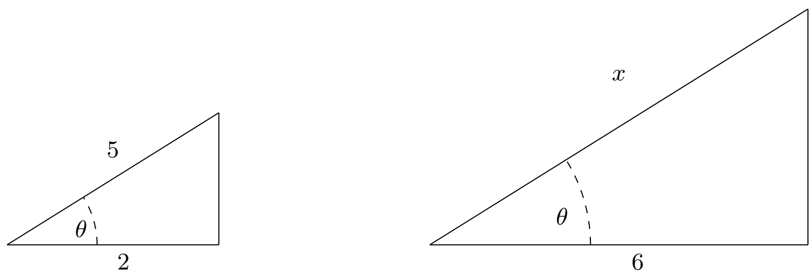
Trigonometric Functions

Trigonometric functions are functions whose inputs are angles, and whose outputs are ratios. They're especially useful for modeling periodic behavior. Let's figure out exactly what they are.

1. Remember similar triangles? Let's say you have the two right triangles below, and they have the same angles. What is x?



2. Then what's the relationship between $\frac{5}{2}$ and $\frac{x}{6}$?

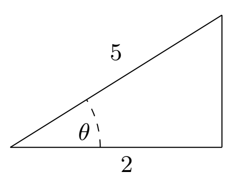
Okay, so it seems that if you take a ratio of the two sides, the actual size of the triangle doesn't matter, just the angles involved. Thus, we can create functions that have as input an angle in a right triangle, and that output a ratio of two sides. To describe these functions, we can name the sides: the *opposite side* is the one directly across from the angle, the *hypotenuse* is the longest side, and the *adjacent* side is the remaining side. Then we can define the following trig functions:

$$\sin(x) = \frac{opp}{hyp} \quad \cos(x) = \frac{adj}{hyp} \quad \tan(x) = \frac{opp}{adj}$$

$$\csc(x) = \frac{hyp}{opp} \quad \sec(x) = \frac{hyp}{adj} \quad \tan(x) = \frac{adj}{opp}$$

The names are short for *sine, cosine, tangent, cosecant, secant* and *cotangent.*

3. Given the triangle below, and the angle θ in that triangle, what are the values of the trig functions if you input θ? (Hint: you might need the Pythagorean Theorem to help.)

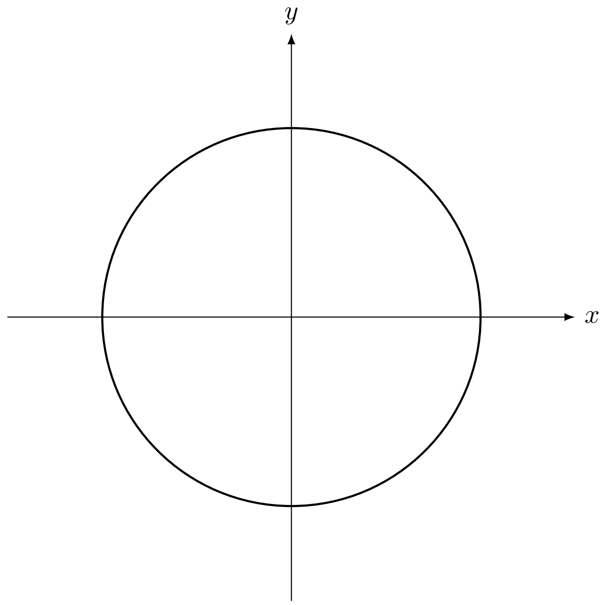


$$\sin(\theta) = \qquad \cos(\theta) = \qquad \tan(\theta) =$$

$$\csc(\theta) = \qquad \sec(\theta) = \qquad \tan(\theta) =$$

Of course, the largest angle that can appear in a right triangle is less than 90 degrees ($\frac{\pi}{2}$ radians). But we have many more angles than that. It would be nice to be able to input *any* angle into a trig function. We can generalize the idea of trig functions to encompass any angle. First, let's look at angles of less than $\frac{\pi}{2}$ radians, and see how we can view them in the unit circle.

4. Let $\theta = \frac{\pi}{3}$. Draw θ in the unit circle.



5. Now connect a vertical line from the point on the circle corresponding to θ down to the x-axis. Together with the radius of the circle and the x-axis, this line forms a triangle. What are the lengths of the sides of this triangle?

6. What is the sine of $\frac{\pi}{3}$? What is the cosine of $\frac{\pi}{3}$?

7. Do the same for $\frac{\pi}{6}$.

8. Can you come up with a pattern that relates the sine and cosine to the coordinates corresponding to an angle?

   In general, the sine of an angle θ is the ______-coordinate of the point corresponding to θ, and the cosine is the ______-coordinate.

That's how we're going to extend the definition of trig functions to any angle. Although only angles less than $\frac{\pi}{2}$ can sit inside a right triangle, every angle on the unit circle has an x-coordinate and a y-coordinate. Thus, we can define the sine and cosine of any angle.

9. What is the sine of $(\frac{2\pi}{3})$?

10. What is $\cos(-\frac{3\pi}{4})$?

11. What is $\sin(\frac{7\pi}{6})$?

12. What is $\cos(\frac{7\pi}{6})$?

13. What is $\cos(\frac{\pi}{2})$?

14. If you know the sine and cosine of an angle, you can say what its tangent is. What is $\tan(\frac{7\pi}{6})$?

15. In fact, you can say what any of the trig functions are on that angle if you just know its sine and cosine. Let's say θ is some angle such that $\sin(\theta) = \frac{1}{3}$, and $\cos(\theta) = \frac{2\sqrt{2}}{3}$. (Note that we're not telling you what θ is...we don't need to!) Then:

$$\tan(\theta) = \qquad \csc(\theta) = \qquad \sec(\theta) = \qquad \tan(\theta) =$$

Now, conceivably, you can calculate trig functions for any angle, as long as you can figure out its coordinates on the unit circle. In general, that's a tricky problem, which is partly why we memorize the coordinates for a few special angles and deal mostly with them, so we can focus on other parts of the theory.

16. What is the cosecant of $\frac{-11\pi}{3}$?

17. What is the tangent of $\frac{3\pi}{2}$?

18. Hopefully it's also clear why trig functions are good for modeling periodic behavior. How do you know that all the trig functions are periodic?