Exponential Growth and Decay

Although the term is used loosely, many real-world phenomena grow or decay at an exponential rate. Population growth can be roughly modeled as an exponential growth process (populations of people, animals, bacteria, and more). Radioactive decay, used for carbon dating, is an example of exponential decay. Even heating and cooling, as we'll see, can be modeled by exponential functions.

1. Say you have a pair of very active rabbits. Every month they have two new rabbits. Ignoring the obvious biological issues with this scenario, suppose also that each pair of their children also have two new rabbits every month (and don't need time to mature). Let's find a function that models the number of rabbits after any number of months.
   (a) How many rabbits are there to start with?
   (b) How many rabbits are there after one month?
   (c) How many rabbits are there after two months?
   (d) How many rabbits are there after three months?
   (e) How many rabbits are there after four months?
   (f) What is a function that expresses the number of rabbits after t months?

$$f(t) =$$

So we have an exponential function that models a very abstract population growth problem. Let's make an adjustment. You saw in our section on change of base formulas that we can really pick any base we like – that is, if we have an exponential function with base 2, we could write that same function with base 3, or base 4, or base e, or whatever base we like, as long as we adjust with some constants. In order to standardize things a bit, we'll do all our exponential models from now on with base e.

2. Verify algebraically that the function you found in the first question is equivalent to $P(t) = 2e^{\ln(2)t}$.

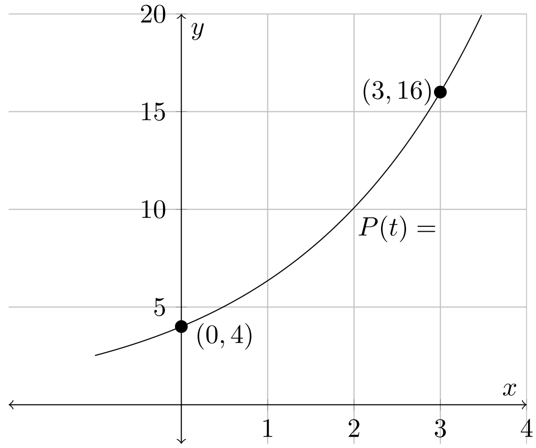
In fact, we'll say that our general form for a function modeling population growth is $P(t) = P_0e^{kt}$. In this situation, P_0 and k are constants that vary, depending on the population we're trying to model. The constant k is sometimes called the *constant of proportionality*. It's a fact that, given any two points not on the same horizontal line, we can find a function of this form that passes through those two points. Let's try that.

3. Say we have the points $(0, 4)$ and $(3, 16)$. Let's find a function of the form $P(t) = P_0e^{kt}$ that passes through those two points. That is, we need to find the values of P_0 and k that "work."
   (a) If $(0, 5)$ is on the graph of the function, that means $P(0) = 4$. Plug that into the formula for our function (P_0e^{kt}) and simplify as much as you can.
   (b) What is P_0?

(c) Now we know P_0! The only thing left is to find k. If $(3, 16)$ is on the graph of the function, then $P(3) = 16$. Plug that into the formula for our function (and this time, plug in the actual value of P_0) and solve for k.

(d) What is k?

(e) You just found the equation for the curve below. Label the curve, just for fun.



4. Find a function of the form $P(t) = P_0e^{kt}$ passing through the points $(0, 7)$ and $(3, 9)$.

5. If you have a function of the form $P(t) = P_0e^{kt}$, what is the "y-intercept" of this function? (We put "y-intercept" in quotes, since our variables are now t and P instead of x and y, but the vertical axis is so commonly referred to as the "y-axis" that we'll just go with it.)

Be careful, however. The pattern you found regarding P_0 and the y-intercept will hold for most of our modeling situations, but not always!

We can do this process even if neither of the two points given to us is the y-intercept. For example, if we're given the two points $(3, 5)$ and $(7, 9)$, we could still find a function of the form $P(t) = P_0 e^{kt}$ passing through these points; it'd just be a bit more work. These two points tell us that

$$5 = P_0 e^{3k} \qquad \text{and} \qquad 9 = P_0 e^{7k}.$$

That means that $\frac{5}{9}$ is the same as $\frac{P_0 e^{3k}}{P_0 e^{7k}}$. So we have another equation, and in particular, one where the P_0's cancel:

$$\frac{5}{9} = \frac{P_0 e^{3k}}{P_0 e^{7k}} = \frac{e^{3k}}{e^{7k}} = e^{-4k}.$$

This is straightforward to solve to find that $-\dfrac{\ln(\frac{5}{9})}{4} = k$. Then, we could plug in for k into either original equation and solve for P_0:

$$5 = P_0 e^{-3\frac{\ln(\frac{5}{9})}{4}} \implies P_0 = \frac{5}{e^{-3\frac{\ln(\frac{5}{9})}{4}}}.$$

We won't do too much of this, mostly because it's messy (as you can see above). And in real-world applications, we can always choose what time we want to be $t = 0$. For example, see the next question.

6. In the year 1800, the world population was about 1 billion. By the year 1900, it was roughly 1.6 billion. Based on those numbers and assuming that the world population was growing at an exponential rate, when would you expect the population to hit 3 billion people?
   (a) We could list these points as $(1800, 1000000000)$ and $(1900, 1600000000)$. But it might be more convenient to make them:

   (b) Find a function $P(t) = P_0 e^{kt}$ modeling this population growth.

   (c) For what value of t will $P(t) = 3$? Set up an equation and solve.

   (d) If you did this all correctly, you should have gotten an answer that, if plugged into a calculator, would be approximately 233.7. Of course, this would be the year 2033, which is a little bit off from the actual answer of 1960. This may be a good point to note that our population models in this course are fairly crude; as you study more math and statistics, you can get much more accurate (but more complicated) models!

7. The populations of Ant Hill A and and Ant Hill B are both growing exponentially. The following population data was recorded one day:

Ant Hill	12:00pm	5:00pm
A	100	120
B	80	150

(a) Find a function, $A(t) = A_0 e^{at}$, that models the population of Ant Hill A at any point in time.

(b) Find a function, $B(t) = B_0 e^{bt}$, that models the population of Ant Hill B at any point in time.

(c) Find the time at which the population of Ant Hill B first exceed the population of Ant Hill A.

(d) Aren't you glad it's not your job to count ants?

Time to work on some other applications. Let's explore a common application, radioactive decay. Certain radioactive isotopes decay at a predictable rate. For example, if you have 100 grams of Carbon-14, we know that after 5730 years, half of it will have decayed and you will have 50 grams left (assuming you're still around 5730 years later). The initial amount does not matter; however much you have, 5730 years later you'll have half left. And 5730 years after that, you'll have half left (so a quarter of your original amount). And so on. This fact is used for carbon-dating, as we can estimate the original amount of Carbon-14 in an object, verify how much remains, and thus tell approximately how old it is.

Other radioactive isotopes will decay at different rates – the time it takes a given isotope to decay by 50% is called its *half-life*.

8. Say a radioactive isotope has a half-life of 10 years. Today, there are 100 grams of it lying around.

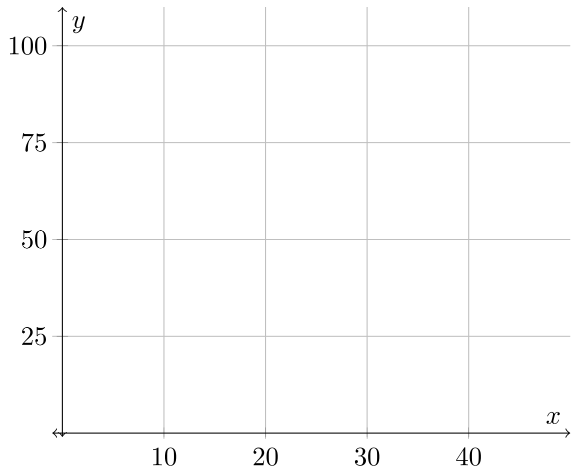
   (a) After 10 years, how much will be left?

   (b) After 20 years, how much will be left?

   (c) After 30 years, how much will be left?

   (d) After 40 years, how much will be left?

   (e) Plot the points you just found on the axes below, where the horizontal axis is time (t = number of years passed) and the vertical is the amount remaining of this isotope.



   (f) Hopefully you got something that looks like an exponential function. In this case, if we write the function in the form $A(t) = A_0 e^{kt}$, we know that k must be

      A. less than 0
      B. between 0 and 1
      C. greater than 1

   (g) That's an interesting fact, but it actually doesn't matter much for finding a function to fit the data. The approach we used for population growth will work here, as well. Find a function of the form $A(t) = A_0 e^{kt}$ to model the amount of radioactive isotope remaining at a given time t. You can pick any two points from the five you plotted above.

9. A scientist isolates 2000 grams of a radioactive isotope. Five hours later, 1800 grams are left.
   (a) Assuming this isotope decays exponentially, find a function of the form $A(t) = A_0 e^{kt}$ to model the amount remaining over time.

   (b) If the scientist returns in 5 more hours, how much of the isotope will remain?

   (c) What is the half-life of this isotope?

Let's move to yet another application of exponential modeling.

10. Suppose we deposit \$500 into a savings account that earns 2% interest, compounded monthly. How much will we have in the account after three years have passed? To answer this, it might be helpful to know that we can compute the future value of an account (F) as follows:

$$F = P(1 + \frac{r}{n})^{nt},$$

where P represents the principal amount (i.e., the amount you start with), r represents the annual interest rate, n represents the number of compounding periods per year, and t represents the number of years.

11. Now something a little more complicated. Suppose we have a savings account that earns 3% interest, compounded monthly. We want to know how long it would take for us to double our money.

    (a) Should the amount of time it takes to double our money depend on how much money we start with? Why or why not?

    (b) Set up an equation to solve, so that the solution will represent the number of years it would take to double our money.

    (c) Solve the equation.

    (d) Express your final answer in terms of the number of months it would take to double your money.

12. Let's finish with a financial prudence lesson. As of now (2015), 3% is a *huge* rate of interest for a savings account – don't expect it. However, interest rates on money owed on credit cards often exceed 20%. Let's suppose you have a credit card with an interest rate of 20%, compounded monthly. Suppose also that you owe \$400 on this credit card. Supposing you don't make any payments, how long will it take for your balance to rise by \$200?

13. Let's explore what happens when you change the number of times you compound per year.

    (a) Suppose you have \$100, and an account that earns 5% interest and is compounded once per year. How much will you have after one year?

    (b) Suppose you have \$100, and an account that earns 5% interest and is compounded twice per year. How much will you have after one year?

    (c) Suppose you have \$100, and an account that earns 5% interest and is compounded 12 times per year. How much will you have after one year?

(d) Suppose you have \$100, and an account that earns 5% interest and is compounded 365 per year. How much will you have after one year?

(e) If you compound more and more often, is the amount of money you can earn unlimited? Explain.

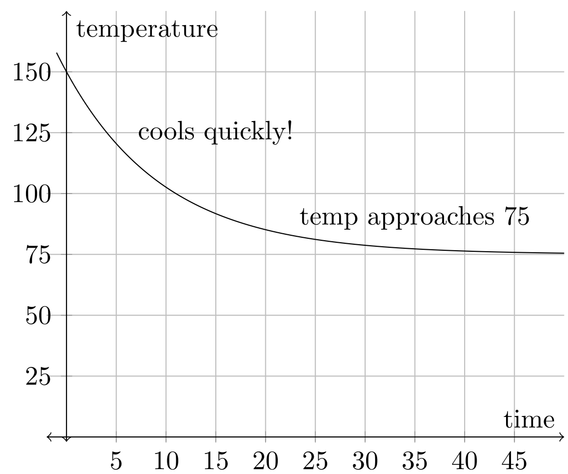
(f) If you took this idea to its limit, kept adding more and more compounding periods, then you would approach what we call *continuous compounding*. That is, compounding not every day, not every minute, not every second, but *continuously*. It turns out that if you have a principal value P, an annual rate r, and you compound continuously for t years, then your future value F is given by

$$F = Pe^{rt}.$$

(There's that e again!) If you have \$100, and an account that earns 5% interest and is compounded continuously, how much will you have after one year?

14. Suppose you have an account that earns 3% interest, compounded continuously. How long would it take for you to double your money?

Now let's explore one last topic: Newton's Law of Heating and Cooling. This is another case where exponential functions are useful. In a simplified form, Newton's Law says that the temperature of an object can be modeled as an exponential curve. For example, if you take a hot potato from the oven and put it into a 75 degree room, the potato's temperature will decay exponentially, falling quickly at first, but then cooling more and more slowly as the temperature of the potato gets closer to 75 degrees. As time goes on, the potato's temperature should pretty much stabilize at 75 degrees. If we were to graph the potato's temperature over time, then, it would look something like:



(Of course, in the real world, the inside of the potato will cool slower than the outside and things are quite a bit more complicated, but we'll stick to the simple model for this class.)

15. What we have, then, is an exponential curve, but one that has been shifted. Write a formula for an exponential curve (any exponential curve) that has a horizontal asymptote at $y = 75$.

In the example above, 75 degrees is the *ambient* temperature. In general, we can model the temperature of an object T with respect to time t via a function of the form:

$$T(t) = T_0 e^{kt} + C,$$

where C is the ambient temperature.

16. Let's work out an example. Suppose a potato comes out of the oven at 160 degrees. We place the potato on the counter in the kitchen, which is 70 degrees. After 10 minutes, we find that the potato is 150 degrees.

    (a) "The potato comes out of the oven at 160 degrees" can be written mathematically as follows (fill in the missing pieces):

    $$T(\quad) = T_0 e^{k \quad} +$$

    (b) Using your equation from above, solve to find T_0.

    (c) Does T_0 represent the initial temperature of the potato?

(d) "After 10 minutes, we find the potato is 150 degrees" can be written mathematically as follows:

(e) Using your equation from above, solve to find k.

(f) What will be the temperature of the potato after 20 minutes?

(g) Your answer to the previous part will be in terms of exponentials and/or logarithms. Supposing you don't have a calculator, we can still use a little common sense. Should your answer to the previous part be a number greater than 140, or less than 140? Why?

(h) How long will it take for the potato to reach 80 degrees?

(i) What about 60 degrees?

17. It's Thanskgiving morning and you have a feast to prepare. You place your thawed, room temperature (75°F) turkey into a preheated oven (375°F) at 10 am. When you check at 11 am, the internal temperature of the turkey has reached 100°F. Will it reach a safe 180°F by 3 pm? (Needs a calculator for final answer...)