

08/29/08

1.2 System of Linear Equations

KEY Concepts	}	Two Eqns
Applications		Two unknowns
Q. & Examples		Decision Making Supply & Demand Equi

1.2 Systems of Linear Equations

APPLICATION

Cost, Revenue, and Profit Models

In Example 3 in the last section we found the cost and revenue equations in the dress-manufacturing industry. Let x be the number of dresses made and sold. Recall that cost and revenue functions were found to be $C(x) = 25x + 80,000$ and $R(x) = 75x$. Find the point at which the profit is zero. See Example 2 for the answer.

We now begin to look at systems of linear equations in many unknowns. In this section we first consider systems of two linear equations in two unknowns. We will see that solutions of such a system have a variety of applications.

✧ Two Linear Equations in Two Unknowns

In this section we will encounter applications that have a unique solution to a system of two linear equations in two unknowns. For example, consider two lines,

$$L_1 : y = m_1x + b_1$$

$$L_2 : y = m_2x + b_2$$

If these two linear equations are not parallel ($m_1 \neq m_2$), then the lines must intersect at a unique point, say (x_0, y_0) as shown in Figure 1.11. This means that (x_0, y_0) is a solution to the two linear equations and must satisfy both of the equations

$$y_0 = m_1x_0 + b_1$$

$$y_0 = m_2x_0 + b_2$$

EXAMPLE 1 Intersection of Two Lines Find the solution (intersection) of the two lines.

$$L_1 : y = 7x - 3$$

$$L_2 : y = -4x + 9$$

Solution Set the two lines equal to each other, $L_1 = L_2$,

$$y_0 = y_0$$

$$7x_0 - 3 = 4x_0 + 9$$

$$11x_0 = 12$$

$$x_0 = \frac{12}{11}$$

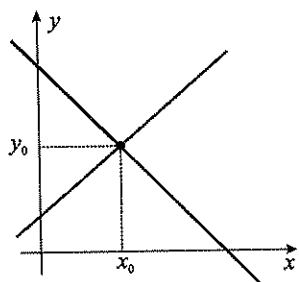


Figure 1.11

$$y = 7x - 3$$

$$m_1 = 7$$

$$y = -4x + 9$$

$$m_2 = -4$$

$$7x - 3 = -4x + 9$$

$$7x + 4x = 9 + 3$$

$$11x = 12$$

$$x = \frac{12}{11}$$

$$y = 7\left(\frac{12}{11}\right) - 3$$

$$= \frac{84}{11} - 3 = \frac{84 - 33}{11}$$

$$= \frac{51}{11}$$

Technology Option

Example 1 is solved using a graphing calculator in Technology Note 1 on page 26

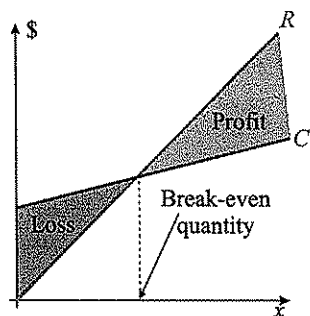


Figure 1.12

Technology Option

Example 2 is solved using a graphing calculator in Technology Note 2 on page 27

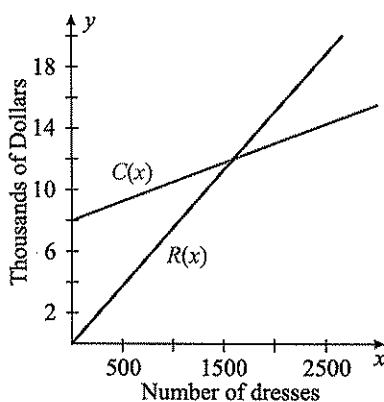


Figure 1.13

To find the value of y_0 , substitute the x_0 value into either equation,

$$y_0 = 7 \left(\frac{12}{11} \right) - 3 = \frac{51}{11}$$

$$y_0 = 4 \left(\frac{12}{11} \right) + 9 = \frac{51}{11}$$

So, the solution to this system is the intersection point $(12/11, 51/11)$. ♦

✧ Decision Analysis

In the last section we considered linear mathematical models of cost, revenue, and profit for a firm. In Figure 1.12 we see the graphs of two typical cost and revenue functions. We can see in this figure that for smaller values of x , the cost line is above the revenue line and therefore the profit P is negative. Thus the firm has losses. As x becomes larger, the revenue line becomes above the cost line and therefore the profit becomes positive. The value of x at which the profit is zero is called the break-even quantity. Geometrically, this is the point of intersection of the cost line and the revenue line. Mathematically, this requires us to solve the equations $y = C(x)$ and $y = R(x)$ simultaneously.

EXAMPLE 2 Finding the Break-Even Quantity In Example 3 in the last section we found the cost and revenue equations in a dress-manufacturing firm. Let x be the number of dresses manufactured and sold and let the cost and revenue be given in dollars. Then recall that the cost and revenue equations were found to be $C(x) = 25x + 80,000$ and $R(x) = 75x$. Find the break-even quantity.

Solution To find the break-even quantity, we need to solve the equations $y = C(x)$ and $y = R(x)$ simultaneously. To do this we set $R(x) = C(x)$. Doing this we have

$$\begin{aligned} R(x) &= C(x) \\ 75x &= 25x + 80,000 \\ 50x &= 80,000 \\ x &= 1600 \end{aligned}$$

Thus, the firm needs to produce and sell 1600 dresses to break even (i.e., for profits to be zero). See Figure 1.13. ♦

REMARK: Notice that $R(1600) = 120,000 = C(1600)$ so it costs the company \$120,000 to make the dresses and they bring in \$120,000 in revenue when the dresses are all sold.

In the following example we consider the total energy consumed by automobile fenders using two different materials. We need to decide how many miles carrying the fenders result in the same energy consumption and which type of fender will consume the least amount of energy for large numbers of miles.

EXAMPLE 3 Break-Even Analysis Saur and colleagues did a careful study of the amount of energy consumed by each type of automobile fender using various materials. The total energy was the sum of the energy needed for production plus the energy consumed by the vehicle used in carrying the fenders. If x is the miles traveled, then the total energy consumption equations for steel and rubber-modified polypropylene (RMP) were as follows:

$$\text{Steel: } E = 225 + 0.012x$$

$$\text{RPM: } E = 285 + 0.007x$$

Graph these equations, and find the number of miles for which the total energy consumed is the same for both fenders. Which material uses the least energy for 15,000 miles?

Source: Saur, Fava, and Spatari 2000

Solution The total energy using steel is $E_1(x) = 225 + 0.012x$ and for RMP is $E_2(x) = 285 + 0.007x$. The graphs of these two linear energy functions are shown in Figure 1.14. We note that the graphs intersect. To find this intersection we set $E_1(x) = E_2(x)$ and obtain

$$E_1(x) = E_2(x)$$

$$225 + 0.012x = 285 + 0.007x$$

$$0.005x = 60$$

$$x = 12,000$$

So, 12,000 miles results in the total energy used by both materials being the same.

Setting $x = 0$ gives the energy used in production, and we note that steel uses less energy to produce these fenders than does RPM. However, since steel is heavier than RPM, we suspect that carrying steel fenders might require more total energy when the number of pair of fenders is large. Indeed, we see in Figure 1.14 that the graph corresponding to steel is above that of RPM when $x > 12,000$. Checking this for $x = 15,000$, we have

$$\text{steel: } E_1(x) = 225 + 0.012x$$

$$\begin{aligned} E_1(15,000) &= 225 + 0.012(15,000) \\ &= 405 \end{aligned}$$

$$\text{RPM: } E_2(x) = 285 + 0.007x$$

$$\begin{aligned} E_2(15,000) &= 285 + 0.007(15,000) \\ &= 390 \end{aligned}$$

So for traveling 15,000 miles, the total energy used by RPM is less than that for steel. ♦

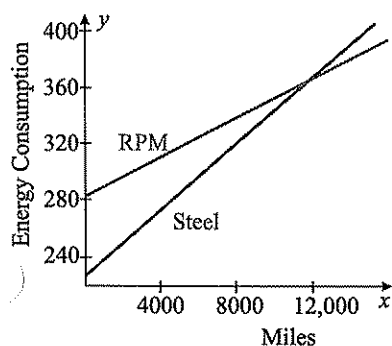


Figure 1.14

✧ Supply and Demand Equilibrium

The best-known law of economics is the law of supply and demand. Figure 1.15 shows a demand equation and a supply equation that intersect.

$$\text{Steel } E = 225 + 0.012X$$

$$\text{RPM } E = 285 + 0.007X$$

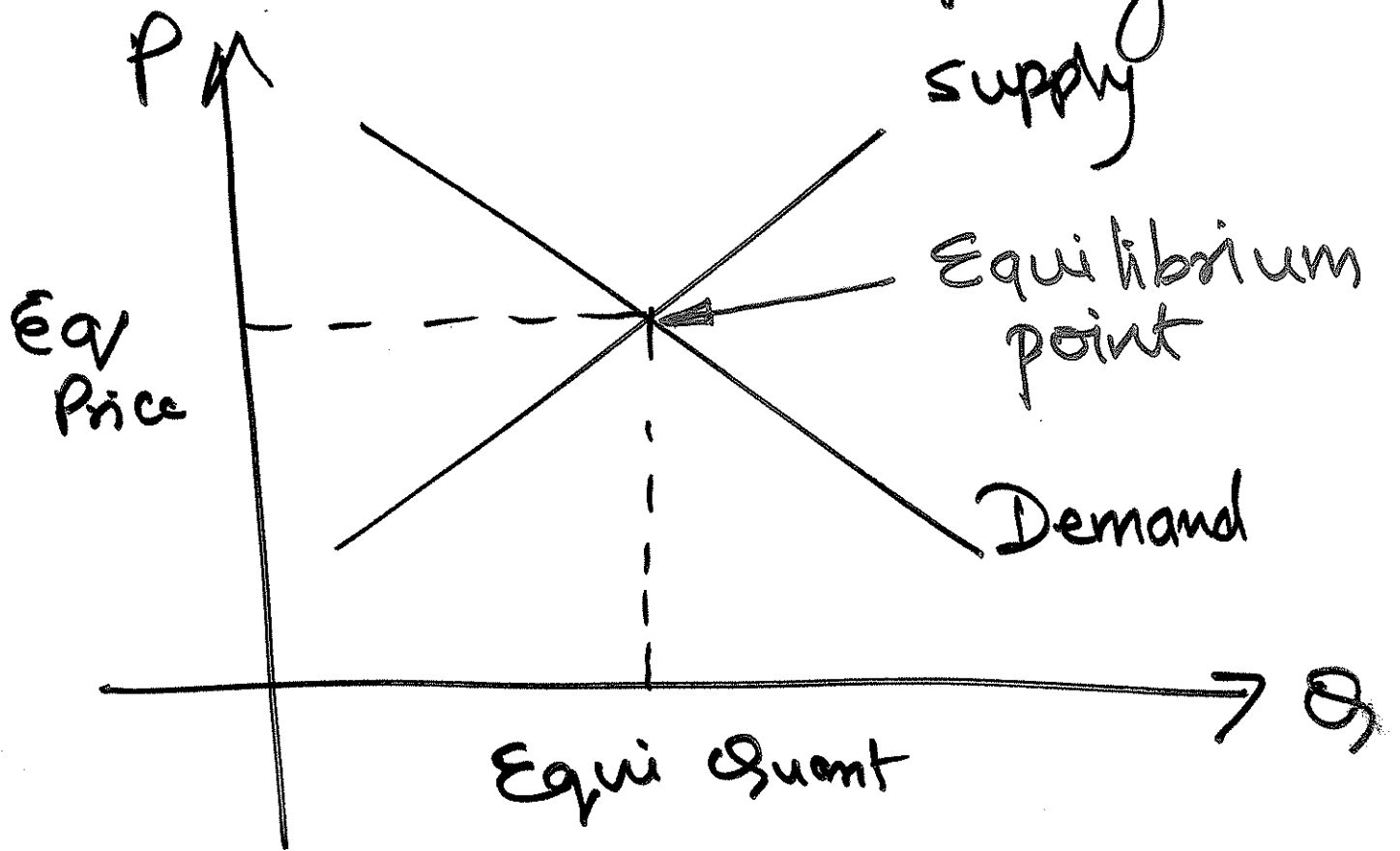
$$225 + 0.012X = 285 + 0.007X$$

$$X = 12,000$$

$$\begin{aligned} 225 + 0.012(15,000) \\ = 405 \end{aligned}$$

$$\begin{aligned} 285 + 0.007(15,000) \\ = 390 \end{aligned}$$

Demand & Supply



The point of intersection or the point at which supply equals demand is called Equilibrium Pt

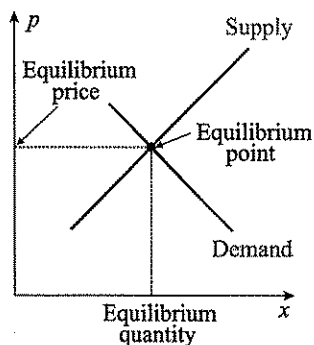


Figure 1.15

The point of intersection, or the point at which supply equals demand, is called the equilibrium point. The x -coordinate of the equilibrium point is called the equilibrium quantity, x_0 , and the p -coordinate is called the equilibrium price, p_0 . In other words, at a price p_0 , the consumer is willing to buy x_0 items and the producer is willing to supply x_0 items.

EXAMPLE 4 Finding the Equilibrium Point Tauer determined demand and supply curves for milk in this country. If x is billions of pounds of milk and p is in dollars per hundred pounds, he found that the demand function for milk was $p = D(x) = 56 - 0.3x$ and the supply function was $p = S(x) = 0.1x$. Graph the demand and supply equations. Find the equilibrium point.

Source: Tauer 1994

Solution The demand equation $p = D(x) = 56 - 0.3x$ is a line with negative slope -0.3 and y -intercept 56 and is graphed in Figure 1.16. The supply equation $p = S(x) = 0.1x$ is a line with positive slope 0.1 with y -intercept 0. This is also graphed in Figure 1.16.

To find the point of intersection of the demand curve and the supply curve, set $S(x) = D(x)$ and solve:

$$\begin{aligned} S(x) &= D(x) \\ 0.1x &= 56 - 0.3x \\ 0.4x &= 56 \\ x &= 140 \end{aligned}$$

Then since $p(x) = 0.1x$,

$$p(140) = 0.1(140) = 14$$

We then see that the equilibrium point is $(x, p) = (140, 14)$. That is, 140 billions pounds of milk at \$14 per hundred pounds of milk. ♦

EXAMPLE 5 Supply and Demand Refer to Example 4. What will consumers and suppliers do if the price is $p_1 = 25$ shown in Figure 1.17a? What if the price is $p_2 = 5$ as shown in Figure 1.17b?

Solution If the price is at $p_1 = 25$ shown in Figure 1.17a, then let the supply of milk be denoted by x_S . Let us find x_S .

$$\begin{aligned} p &= S(x_S) \\ 25 &= 0.1x_S \\ x_S &= 250 \end{aligned}$$

That is, 250 billion pounds of milk will be supplied. Keeping the same price of $p_1 = 25$ shown in Figure 1.17a, then let the demand of milk be denoted by x_D . Let us find x_D . Then

$$\begin{aligned} p &= D(x_D) \\ 25 &= 56 - 0.3x_D \\ x_D &\approx 103 \end{aligned}$$

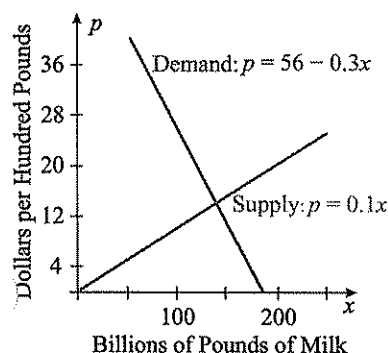


Figure 1.16

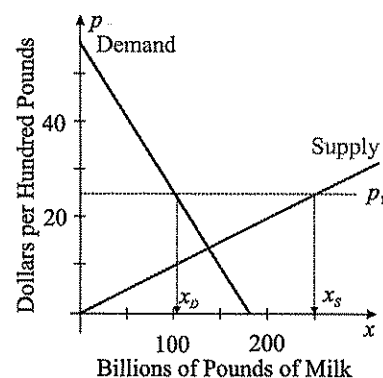


Figure 1.17a

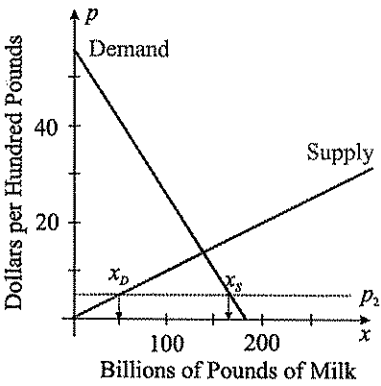


Figure 1.17b

So only 103 billions of pounds of milk are demanded by consumers. There will be a surplus of $250 - 103 = 147$ billions of pounds of milk. To work off the surplus, the price should fall toward the equilibrium price of $p_0 = 14$.

If the price is at $p_2 = 5$ shown in Figure 1.17b, then let the supply of milk be denoted by x_S . Let us find x_S . Then

$$\begin{aligned} p &= S(x_S) \\ 5 &= 0.1x_S \\ x_S &= 50 \end{aligned}$$

That is, 50 billion pounds of milk will be supplied. Keeping the same price of $p_2 = 5$ shown in Figure 1.17b, then let the demand of milk be denoted by x_D . Let us find x_D . Then

$$\begin{aligned} p &= D(x_D) \\ 5 &= 56 - 0.3x_D \\ x_D &\approx 170 \end{aligned}$$

So 170 billions of pounds of milk are demanded by consumers. There will be a shortage of $170 - 50 = 120$ billions of pounds of milk, and the price should rise toward the equilibrium price. ♦

CONNECTION
Demand for Steel
Outpaces Supply

In early 2002 President George W. Bush imposed steep tariffs on imported steel to protect domestic steel producers. As a result millions of tons of imported steel were locked out of the country. Domestic steelmakers announced on March 27, 2002, that they had been forced to ration steel to their customers and boost prices because demand has outpaced supply.
Source: Wall Street Journal

✧ **Enrichment: Decision Analysis Complications**

In the following example we look at the cost of manufacturing automobile fenders using two different materials. We determine the number of pairs of fenders that will be produced by using the same cost. However, we must keep in mind that we do not produce fractional numbers of fenders, but rather only whole numbers. For example, we can produce one or two pairs of fenders, but not 1.43 pairs.

EXAMPLE 6 Decision Analysis for Manufacturing Fenders Saur and colleagues did a careful study of the cost of manufacturing automobile fenders using two different materials: steel and a rubber-modified polypropylene blend (RMP). The following table gives the fixed and variable costs of manufacturing each pair of fenders.

Variable and Fixed Costs of Pairs of Fenders		
Costs	Steel	RMP
Variable	\$5.26	\$13.19
Fixed	\$260,000	\$95,000

(30) page 29

$$p = 55.9867 - 0.3249x$$

demand

$$p = 0.07958x$$

Supply

$$55.9867 - 0.3249x = 0.07958x$$

$$55.9867 = 0.40448x$$

$$x = \frac{55.9867}{0.40448} = \underline{\underline{138.42}}$$

$$p = 0.07958(138.42) = 11.0152$$