

Chapter 1

System of Linear Equations and Models

So

- Key Concepts
 - Applications
 - H.W problems
- } 1.1
Mathematical
Models

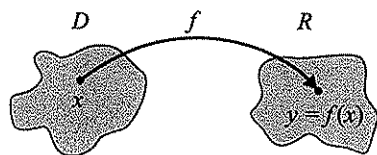
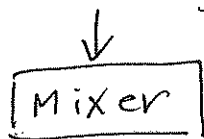


Figure 1.1
A function as a mapping

Food Processor.
Juice maker
as function

Put orange



orange juice

Definition of a Function

A function f from D to R is a rule that assigns to each element x in D one and only one element $y = f(x)$ in R . See Figure 1.1.

The set D in the definition is called the domain of f . We might think of the domain as the set of inputs. We then can think of the values $f(x)$ as outputs. The set of outputs, R is called the range of f .

Another helpful way to think of a function is shown in Figure 1.2. Here the function f accepts the input x from the conveyor belt, operates on x , and outputs (assigns) the new value $f(x)$.

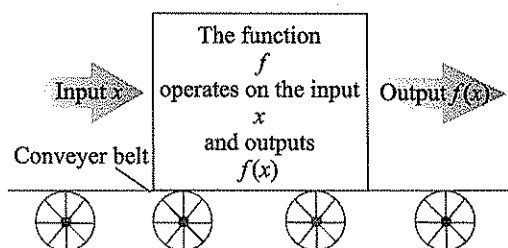


Figure 1.2
A function as a process

The letter representing elements in the domain is called the independent variable and the letter representing the elements in the range is called the dependent variable. Thus, if $y = f(x)$, x is the independent variable, and y is the dependent variable, since the value of y depends on x . In the equation $d = 40t$, we can write $d = f(t) = 40t$ with t as the independent variable. The dependent variable is d , since the distance depends on the spent time t traveling. We are free to set the independent variable t equal to any number of values in the domain. The domain for this function is $t \geq 0$ since only nonnegative time is allowed.

REMARK: The domain in an application problem will always be those values that are allowed for the independent variable in the particular application. This often means that we are restricted to nonnegative values or perhaps we will be limited to the case of whole numbers only, as in the next example.

EXAMPLE 1 Steak Specials A restaurant serves a steak special for \$12. Write a function that models the amount of revenue made from selling these specials. How much revenue will 10 steak specials earn?

Solution We first need to decide if the independent variable is the price of the steak specials, the number of specials sold, or the amount of revenue earned. Since the price is fixed at \$12 per special and revenue depends on the number of specials sold, we choose the independent variable, x , to be the number of specials sold and the dependent variable, $R = f(x)$ to be the amount of revenue. Our rule will be $R = f(x) = 12x$, where x is the number of steak specials sold and R is the revenue from selling these

Technology Option

Example 1 is solved using a graphing calculator in Technology Note 1 on page 12.

specials in dollars. Note that x must be a whole number, so the domain is $x = 0, 1, 2, 3, \dots$. To determine the revenue made on selling 10 steak specials, plug $x = 10$ into the model:

$$R = f(10) = 12(10) = 120$$

So the revenue is \$120. ♦

Recall that lines satisfy the equation $y = mx + b$. Actually, we can view this as a function. We can set $y = f(x) = mx + b$. Given any number x , $f(x)$ is obtained by multiplying x by m and adding b . More specifically, we call the function $y = f(x) = mx + b$ a linear function.

Definition of Linear Function

A linear function f is any function of the form,

$$y = f(x) = mx + b$$

where m and b are constants.

EXAMPLE 2 Linear Functions Which of the following functions are linear?

a. $y = -0.5x + 12$

b. $5y - 2x = 10$

c. $y = \frac{1}{x} + 2$

d. $y = x^2$

Solution

a. This is a linear function. The slope is $m = -0.5$ and the y -intercept is $b = 12$.

b. Rewrite this function first as,

$$5y - 2x = 10$$

$$5y = 2x + 10$$

$$y = (2/5)x + 2$$

known as
standard form

Now we see it is a linear function with $m = 2/5$ and $b = 2$.

c. This is not a linear function. Rewrite $1/x$ as x^{-1} and this shows that we do not have a term mx and so this is not a linear function.

d. x is raised to the second power and so this is not a linear function. ♦

Technology Option

You can graph the functions on a calculator to verify your results. Linear functions will be a straight line in any window.

X^0 — zeroth order degree=0

X^1 — 1st order — linear

X^2 — 2nd " — Quadratic

X^3 — 3rd Order Cubic

$$5y - 2x = 10$$

$$y = mx + b$$

$$5y - 2x = 10$$

$$5y - \cancel{2x} + \cancel{2x} = 10 + 2x$$

$$5y = 10 + 2x$$

$$y = \frac{10}{5} + \frac{2}{5}x$$

$$y = 2 + \frac{2}{5}x$$

$$y = \left(\frac{2}{5}\right)x + 2$$

✧ Mathematical Modeling

When we use mathematical modeling we are attempting to describe some part of the real world in mathematical terms, just as we have done for the distance traveled and the revenue from selling meals. There are three steps in mathematical modeling: formulation, mathematical manipulation, and evaluation.



Formulation

First, on the basis of observations, we must state a question or formulate a hypothesis. If the question or hypothesis is too vague, we need to make it precise. If it is too ambitious, we need to restrict it or subdivide it into manageable parts. Second, we need to identify important factors. We must decide which quantities and relationships are important to answer the question and which can be ignored. We then need to formulate a mathematical description. For example, each important quantity should be represented by a variable. Each relationship should be represented by an equation, inequality, or other mathematical construct. If we obtain a function, say, $y = f(x)$, we must carefully identify the input variable x and the output variable y and the units for each. We should also indicate the interval of values of the input variable for which the model is justified.



Mathematical Manipulation

After the mathematical formulation, we then need to do some mathematical manipulation to obtain the answer to our original question. We might need to do a calculation, solve an equation, or prove a theorem. Sometimes the mathematical formulation gives us a mathematical problem that is impossible to solve. In such a case, we will need to reformulate the question in a less ambitious manner.



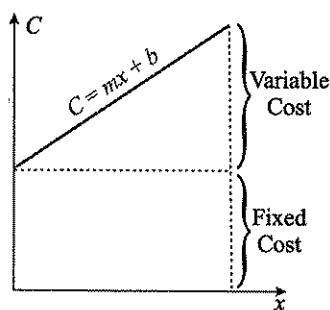
Evaluation

Naturally, we need to check the answers given by the model with real data. We normally expect the mathematical model to describe only a very limited aspect of the world and to give only approximate answers. If the answers are wrong or not accurate enough for our purposes, then we will need to identify the sources of the model's shortcomings. Perhaps we need to change the model entirely, or perhaps we need to just make some refinements. In any case, this requires a new mathematical manipulation and evaluation. Thus, modeling often involves repeating the three steps of formulation, mathematical manipulation, and evaluation.

We will next create linear mathematical models by finding equations that relate cost, revenue, and profits of a manufacturing firm to the number of units produced and sold.

✧ Cost, Revenue, and Profit

Any manufacturing firm has two types of costs: fixed and variable. Fixed costs are those that do not depend on the amount of production. These costs include real estate taxes, interest on loans, some management salaries, certain minimal maintenance, and protection of plant and equipment. Variable costs depend on the amount of production. They include the cost of

**Figure 1.3**

A linear cost function

HW.

1,

material and labor. Total cost, or simply cost, is the sum of fixed and variable costs:

$$\text{cost} = \text{variable cost} + \text{fixed cost}$$

Let x denote the number of units of a given product or commodity produced by a firm. (Notice that we must have $x \geq 0$.) The units could be bales of cotton, tons of fertilizer, or number of automobiles. In the linear cost model we assume that the cost m of manufacturing one unit is the same no matter how many units are produced. Thus, the variable cost is the number of units produced times the cost of each unit:

$$\begin{aligned} \text{variable cost} &= (\text{cost per unit}) \times (\text{number of units produced}) \\ &= mx \end{aligned}$$

If b is the fixed cost and $C(x)$ is the cost, then we have the following:

$$\begin{aligned} C(x) &= \text{cost} \\ &= (\text{variable cost}) + (\text{fixed cost}) \\ &= mx + b \end{aligned}$$

Notice that we must have $C(x) \geq 0$. In the graph shown in Figure 1.3, we see that the y -intercept is the fixed cost and the slope is the cost per item.

CONNECTION**What Are Costs?**

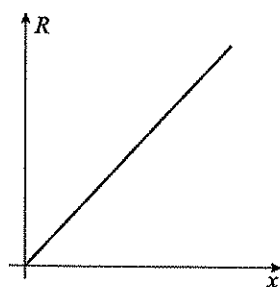
Isn't it obvious what the costs to a firm are? Apparently not. On July 15, 2002, Coca-Cola Company announced that it would begin treating stock-option compensation as a cost, thereby lowering earnings. If all companies in the Standard and Poor's 500 stock index were to do the same, the earnings for this index would drop by 23%.

Source: The Wall Street Journal, July 16, 2002

In the linear revenue model we assume that the price p of a unit sold by a firm is the same no matter how many units are sold. (This is a reasonable assumption if the number of units sold by the firm is small in comparison to the total number sold by the entire industry.) Revenue is always the price per unit times the number of units sold. Let x be the number of units sold. For convenience, we always assume that the number of units sold equals the number of units produced. Then, if we denote the revenue by $R(x)$,

$$\begin{aligned} R(x) &= \text{revenue} \\ &= (\text{price per unit}) \times (\text{number sold}) \\ &= px \end{aligned}$$

Since $p > 0$, we must have $R(x) \geq 0$. Notice in Figure 1.4 that the straight line goes through $(0, 0)$ because nothing sold results in no revenue. The slope is the price per unit.

**Figure 1.4**

A linear revenue function

HW 2

CONNECTION

What Are Revenues?

The accounting practices of many telecommunications companies, such as Cisco and Lucent, have been criticized for what the companies consider revenues. In particular, these companies have loaned money to other companies, which then use the proceeds of the loan to buy telecommunications equipment from Cisco and Lucent. Cisco and Lucent then book these sales as “revenue.” But is this revenue?

Regardless of whether our models of cost and revenue are linear or not, profit P is always revenue less cost. Thus

$$\begin{aligned} P &= \text{profit} \\ &= (\text{revenue}) - (\text{cost}) \\ &= R - C \end{aligned}$$

Recall that both cost $C(x)$ and revenue $R(x)$ must be nonnegative functions. However, the profit $P(x)$ can be positive or negative. Negative profits are called losses.

Let's now determine the cost, revenue, and profit equations for a dress-manufacturing firm.

EXAMPLE 3 Cost, Revenue, and Profit Equations A firm has weekly fixed costs of \$80,000 associated with the manufacture of dresses that cost \$25 per dress to produce. The firm sells all the dresses it produces at \$75 per dress.

- Find the cost, revenue, and profit equations if x is the number of dresses produced per week.
- Make a table of values for cost, revenue, and profit for production levels of 1000, 1500, and 2000 dresses and discuss what the table means.

Solution

- The fixed cost is \$80,000 and the variable cost is $25x$. So

$$\begin{aligned} C &= (\text{variable cost}) + (\text{fixed cost}) \\ &= mx + b \\ &= 25x + 80,000 \end{aligned}$$

See Figure 1.5a. Notice that $x \geq 0$ and $C(x) \geq 0$.

- The revenue is just the price \$75 that each dress is sold for multiplied by the number x of dresses sold. So

$$\begin{aligned} R &= (\text{price per dress}) \times (\text{number sold}) \\ &= px \\ &= 75x \end{aligned}$$

See Figure 1.5b. Notice that $x \geq 0$ and $R(x) \geq 0$. Also notice that if there are no sales, then there is no revenue, that is, $R(0) = 0$.

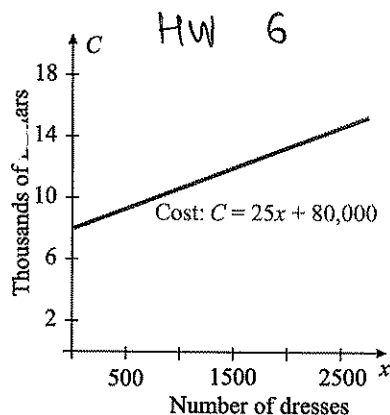


Figure 1.5a

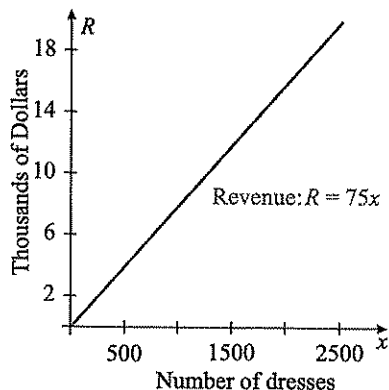


Figure 1.5b

$$\textcircled{2} \text{ cost/item} = 6\$$$

$$\text{fixed cost} = 14,000 \$$$

$$6x + 14,000$$

$\textcircled{4}$

R - revenue

X - no items

$$0.10 \$$$

$$R(x) = 0.1x$$

$$P = 50x - 80000$$

$$x = 1600$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

(x, p)
↑
are feet
p - price

$$(y - y_1) = m(x - x_1)$$

point - slope form

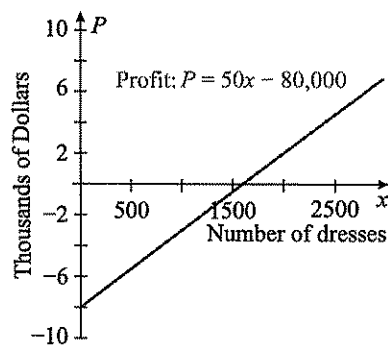


Figure 1.5c

Profit is always revenue less cost. So

$$\begin{aligned}
 P &= (\text{revenue}) - (\text{cost}) \\
 &= R - C \\
 &= (75x) - (25x + 80,000) \\
 &= 50x - 80,000
 \end{aligned}$$

See Figure 1.5c. Notice in Figure 1.5c that profits can be negative.

c. When 1000 dresses are produced and sold $x = 1000$ so we have

$$C(1000) = 25(1000) + 80,000 = 105,000$$

$$R(1000) = 75(1000) = 75,000$$

$$P(1000) = 75,000 - 105,000 = -30,000$$

Thus, if 1000 dresses are produced and sold, the cost is \$105,000, the revenue is \$75,000, and there is a negative profit or loss of \$30,000. Doing the same for 1500 and 2000 dresses, we have the results shown in Table 1.1.

Number of Dresses Made and Sold	1000	1500	2000
Cost in dollars	105,000	117,500	130,000
Revenue in dollars	75,000	112,500	150,000
Profit (or loss) in dollars	-30,000	-5,000	20,000

Table 1.1

We can see in Figure 1.5c or in Table 1.1, that for smaller values of x , $P(x)$ is negative; that is, the firm has losses as their costs are greater than their revenue. For larger values of x , $P(x)$ turns positive and the firm has (positive) profits. ♦

✧ Supply and Demand

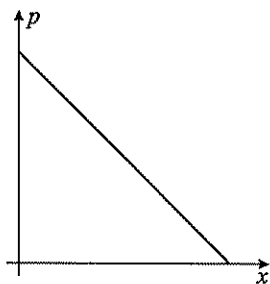
In the previous discussion we assumed that the number of units produced and sold by the given firm was small in comparison to the number sold by the industry. Under this assumption it was reasonable to conclude that the price, p , was constant and did not vary with the number x sold. But if the number of units sold by the firm represented a large percentage of the number sold by the entire industry, then trying to sell significantly more units could only be accomplished by lowering the price of each unit. Since we just stated that the price affects the number sold, you would expect the price to be the independent variable and thus graphed on the horizontal axis. However, by custom, the price is graphed on the vertical axis and the quantity x on the horizontal axis. This convention was started by English economist Alfred Marshall (1842–1924) in his important book, *Principles of Economics*. We will abide by this custom in this text.

For most items the relationship between quantity and price is a decreasing function (there are some exceptions to this rule, such as certain luxury goods, medical care, and higher education, to name a few). That is, for the

HISTORICAL NOTE

Adam Smith (1723–1790)

Adam Smith was a Scottish political economist. His *Inquiry into the Nature and Causes of the Wealth of Nations* was one of the earliest attempts to study the development of industry and commerce in Europe. That work helped to create the modern academic discipline of economics. In the Western world, it is arguably the most influential book on the subject ever published.

**Figure 1.6**

A linear demand function

number of items to be sold to increase, the price must decrease. We assume now for mathematical convenience that this relationship is linear. Then the graph of this equation is a straight line that slopes downward as shown in Figure 1.6.

We assume that x is the number of units produced and sold by the entire industry during a given time period and that $p = D(x) = -cx + d$, $c > 0$, is the price of one unit if x units are sold; that is, $p = -cx + d$ is the price of the x^{th} unit sold. We call $p = D(x)$ the demand equation and the graph the demand curve.

Estimating the demand equation is a fundamental problem for the management of any company or business. In the next example we consider the situation when just two data points are available and the demand equation is assumed to be linear.

EXAMPLE 4 Finding the Demand Equation Timmins estimated the municipal water demand in Delano, California. He estimated the demand x , measured in acre-feet (the volume of water needed to cover one acre of ground at a depth of one foot), with price p per acre-foot. He indicated two points on the demand curve, $(x, p) = (1500, 230)$ and $(x, p) = (5100, 50)$. Use this data to estimate the demand curve using a linear model. Estimate the price when the demand is 3000 acre-feet.

Source: Timmins 2002

Solution Figure 1.7 shows the two points $(x, p) = (1500, 230)$ and $(x, p) = (5100, 50)$ that lie on the demand curve. We are assuming that the demand curve is a straight line. The slope of the line is

$$m = \frac{50 - 230}{5100 - 1500} = -0.05$$

Now using the point-slope equation for a line with $(1500, 230)$ as the point on the line, we have

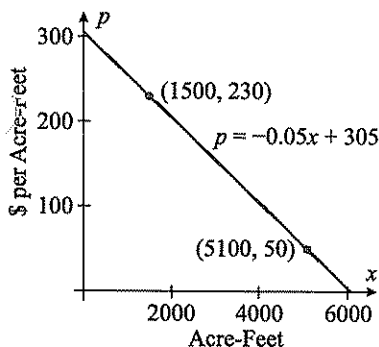
$$\begin{aligned} p - 230 &= m(x - 1500) \\ &= -0.05(x - 1500) \\ p &= -0.05x + 75 + 230 \\ &= -0.05x + 305 \end{aligned}$$

When demand is 3000 acre-feet, then $x = 3000$, and

$$p = -0.05(3000) + 305 = 155$$

or \$155 per acre-foot. Thus, according to this model, if 3000 acre-feet is demanded, the price of each acre-foot will be \$155. ♦

HW
3

**Figure 1.7**

CONNECTION Demand for Apartments

The figure shows that during the minor recession of 2001, vacancy rates for apartments increased, that is, the demand for apartments decreased. Also notice from the figure that as demand for apartments decreased, rents also decreased. For example, in San Francisco's South Beach area, a two-bedroom apartment that had rented for \$3000 a month two years before saw the rent drop to \$2100 a month.

Source: Wall Street Journal, April 11, 2002

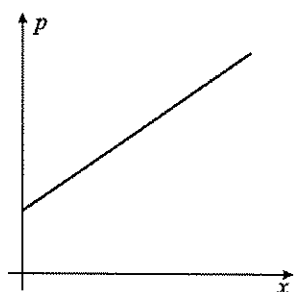
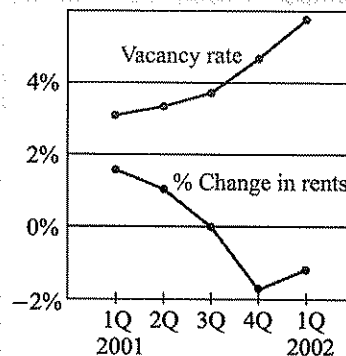


Figure 1.8
A supply equation

HW
4

EXAMPLE 5 Finding the Supply Equation Antle and Capalbo estimated a spring wheat supply curve. Use a mathematical model to determine a linear curve using their estimates that the supply of spring wheat will be 50 million bushels at a price of \$2.90 per bushel and 100 million bushels at a price of \$4.00 per bushel. Estimate the price when 80 million bushels is supplied.

Source: Antle and Capalbo 2001

Solution Let x be in millions of bushels of wheat. We are then given two points on the linear supply curve, $(x, p) = (50, 2.9)$ and $(x, p) = (100, 4)$. The slope is

$$m = \frac{4 - 2.9}{100 - 50} = 0.022$$

The equation is then given by

$$p - 2.9 = 0.022(x - 50)$$

or $p = 0.022x + 1.8$. See Figure 1.9 and note that the line rises.

When supply is 80 million bushels, $x = 80$, and we have

$$p = 0.022(80) + 1.8 = 3.56$$

This gives a price of \$3.56 per bushel. ♦

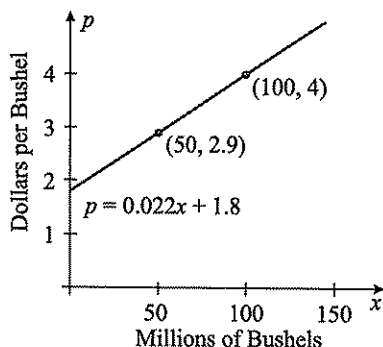


Figure 1.9

CONNECTION Supply of Cotton

On May 2, 2002, the U.S. House of Representatives passed a farm bill that promises billions of dollars in subsidies to cotton farmers. With the prospect of a greater supply of cotton, cotton prices dropped 1.36 cents to 33.76 cents per pound.

Source: The Wall Street Journal, May 3, 2002.

✧ Straight-Line Depreciation

H.W. 7

Many assets, such as machines or buildings, have a finite useful life and furthermore depreciate in value from year to year. For purposes of determining profits and taxes, various methods of depreciation can be used. In straight-line depreciation we assume that the value V of the asset is given by a linear equation in time t , say, $V = mt + b$. The slope m must be negative since the value of the asset decreases over time. The y -intercept is the initial value of the item and the slope gives the rate of depreciation (how much the item decreases in value per time period).

EXAMPLE 6 Straight-Line Depreciation A company has purchased a new grinding machine for \$100,000 with a useful life of 10 years, after which it is assumed that the scrap value of the machine is \$5000. Use straight-line depreciation to write an equation for the value V of the machine, where t is measured in years. What will be the value of the machine after the first year? After the second year? After the ninth year? What is the rate of depreciation?

Solution We assume that $V = mt + b$, where m is the slope and b is the V -intercept. We then must find both m and b . We are told that the machine is initially worth \$100,000, that is, when $t = 0$, $V = 100,000$. Thus, the point $(0, 100,000)$ is on the line, and 100,000 is the V -intercept, b . See Figure 1.10 and note the domain of t is $0 \leq t \leq 10$.

Since the value of the machine in 10 years will be \$5000, this means that when $t = 10$, $V = 5000$. Thus, $(10, 5000)$ is also on the line. From Figure 1.10, the slope can then be calculated since we now know that the two points $(0, 100,000)$ and $(10, 5000)$ are on the line. Then

$$m = \frac{5000 - 100,000}{10 - 0} = -9500$$

Then, using the point-slope form of a line,

$$V = -9500t + 100,000$$

where the time t is in years since the machine was purchased and V is the value in dollars. Now we can find the value at different time periods,

$$V(1) = -9500(1) + 100,000 = 90,500 \text{ or } \$90,500$$

$$V(2) = -9500(2) + 100,000 = 81,000 \text{ or } \$81,000$$

$$V(9) = -9500(9) + 100,000 = 14,500 \text{ or } \$14,500$$

The rate of depreciation is the slope of the line, $-\$9500/\text{year}$. ✧

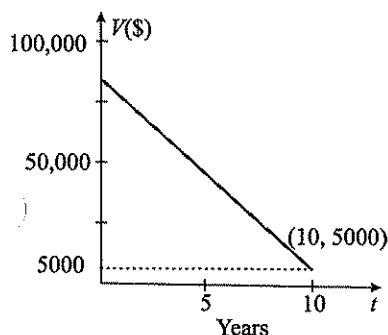


Figure 1.10

Technology Option

Example 6 is solved using a graphing calculator in Technology Note 2 on page 12