, \ x = -2$